ON THE SEMIGROUP OF BOUNDED C^1-MAPPINGS

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Let E be a real Banach space. If $f: E \to E$ is (Fréchet-) differentiable at every point of E, the derivative of f at x is denoted by $f'(x)$, which is an element of the Banach algebra $\mathscr{L} = \mathscr{L}(E)$ of all linear continuous mappings of E into itself with the usual upper bound norm, and, if we put

$$r(f, x, y) = f(x+y) - f(x) - f'(x)(y),$$

we have

$$\lim_{\|y\| \to 0} \|y\|^{-1} \|r(f, x, y)\| = 0.$$

If $f: E \to E$ is differentiable at every point and the mapping $f': E \to \mathscr{L}$ is continuous, f is called a C^1-mapping. If, moreover, f' is a bounded mapping (i.e., maps every bounded subset of E into a bounded subset of $\mathscr{L}$), then f is called a BC^1-mapping. Evidently, if E is finite-dimensional, every C^1-mapping is a BC^1-mapping. *The set of all BC^1-mappings of E into itself is denoted by $\mathscr{BC}^1$ or $\mathscr{BC}^1(E)$* [5, p. 777].

We give to $\mathscr{BC}^1$ the metric topology defined by the sequence of semi-norms:

$$\|f\|_n = \sup_{\|x\| \leqq n} \{\|f(x)\| + \|f'(x)\|\}$$

for $n = 1, 2, \cdots$. A proof of the completeness of this metric topology can be found in [1, p. 24], where $\mathscr{BC}^1$ is denoted by $\mathscr{B}^1$.

For $f_i \in \mathscr{BC}^1$ $(i = 0, 1, 2, \cdots)$, we write $f_i \Rightarrow f_0$ if $\{f_i\}_{i=1,2,\ldots}$ converges to f_0 in this metric topology.

Obviously, $\mathscr{BC}^1$ is a *semigroup* with respect to the product

$$(fg)(x) = f(g(x)) \text{ for every } x \in E.$$

The purpose of this paper is to prove the following theorem.

THEOREM. *Every topological automorphism ϕ of the semigroup $\mathscr{BC}^1$ is inner, that is, there exists $h \in \mathscr{BC}^1$ such that $h^{-1} \in \mathscr{BC}^1$ and*

$$\phi(f) = hfh^{-1} \text{ for every } f \in \mathscr{BC}^1.$$

REMARK 1. Eidelheit [6] has proved that every continuous automorphism of the *semigroup* $\mathscr{L}$ is inner. He has also proved that every automorphism of the *ring* $\mathscr{L}$ is inner. At the end of this paper we shall prove the corresponding fact for the *near-ring* $\mathscr{BC}^1$.

REMARK 2. It is easy to see that the following facts can be proved in the same way: *Let E_1 and E_2 be real Banach spaces, and $\mathscr{BC}^1(E_1)$ and $\mathscr{BC}^1(E_2)$ be the corresponding sets of BC^1-mappings. Then,* (1) *if the semigroups $\mathscr{BC}^1(E_1)$ and $\mathscr{BC}^1(E_2)$ are homeomorphic, then E_1 and E_2 are BC^1-diffeomorphic*; (2) *if the near-rings $\mathscr{BC}^1(E_1)$ and $\mathscr{BC}^1(E_2)$ are (algebraically) isomorphic, then E_1 and E_2 are topologically linearly isomorphic.*

REMARK 3. In the case of the semigroup $\mathscr{D}$ of all differentiable mappings of E into itself, Magill, [7] has shown that $\mathscr{D}$ has the property that every automorphism is inner, when E is one-dimensional. In [8], a necessary and sufficient condition for $\mathscr{D}$ to have this property for general E has been given.

Proof of the theorem

We assume that ϕ is a bicontinuous bijection of the metric semigroup $\mathscr{BC}^1 = \mathscr{BC}^1(E)$ such that

$$\phi(fg) = \phi(f)\phi(g) \text{ for } f, g \in \mathscr{BC}^1.$$

Since we can start with ϕ^{-1} instead of ϕ, we shall make free use of the fact that any fact established for ϕ is enjoyed also by ϕ^{-1}.

In the following, the Greek letters $\alpha, \beta, \varepsilon, \xi$ and η denote real numbers.

1. *The existence of the bijection h.*

For any $a \in E$, the *constant* mapping whose value is a is denoted by c_a: $c_a(x) = a$ for every $x \in E$. Obviously, $c_a \in \mathscr{BC}^1$ for any $a \in E$. Therefore, as in the proof of Magill's theorem [7], we can prove that there exists a bijection $h: E \to E$ such that $\phi(c_a) = c_{h(a)}$ and

$$\phi(f) = hfh^{-1} \text{ for any } f \in \mathscr{BC}^1. \tag{1}$$

By the same reason as in [8, p. 505], *we can assume that* $h(0) = 0$.

2. *h is continuous.*

If $x_\iota \to x_0$, then $c_{x_\iota} \Rightarrow c_{x_0}$. Since ϕ is continuous, $\phi(c_{x_\iota}) \Rightarrow \phi(c_{x_0})$, which implies that $h(x_\iota) \to h(x_0)$.

3. *The limit* $\lim_{\varepsilon \to 0} \varepsilon^{-1}[h((1+\varepsilon)a) - h(a)]$ *exists for every* $a \in E$.

If we regard ε as the mapping $x \to \varepsilon x$, then $\varepsilon \in \mathscr{BC}^1$ and the existence of this limit is equivalent to the existence of the limit:

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$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[\phi(e^{\varepsilon})-1](a)$$

for any $a \in E$, because, if we put $e^{\varepsilon}-1=\eta$ and $h^{-1}(a)=b$, it follows from (1) that

$$\varepsilon^{-1}[\phi(e^{\varepsilon})-1](a)=\varepsilon^{-1}(e^{\varepsilon}-1)\eta^{-1}[h((1+\eta)b)-h(b)].$$

Therefore, we meet here a *one-parameter group of diffromorphisms* $\{\phi(e^{\varepsilon})\}$ and what we need is the differentiability with respect to the parameter. On this subject in the finite-dimensional spaces, we have the classical Bochner-Montgomery theorem [2], in which the mean-value theorem played an essential role. Since the infinite-dimensional mean-value theorem is different from the finite-dimensional one, it seems to be impossible to apply directly the Bochner-Montgomery theorem to our case. However, in the following, we shall show that Dorroh's ingenious method [4] enables us to by-pass this difficulty. Except for minor changes, we shall reproduce Dorroh's argument. *We denote* $\phi(e_{\xi})$ *by* $\psi(\xi)$.

Let $a \in E$ be fixed. If $\xi \to \alpha$, then, since $\xi \Rightarrow \alpha$, $\psi(\xi) \Rightarrow \psi(\alpha)$, which implies that $\psi(\xi)'(a) \to \psi(\alpha)'(a)$. Therefore, $\psi(\xi)'(a)$ is continuous with respect to ξ. Hence, we can find $\alpha > 0$ such that $0 \leqq \xi \leqq \alpha$ implies

(2) $$\|\psi(\xi)'(a)-1\| < \tfrac{1}{2} \text{ and } \|\psi(\xi)(a)-a\| < 1.$$

If we put

$$u=\alpha^{-1}\int_0^{\alpha}\psi(\xi)'(a)d\xi,$$

then $u \in \mathscr{L}$ and $\|u-1\| < \frac{1}{2}$. Therefore, u is invertible and $\|u(x)\| \geqq \frac{1}{2}\|x\|$ for $x \in E$. Now, if we put

$$\Phi(\varepsilon)=\varepsilon^{-1}(\psi(\varepsilon)-1),$$

we have

$$\begin{aligned}
a_{\alpha,\varepsilon} &\equiv \alpha^{-1}\int_0^{\alpha}\Phi(\varepsilon)\,\psi(\xi)(a)d\xi=(\alpha\varepsilon)^{-1}\int_0^{\alpha}(\psi(\xi+\varepsilon)(a)-\psi(\xi)(a))d\xi\\
&= (\alpha\varepsilon)^{-1}\left(\int_{\varepsilon}^{\alpha+\varepsilon}-\int_0^{\alpha}\right)\psi(\xi)(a)d\xi\\
&= (\alpha\varepsilon)^{-1}\int_0^{\varepsilon}(\psi(\xi+\alpha)(a)-\psi(\xi)(a))d\xi\\
&= \varepsilon^{-1}\int_0^{\varepsilon}\Phi(\alpha)\psi(\xi)(a)d\xi \to \Phi(\alpha)(a) \qquad \text{if} \quad \varepsilon\to 0.
\end{aligned}$$

Therefore,

(3) $$a_{\alpha,\varepsilon} \to \Phi(\alpha)(a) \qquad \text{if } \varepsilon \to 0,$$

and, hence, for sufficiently small ε, we have, by (2),

$$\| a_{\alpha,\varepsilon} \| < \| \Phi(x)(a) \| + 1 < \alpha^{-1} + 1. \tag{4}$$

On the other hand, since [4, p. 318]

$$\| \Phi(\varepsilon)\psi(\xi)(a) - \psi(\xi)'(a)\Phi(\varepsilon)(a) \| \leqq \| \Phi(\varepsilon)(a) \| \sup_{x \in [a, \Psi(\varepsilon)(a)]} \| \psi(\xi)'(x) - \psi(\xi)'(a) \|,$$

where $[a, \psi(\varepsilon)(a)]$ is the segment connecting a and $\psi(\varepsilon)(a)$, we have

$$\| a_{\alpha,\varepsilon} - u(\Phi(\varepsilon)a) \| = \| \alpha^{-1} \int_0^\alpha [\Phi(\varepsilon)\psi(\xi)(a) - \psi(\xi)'(a)\Phi(\varepsilon)(a)] d\xi$$

$$\leqq \| \Phi(\varepsilon)(a) \| \sup_{0 \leqq \xi \leqq \alpha} \sup_{x \in [a, \psi(\varepsilon)(a)]} \| \psi(\xi)'(x) - \psi(\xi)'(a) \|. \tag{5}$$

Moreover,

$$\sup_{0 \leqq \xi \leqq \alpha} \sup_{x \in [a, \psi(\varepsilon)(a)]} \| \psi(\xi)'(x) - \psi(\xi)'(a) \| \to 0 \quad \text{if } \varepsilon \to 0. \tag{6}$$

In fact, if this is not true, there exist $\varepsilon_i, \beta > 0$ and ξ_i such that $\varepsilon_i \to 0, 0 \leqq \xi_i \leqq \alpha$ and

$$\sup_{x \in [a, \psi(\varepsilon_i)(a)]} \| \psi(\xi_i)'(x) - \psi(\xi_i)'(a) \| > \beta.$$

Taking a subsequence if necessary, we can assume that $\xi_i \to \xi_0$, and we can find $x_i \in [a, \psi(\varepsilon_i)(a)]$ such that

$$\| \psi(\xi_i)'(x_i) - \psi(\xi_i)'(a) \| > \beta.$$

Since $\psi(\varepsilon_i) \to 1$ if $i \to \infty$, we have $x_i \to a$ if $i \to \infty$, and, for n such that $\| x_i \|$, $\| a \| \leqq n$, we have

$$\begin{aligned} &\| \psi(\xi_i)'(x_i) - \psi(\xi_i)'(a) \| \\ &\leqq \| \psi(\xi_i)'(x_i) - \psi(\xi_0)'(x_i) \| + \| \psi(\xi_0)'(x_i) - \psi(\xi_0)'(a) \| + \| \psi(\xi_0)'(a) - \psi(\xi_i)'(a) \| \\ &\leqq 2 \sup_{\|x\| \leqq n} \| \psi(\xi_i)'(x) - \psi(\xi_0)'(x) \| + \| \psi(\xi_0)'(x_i) - \psi(\xi_0)'(a) \| \\ &\to 0 \quad \text{if } i \to \infty, \end{aligned}$$

because $\xi_i \Rightarrow \xi_0$ and $\psi(\xi_0)$ is a C^1-mapping. Therefore, we have (6). Thus, from (5) it follows that

$$\| \Phi(\varepsilon)(a) \|^{-1} \| a_{\alpha,\varepsilon} - u(\Phi(\varepsilon)(a)) \| \to 0 \quad \text{if } \varepsilon \to 0, \tag{7}$$

or, for sufficiently small ε, we have

$$\| \Phi(\varepsilon)(a) \|^{-1} \| a_{\alpha,\varepsilon} - u(\Phi(\varepsilon)(a)) \| < \tfrac{1}{4},$$

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hence it follows that

$$\tfrac{1}{4}\|\Phi(\varepsilon)(a)\| > \|u(\Phi(\varepsilon)(a))\| - \|a_{\alpha,\varepsilon}\| \geqq \tfrac{1}{2}\|\Phi(\varepsilon)(a)\| - \|a_{\alpha,\varepsilon}\|,$$

and, by (4),

$$\|\Phi(\varepsilon)(a)\| \leqq 4\|a_{\alpha,\varepsilon}\| < 4(\alpha^{-1}+1).$$

Therefore, from (7),

$$\|a_{\alpha,\varepsilon} - u(\Phi(\varepsilon)(a))\| \to 0 \qquad \text{if } \varepsilon \to 0.$$

Then, by (3), we have

$$u(\Phi(\varepsilon)(a)) \to \Phi(\alpha)(a) \qquad \text{if } \varepsilon \to 0,$$

and, since u is invertible,

$$\Phi(\varepsilon)(a) \to u^{-1}(\Phi(\alpha)(a)).$$

Thus, the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[\phi(e^{\varepsilon})(a) - a]!$$

exists for every $a \in E$, and hence the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[h((1+\varepsilon)a) - h(a)]$$

exists for every $a \in E$. *We denote this limit by* $h^*(a)(a)$.

4. *The limit* $\lim_{\varepsilon\to 0}\varepsilon^{-1}[h(b+\varepsilon a) - h(b)]$ *exists for any* $a \in E$ *and any* $b \in E$.

For any $a \in E$ and for any $b \in E$, if we put

$$t = (1 + c_b)(1 - c_a),$$

then $t \in \mathscr{BC}^1$ and

$$h(b+\varepsilon a) - h(b) = ht((1+\varepsilon)a) - ht(a) = \phi(t)h((1+\varepsilon)a) - \phi(t)h(a)$$
$$= \phi(t)'(h(a))[h((1+\varepsilon)a) - h(a)] + r(\phi(t), h(a), h((1+\varepsilon)a) - h(a)).$$

Therefore,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}[h(b+\varepsilon a) - h(b)] = \phi(t)'(h(a))(h^*(a)(a)). \tag{8}$$

We denote this limit by $h^*(b)(a)$.

5. $h^*(\xi a)(a)$ *is continuous with respect to* ξ.

If $\xi_i \to \xi_0$, since $C_{\xi_i a} \Rightarrow C_{\xi_0 a}$, for

$$t_i = (1 + c_{\xi_i a})(1 - c_a) \qquad (i = 0, 1, 2, \ldots),$$

we have $t_i \Rightarrow t_0$ and, since ϕ is continuous, $\phi(t_i) \Rightarrow \phi(t_0)$. Therefore, by (8). $h^*(\xi_i a)(a) \to h^*(\xi_0 a)(a)$.

For $a \in E$ and $\bar{a} \in \bar{E}$ (the conjugate space of E), we denote by $a \otimes \bar{a}$ the mapping defined by

$$(a \otimes \bar{a})(x) = \langle x, \bar{a} \rangle a,$$

where $\langle x, \bar{a} \rangle$ is the value of $\bar{a}$ at x.

6. *$h(a \otimes \bar{a}) \in \mathscr{BC}^1$ for any $a \otimes \bar{a}$.*

In fact, since, by (8),

$$\begin{aligned} &\|y\|^{-1} \| h(a \otimes \bar{a})(x+y) - h(a \otimes \bar{a})(x) - \langle y, a \rangle h^*(\langle x, \bar{a} \rangle a)(a) \| \\ = &\|y\|^{-1} \| h(\langle x, \bar{a} \rangle a + \langle y, \bar{a} \rangle a) - h(\langle x, \bar{a} \rangle a) - \langle y, \bar{a} \rangle h^*(\langle x, \bar{a} \rangle a)(a) \| \\ = &\|y\|^{-1} |\langle y, \bar{a} \rangle| \; \| \langle y, \bar{a} \rangle^{-1} \{ h(\langle x, \bar{a} \rangle a + \langle y, \bar{a} \rangle a) - h(\langle x, \bar{a} \rangle a) \} \\ &\qquad - h^*(\langle x, \bar{a} \rangle a)(a) \| \to 0 \qquad \text{if } \|y\| \to 0, \end{aligned}$$

the mapping $h(a \otimes \bar{a})$ is differentiable and

$$(h(a \otimes \bar{a}))'(x)(y) = \langle y, \bar{a} \rangle h^*(\langle x, \bar{a} \rangle a)(a).$$

To prove that $h(a \otimes \bar{a})$ is *continuously* differentiable, assume that $x_i \to x_0$. Then, by 5

$$\begin{aligned} &\| (h(a \otimes \bar{a}))'(x_i) - (h(a \otimes \bar{a}))'(x_0) \| \\ = &\sup_{\|y\| \leqq 1} \| \{ (h(a \otimes \bar{a}))'(x_i) - (h(a \otimes \bar{a}))'(x_0) \} (y) \| \\ = &\sup_{\|y\| \leqq 1} |\langle y, \bar{a} \rangle| \; \| \{ h^*(\langle x_i, \bar{a} \rangle) - h^*(\langle x_0, \bar{a} \rangle a) \} (a) \| \\ = &\|\bar{a}\| \; \| h^*(\langle x_i, \bar{a} \rangle a)(a) - h^*(\langle x_0, \bar{a} \rangle a)(a) \| \to 0 \qquad \text{if } i \to \infty. \end{aligned}$$

Moreover, for each n,

$$\sup_{\|x\| \leqq n} \| (h(a \otimes \bar{a}))'(x) \| = \sup_{\|x\| \leqq n} \|\bar{a}\| \; \| h^*(\langle x, \bar{a} \rangle a)(a) \| < \infty,$$

because $\| h^*(\langle x, \bar{a} \rangle a)(a) \|$ is continuous with respect to x and $|\langle x, \bar{a} \rangle| \leqq n \|\bar{a}\|$.

7. *$(a \otimes \bar{a})h \in \mathscr{BC}^1$ for any $a \otimes \bar{a}$.*

This follows from 6 and $(a \otimes \bar{a})h = \phi^{-1}(h(a \otimes \bar{a}))$.

8. *$h^*(x) \in \mathscr{L}$ for any $x \in E$.*

From 7, we have

$$((a \otimes \bar{a})h)'(x)(y) = \langle h^*(x)(y), \bar{a} \rangle a. \tag{9}$$

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Therefore, $h^*(x)(y)$ is linear with respect to y and

$$\sup_{\|y\| \leqq 1} |\langle h^*(x)(y), \bar{a}\rangle| < \infty$$

for any $\bar{a} \in \bar{E}$. Therefore, $h^*(x) \in \mathscr{L}$ for any $x \in E$.

9. $h \in \mathscr{BC}^1$.

Because of [3, Problem 1, p. 169], to prove that h is a C^1-mapping, we have only to show that h^* is continuous as a mapping of E into $\mathscr{L}$. To do this, we use the following equality:

$$h^*(x) = \phi(1 + c_x)'(0)h^*(0),$$

which follows immediately from the definition of the derivatives. Now, assume that $x_i \to x_0$. Then, since $c_{x_i} \Rightarrow c_{x_0}$, we have $\phi(1 + c_{x_i}) \Rightarrow \phi(1 + c_{x_0})$, which implies $\phi(1 + c_{x_i})'(0) \to \phi(1 + c_{x_0})'(0)$. Therefore,

$$\| h^*(x_i) - h^*(x_0) \| \leqq \| \phi(1 + c_{x_i})'(0) - \phi(1 + c_{x_0})'(0)\| \; \| h^*(0) \| \to 0$$

if $i \to \infty$. Moreover, from (9) it follows that, for each n and each $\bar{a} \in \bar{E}$,

$$\sup_{\|x\| \leqq n} \left\{ \sup_{\|y\| \leqq 1} |\langle h'(x)(y), \bar{a}\rangle| \right\} < \infty,$$

which implies that

$$\sup_{\|x\| \leqq n} \| h'(x) \| < \infty.$$

Thus the proof is completed.

BC^1 as a near-ring

In addition to the product, by

$$(f + g)(x) = f(x) + g(x) \qquad \text{for every } x \in E,$$

we can define the addition in $\mathscr{BC}^1$, and, with these two operations, $\mathscr{BC}^1$ is a near-ring.

Let ϕ be a near-ring automorphism of $\mathscr{BC}^1$, i.e., ϕ is a bijection of $\mathscr{BC}^1$ such that

$$\phi(fg) = \phi(f)\phi(g) \text{ and } \phi(f + g) = \phi(f) + \phi(g) \qquad \text{for } f, g \in \mathscr{BC}^1.$$

We shall prove that *every near-ring automorphism of the near-ring* $\mathscr{BC}^1$ *is inner and the mapping* h *is a topological linear isomorphism of* E.

To prove this, we first see that we have a bijection h such that the condition (1) is satisfied, by the same reason as in the case of the semigroup theory.

Secondly, as we have proved in [8], h is weakly continuous.

Thirdly, h is linear, because

$$\begin{aligned} h(\alpha x + \beta y) &= h(c_{\alpha x + \beta y})(z) = h(\alpha c_x + \beta c_y)(z) \\ &= \phi(\alpha c_x + \beta c_y)h(z) = [\phi(\alpha)\phi(c_x) + \phi(\beta)\phi(c_y)]h(z) \\ &= \phi(\alpha)h(x) + \phi(\beta)h(y), \end{aligned}$$

where, since the equation $\phi(\xi + \eta) = \phi(\xi) + \phi(\eta)$ implies that $\phi(\xi) = \xi$ for rational numbers ξ, from the weak continuity of h it follows that $\phi(\xi) = \xi$ for all real numbers ξ.

Thus, we have only to prove the continuity of h. Since $\mathscr{L} \subset \mathscr{BC}^1$, we can consider the mapping $\psi: \mathscr{L} \to \mathscr{L}$ defined by

$$\psi(u) = \phi(u)'(0) \qquad \text{for } u \in \mathscr{L}.$$

Then, for any $u \in \mathscr{L}$, since h is linear,

$$\psi(u)(x) = \lim_{\varepsilon \to 0} \varepsilon^{-1} huh^{-1}(\varepsilon x) = huh^{-1}(x) = \phi(u)(x).$$

Therefore, ψ is injective. Moreover, it is surjective, because, for any $u \in \mathscr{L}$, if we take $f \in \mathscr{BC}^1$ such that $\phi(f) = u$, then

$$\begin{aligned} u(x) &= \phi(f)'(0)(x) = \lim_{\varepsilon \to 0} \varepsilon^{-1} hfh^{-1}(\varepsilon x) \\ &= \text{weak} - \lim_{\varepsilon \to 0} h(\varepsilon^{-1} f \varepsilon)h^{-1}(x) = hf'(0)h^{-1}(x) \\ &= \psi(f'(0))(x) \end{aligned}$$

for every $x \in E$. Therefore, by the theorem of Eidelheit [6] there exists a topological linear isomorphism $h_0: E \to E$ such that

$$\psi(u) = \phi(u) = h_0 u h_0^{-1} \qquad \text{for } u \in \mathscr{L},$$

from which it follows that $h_0 = h$, and $h \in \mathscr{L}$.

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