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A Mahler Measure of a $K3$ Surface Expressed as a Dirichlet L-Series

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Abstract. We present another example of a 3-variable polynomial defining a $K3$-hypersurface and having a logarithmic Mahler measure expressed in terms of a Dirichlet L-series.

1 Introduction

The logarithmic Mahler measure $m(P)$ of a Laurent polynomial $P \in \mathbb{C}[X_1^{\pm}, \dots, X_n^{\pm}]$ is defined by

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1^{\pm}, \dots, x_n^{\pm})| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n$ is the n-torus $\{(x_1, \dots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}$.

For $n = 2$ and polynomials P defining elliptic curves E, conjectures have been made, with proofs in the CM case, by various authors [6, 13]. These conjectures give conditions on the polynomial P for getting explicit expressions of $m(P)$ in terms of the L-series of E. A crucial condition for P is to be "tempered," that is, the roots of the polynomials of the faces of its Newton polygon are only roots of unity. This condition is related to the link between $m(P)$ and the second group of K-theory, [1, 13].

We are trying to generalize these results for $n = 3$. Since both elliptic curves and $K3$ surfaces are Calabi–Yau varieties, we take polynomials in three variables defining $K3$ surfaces. So we consider two families (P_k) and (Q_k):

$$\begin{aligned} P_k &= X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} - k \\ Q_k &= X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} \\ &\qquad + XY + \frac{1}{XY} + ZY + \frac{1}{ZY} + XYZ + \frac{1}{XYZ} - k. \end{aligned}$$

The families (P_k) and (Q_k) are respectively generalizations of the modular families of elliptic curves

$$X + \frac{1}{X} + Y + \frac{1}{Y} - k \quad \text{and} \quad X + \frac{1}{X} + Y + \frac{1}{Y} + XY + \frac{1}{XY} - k$$

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studied variously by Bertin [2], Boyd [6], Lalin and Rogers [9], Rodriguez-Villegas [13].

We recall the main results obtained in [3–5]. If Y_k (resp. Z_k) denotes the $K3$ surface defined by the polynomial P_k (resp. Q_k) with transcendental lattice $T(Y_k)$ (resp. $T(Z_k)$) and L-series $L(Y_k, s)$ (resp. $L(Z_k, s)$), then

$$\begin{aligned}
m(P_0) &= d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),\\
m(P_2) &= \frac{\mid T(Y_2) \mid^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3),\\
m(P_6) &= \frac{\mid T(Y_6) \mid^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3),\\
m(P_{10}) &= \frac{\mid T(Y_{10}) \mid^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3,\\
m(Q_0) &= \frac{\mid T(Z_0) \mid^{3/2}}{2\pi^3} L(Z_0, 3) = \frac{12\sqrt{12}}{2\pi^3} L(f_{12}, 3),\\
m(Q_{12}) &= 2\frac{\mid T(Z_{12} \mid^{3/2}}{\pi^3} L(Z_{12}, 3) = 2\frac{12\sqrt{12}}{\pi^3} L(f_{12}, 3).
\end{aligned}$$

Here f_N denotes the unique, up to twist, CM-newform, CM by $\mathbb{Q}(\sqrt{-N})$, of weight 3 and level N with rational coefficients [15].

Other formulae of the same type are under preparation for $m(P_3)$ and $m(P_{18})$.

All these results concern "singular" $K3$ surfaces, that is, $K3$ surfaces with Picard number 20. The corresponding k were computed by Boyd.[1] Very recently, Elkies and Schütt [7] wrote an algorithm for finding "singular" $K3$ surfaces in a family of $K3$ surfaces of generic Picard number 19. Their computations agreed with Boyd's and gave only two extra values for k^2 in the first family.

The reason why we may expect the Mahler measure $m(P)$ to be related to L-functions or modular forms is the following. Let $P \in \mathbb{C}[x, y, z]$ and define the differential form $\eta(x, y, z)$ on the surface $S = \{P(x, y, z) = 0\}$ minus the set Z of zeros and poles of x, y, and z

$$\begin{aligned}
\eta(x, y, z) := \log|x|(\tfrac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| - \mathrm{d}\arg y \wedge \mathrm{d}\arg z)\\
\log|y|(\tfrac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| - \mathrm{d}\arg z \wedge \mathrm{d}\arg x)\\
\log|z|(\tfrac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| - \mathrm{d}\arg x \wedge \mathrm{d}\arg y).
\end{aligned}$$

We can express the Mahler measure of P as

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x, y, z),$$

[1] Personal communication.

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where P^* is the leading coefficient of the polynomial $P \in \mathbb{C}[x, y][z]$ and

$$\Gamma = \{P(x, y, z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}.$$

In fact, η is a closed form on $S \backslash Z$. But in our situation concerning polynomials P_k and Q_k, the form is not exact: the set Γ consists of closed subsets and the integral can be computed by residues. We are led to instances of Beilinson's conjectures that produce special values of L-functions of surfaces. The fact that these L-functions are also the Mellin transforms of CM-newforms of weight 3 [15] derives from Livné's theorem of modularity of "singular" $K3$-surfaces defined over $\mathbb{Q}$ [11, 17].

These examples and the future ones are extremely important to help us answer the following question: for which class of polynomials in three variables defining $K3$ surfaces, can the Mahler measure be expressed in terms of the L-series of the variety plus a Dirichlet L-series? How important are the faces of the Newton polyhedron of the polynomial and the fact that the polynomial does not intersect the torus?

The result obtained in the following theorem is the second example where the Mahler measure of a polynomial defining a $K3$ surface is expressed only in terms of a Dirichlet L-series, that is, only in terms of the measure of faces.

Theorem 1.1 *With the above notations we get* $m(Q_{-3}) = \frac{8}{5} d_3$.

In this theorem, the evaluation of the modular part needs the use of Serre–Livné's criterion [10], since we must compare two l-adic representations, and also recent results about Dirichlet L-series [19].

2 Some Facts

The polynomial Q_{-3} belongs to the family of polynomials Q_k whose Mahler measure has been studied in a previous paper [3].

Theorem 2.1

$$Q_k = X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} + XY + \frac{1}{XY} + ZY + \frac{1}{ZY} + XYZ + \frac{1}{XYZ} - k.$$

Let $k = -(t + \frac{1}{t}) - 2$ *and define*

$$t = \frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^8 \eta(6\tau)^{12}},$$

where η *denotes the Dedekind eta function*

$$\eta(\tau) = e^{(\pi i \tau)/12} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}).$$

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Then

$$\begin{aligned} m(Q_k) = \frac{\Im\tau}{8\pi^3}\Big\{ \sum_{m,\kappa}{}' \Big(& 2\Big(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar\tau+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar\tau+\kappa)^2}\Big) \\ & - 32\Big(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar\tau+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar\tau+\kappa)^2}\Big) \\ & - 18\Big(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar\tau+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar\tau+\kappa)^2}\Big) \\ & + 288\Big(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar\tau+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar\tau+\kappa)^2}\Big)\Big)\Big\}. \end{aligned}$$

Let us now recall the following results.

Given a normalised Hecke eigenform f of some level N and weight $k = 3$, we can associate a Galois representation [8, 14] $\rho_f\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(2, \mathbb{Q}_l)$.

To a normalised Hecke newform f can also be associated an L-function $L(f, s)$ by $L(f, s) := L(\rho_f, s)$ (the L-series of the Galois representation ρ_f). Equivalently, if f has a Fourier expansion $f = \sum_n b_n q^n$, then $L(f, s)$ is also the Mellin transform of f

$$L(f, s) = \sum_n \frac{b_n}{n^s}.$$

Moreover, the series $L(f, s)$ has a product expansion

$$L(f, s) = \sum_{n\geq 1} \frac{b_n}{n^s} = \prod_p \frac{1}{1 - b_p p^{-s} + \chi(p)p^{k-1-2s}}$$

where $\chi(p) = 0$ if $p \mid N$.

Concerning the comparison between l-adic representations, Serre's and Livné's result can be found, for example, in [12, 17].

Lemma 2.2 *Let $\rho_l, \rho_l'\colon G_{\mathbb{Q}} \to \mathrm{Aut}\, V_l$ two rational l-adic representations with $\mathrm{Tr}\, F_{p,\rho_l} = \mathrm{Tr}\, F_{p,\rho_l'}$ for a set of primes p of density one,* i.e., *for all but finitely many primes. If ρ_l and ρ_l' fit into two strictly compatible systems, the L-functions associated with these systems are the same.*

Then the great idea in [10] is to replace this set of primes of density one by a finite set.

Definition 2.3 A finite set T of primes is said to be an effective test set for a rational Galois representation $\rho_l\colon G_{\mathbb{Q}} \to \mathrm{Aut}\, V_l$ if the previous lemma holds with the set of density one replaced by T.

Definition 2.4 Let $\mathcal{P}$ denote the set of primes, S a finite subset of $\mathcal{P}$ with r elements, $S' = S \cup \{-1\}$. Define for each $t \in \mathcal{P}$, $t \neq 2$ and each $s \in S'$ the function

$$f_s(t) := \frac{1}{2}\Big(1 + \Big(\frac{s}{t}\Big)\Big)$$

and if $T \subset \mathcal{P}$, $T \cap S = \varnothing$, $f\colon T \to \big(\mathbb{Z}/2\mathbb{Z}\big)^{r+1}$ such that $f(t) = (f_s(t))_{s\in S'}$.

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Theorem 2.5 (Livné's criterion) *Let ρ and ρ' be two 2-adic $G_{\mathbb{Q}}$-representations which are unramified outside a finite set S of primes, satisfying*

$$\operatorname{Tr} F_{p,\rho} \equiv \operatorname{Tr} F_{p,\rho'} \equiv 0 \pmod 2 \quad \text{and} \quad \det F_{p,\rho} \equiv \det F_{p,\rho'} \pmod 2$$

for all $p \notin S \cup \{2\}$.

Any finite set T of rational primes disjoint from S with $f(T) = (\mathbb{Z}/2\mathbb{Z})^{r+1} \backslash \{0\}$ is an effective test set for ρ with respect to ρ'.

The $K3$ surface $\tilde{X}$ defined by the polynomial Q_{-3} has been studied by Peters, Top and van der Vlugt [12]. In particular they proved the theorem.

Theorem 2.6 *There is a system $\rho = (\rho_l)$ of 2-dimensional l-adic representations of $G_{\mathbb{Q}} = \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ $\rho_l \colon G_{\mathbb{Q}} \to \operatorname{Aut} H^2_{\mathrm{trc}}(\tilde{X}, \mathbb{Q}_l)$. The system $\rho = (\rho_l)$ has an L-function*

$$L(s, \rho) = \prod_{p \neq 3,5} \frac{1}{1 - A_p p^{-s} + \left(\frac{p}{15}\right) p^2 p^{-2s}}.$$

This L-function is the L-function of the modular form $f^+ = g\theta_1 \in S_3(15, (\frac{\cdot}{15}))$, where

$$\theta_1 = \sum_{m,n \in \mathbb{Z}} q^{m^2 + mn + 4n^2} \qquad g = \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

and η is the Dedekind eta function. The Mellin transform $\sum \frac{b_n}{n^s}$ of f^+ satisfies $b_p = A_p$ for $p \neq 3, 5$, where A_p can be computed as follows.

- *If $p \equiv 1$ or $4 \mod 15$, find an integral solution of the equation $x^2 + xy + 4y^2 = p$. Then $A_p = 2x^2 - 7y^2 + 2xy$.*
- *If $p \equiv 2$ or $8 \mod 15$, find an integral solution of the equation $2x^2 + xy + 2y^2 = p$. Then $A_p = x^2 + 8xy + y^2$.*

3 Proof of Theorem 1.1

The proof follows from three propositions.

Proposition 3.1

$$\begin{aligned} m(Q_{-3}) = \frac{3\sqrt{15}}{\pi^3} \sum_{m',\kappa}{}' \Big(&\frac{15k^2 - m'^2}{(m'^2 + 15\kappa^2)^3} + \frac{-5k^2 + 3m'^2}{(3m'^2 + 5\kappa^2)^3} \Big) \\ &+ \Big(\frac{1}{2} \frac{2m'^2 + 2m'\kappa - 7\kappa^2}{(m'^2 + m'\kappa + 4\kappa^2)^3} + \frac{1}{2} \frac{m'^2 + 8m'\kappa + \kappa^2}{(2m'^2 + m'\kappa + 2\kappa^2)^3} \Big) \\ &+ \frac{6\sqrt{15}}{\pi^3} \sum_{m',\kappa}{}' \Big(\frac{1}{(m'^2 + 15\kappa^2)^2} - \frac{1}{(3m'^2 + 5\kappa^2)^2} \Big) \\ &+ \Big(\frac{1}{(2m'^2 + m'\kappa + 2\kappa^2)^2} - \frac{1}{(m'^2 + m'\kappa + 4\kappa^2)^2} \Big). \end{aligned}$$

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Proof Define $D_{j\tau} = (mj\tau + \kappa)(mj\bar{\tau} + \kappa)$. So

$$m(Q_k) = \frac{\Im\tau}{8\pi^3} \sum_{m,\kappa}{}' \Big[2\frac{(m(\tau+\bar{\tau})+2\kappa)^2}{D_\tau^3} + \frac{-2}{D_\tau^2} - 32\frac{(2m(\tau+\bar{\tau})+2\kappa)^2}{D_{2\tau}^3} + \frac{32}{D_{2\tau}^2}$$
$$- 18\frac{(3m(\tau+\bar{\tau})+2\kappa)^2}{D_{3\tau}^3} + \frac{18}{D_{3\tau}^2}$$
$$+ 288\frac{(6m(\tau+\bar{\tau})+2\kappa)^2}{D_{6\tau}^3} - \frac{288}{D_{6\tau}^2}\Big]$$

If $k = -3$, then $\tau = \frac{-3+\sqrt{-15}}{24}$ and

$$D_\tau = \frac{1}{24}(m^2 - 6m\kappa + 24\kappa^2) = \frac{1}{24}(m'^2 + 15\kappa^2) \quad \text{with } m' = m - 3\kappa,$$
$$D_{2\tau} = \frac{1}{6}(m^2 - 3m\kappa + 6\kappa^2) = \frac{1}{6}(m'^2 + m'\kappa + 4\kappa^2) \quad \text{with } m' = m - 2\kappa,$$
$$D_{3\tau} = \frac{1}{8}(3m^2 - 6m\kappa + 8\kappa^2) = \frac{1}{8}(3m'^2 + 5\kappa^2) \quad \text{with } m' = m - \kappa,$$
$$D_{6\tau} = \frac{1}{2}(3m^2 - 3m\kappa + 2\kappa^2) = \frac{1}{2}(2m^2 + m\kappa + 2\kappa'^2) \quad \text{with } \kappa' = \kappa - m.$$

Thus

$$m(Q_{-3}) = \frac{\sqrt{15}}{24 \times 8\pi^3} \sum_{m',\kappa}{}' (A_1 + A_2 + A_3 + A_4).$$

Now A_1 can be written

$$A_1 = (24)^2 \Big(\frac{-m'^2 + 15\kappa^2 - 30m'\kappa}{(m'^2 + 15\kappa^2)^3} + \frac{2}{(m'^2 + 15\kappa^2)^2} \Big)$$

and

$$\sum_{m',\kappa}{}' A_1 = (24)^2 \sum_{m',\kappa}{}' \Big(\frac{15k^2 - m'^2}{(m'^2 + 15\kappa^2)^3} + \frac{2}{(m'^2 + 15\kappa^2)^2} \Big).$$

Then we get

$$A_2 = (24)^2 \Big(\frac{m'^2 + 16m'\kappa + 4\kappa^2}{(m'^2 + m'\kappa + 4\kappa^2)^3} - \frac{2}{(m'^2 + m'\kappa + 4\kappa^2)^2} \Big)$$

Now with the change of variable $\kappa = \kappa' - m'$ we make the denominators of A_2 symmetric with respect to m' and κ'. So

$$A_2 = (24)^2 \Big(\frac{-11m'^2 + 8m'\kappa' + 4\kappa'^2}{(4m'^2 - 7m'\kappa' + 4\kappa'^2)^3} - \frac{2}{(4m'^2 - 7m'\kappa' + 4\kappa^2)^2} \Big),$$

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that is,

$$A_2 = (24)^2 \Big(\frac{1}{2} \frac{-7m'^2 + 16m'\kappa' - 7\kappa'^2}{(4m'^2 - 7m'\kappa' + 4\kappa'^2)^3} - \frac{2}{(4m'^2 - 7m'\kappa' + 4\kappa^2)^2} \Big)$$

and coming back to variables m' and κ,

$$A_2 = (24)^2 \Big(\frac{1}{2} \frac{2m'^2 + 2m'\kappa - 7\kappa^2}{(m'^2 + m'\kappa + 4\kappa^2)^3} - \frac{2}{(m'^2 + m'\kappa + 4\kappa^2)^2} \Big).$$

The same way we obtain,

$$A_3 = (24)^2 \Big(\frac{3m'^2 + 30m'\kappa - 5\kappa^2}{(3m'^2 + 5\kappa^2)^3} - \frac{2}{(3m'^2 + 5\kappa^2)^2} \Big)$$

or

$$A_3 = (24)^2 \Big(\frac{3m'^2 - 5\kappa^2}{(3m'^2 + 5\kappa^2)^3} - \frac{2}{(3m'^2 + 5\kappa^2)^2} \Big).$$

Finally, using the same tricks as for A_2, we obtain

$$A_4 = (24)^2 \Big(\frac{1}{2} \frac{m^2 + 8m\kappa' + \kappa'^2}{(2m^2 + m\kappa' + 2\kappa'^2)^3} + \frac{2}{(2m^2 + m\kappa' + 2\kappa'^2)^2} \Big). \qquad \blacksquare$$

From Proposition 3.1, we notice that the Mahler measure is expressed as a sum of a modular part

$$\frac{3\sqrt{15}}{\pi^3} \sum_{m',\kappa}{}' \Big(\frac{15k^2 - m'^2}{(m'^2 + 15\kappa^2)^3} + \frac{-5k^2 + 3m'^2}{(3m'^2 + 5\kappa^2)^3} \Big) \\ + \Big(\frac{1}{2} \frac{2m'^2 + 2m'\kappa - 7\kappa^2}{(m'^2 + m'\kappa + 4\kappa^2)^3} + \frac{1}{2} \frac{m'^2 + 8m'\kappa + \kappa^2}{(2m'^2 + m'\kappa + 2\kappa^2)^3} \Big)$$

and a part related to a Dirichlet L-series

$$+ \frac{6\sqrt{15}}{\pi^3} \sum_{m',\kappa}{}' \Big(\frac{1}{(m'^2 + 15\kappa^2)^2} - \frac{1}{(3m'^2 + 5\kappa^2)^2} \Big) \\ + \Big(\frac{1}{(2m'^2 + m'\kappa + 2\kappa^2)^2} - \frac{1}{(m'^2 + m'\kappa + 4\kappa^2)^2} \Big).$$

To prove that the modular part is 0, we observe first that

$$L(f_1, s) = \frac{1}{2} \sum_{r,s}{}' \frac{5r^2 - 3k^2}{(3r^2 + 5k^2)^s} \quad \text{and} \quad L(f_2, s) = \frac{1}{2} \sum_{r,s}{}' \frac{r^2 - 15k^2}{(r^2 + 15k^2)^s}$$

are the Mellin transforms of the two weight 3 modular forms

$$f_1 = \frac{1}{2}\sum_{r,s\in\mathbb{Z}}(5r^2 - 3k^2)q^{3r^2+5k^2} \qquad f_2 = \frac{1}{2}\sum_{r,s\in\mathbb{Z}}(r^2 - 15k^2)q^{r^2+15k^2}.$$

Then using Theorem 2.6, we know that

$$\sum{}'\Big(\frac{1}{4}\frac{2m'^2+2m'\kappa-7\kappa^2}{(m'^2+m'\kappa+4\kappa^2)^s}+\frac{1}{4}\frac{m^2+8m\kappa'+\kappa'^2}{(2m^2+m\kappa'+2\kappa'^2)^s}\Big) = L(f^+,s)$$

is the L-series attached to the modular $K3$-surface $\tilde{X}$.

Proposition 3.2

$$\sum_{m,k}{}'\Big(\frac{-15k^2+m^2}{(m^2+15k^2)^3}+\frac{5k^2-3m^2}{(3m^2+5k^2)^3}\Big) = \sum_{m,k}{}'\Big(\frac{1}{2}\frac{2m^2+2mk-7k^2}{(m^2+mk+4k^2)^3}+\frac{1}{2}\frac{m^2+8mk+k^2}{(2m^2+mk+2k^2)^3}\Big).$$

Proof Let a be a rational integer and denote $\theta_a = \sum_{n\in\mathbb{Z}} q^{an^2}$ the weight $1/2$ modular form for the congruence group $\Gamma = \Gamma_0(4)$. Denote $f_1 := [\theta_5, \theta_3]$, $f_2 := [\theta_1, \theta_{15}]$ the Rankin–Cohen brackets which are modular forms of weight 3 for Γ.

Recall that if f and g are modular forms of respective weight k and l for a congruence subgroup, then its Rankin–Cohen bracket is the modular form of weight $k+l+2$ defined by $[g,h] := kgh' - lg'h$.

Thus we get the two weight 3 modular forms

$$f_1 = \frac{1}{2}\sum_{r,s\in\mathbb{Z}}(5r^2 - 3k^2)q^{3r^2+5k^2} \qquad f_2 = \frac{1}{2}\sum_{r,s\in\mathbb{Z}}(r^2 - 15k^2)q^{r^2+15k^2}.$$

So to compare $L(f_1,s) + L(f_2,s) = \sum \frac{A_1(n)}{n^s}$ and $L(f^+,s) = \sum \frac{A_2(n)}{n^s}$ we apply Livné's criterion.

First we determine an effective test set T for the respective representations

$$T = \{7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 53, 61, 71, 73, 83\}.$$

Then we compute the corresponding $A_1(p)$ and $A_2(p)$.

p	7	11	13	17	19	23	29	31	41	43	53	61	71	73	83
$A_1(p)$	0	0	0	-14	-22	34	0	2	0	0	-86	-118	0	0	154
$A_2(p)$	0	0	0	-14	-22	34	0	2	0	0	-86	-118	0	0	154

This achieves the proof of the proposition. ∎

Proposition 3.3

$$\frac{6\sqrt{15}}{\pi^3}\sum_{m,k}{}' \frac{1}{(m^2+15k^2)^2} - \frac{1}{(3m^2+5k^2)^2} + \frac{1}{(2m^2+mk+2k^2)^2} - \frac{1}{(m^2+mk+4k^2)^2} = \frac{8}{5}d_3$$

Proof We denote $L_f(s) := L(\chi_f, s)$ the Dirichlet's L-series for the character χ_f attached to the quadratic field $\mathbb{Q}(\sqrt{f})$. The proof follows from a lemma.

Lemma 3.4

$$\sum_{m,k}{}' \Big(\frac{1}{(2m^2+mk+k^2)^s} + \frac{1}{m^2+mk+4k^2)^s}\Big) = 2\zeta(s)L_{-15}(s) \tag{3.1}$$

$$\sum_{m,k}{}' \Big(\frac{1}{(3m^2+5k^2)^s} + \frac{1}{(m^2+15k^2)^s}\Big) = 2\Big(1+\frac{1}{2^{2s-1}} - \frac{1}{2^{s-1}}\Big)\zeta(s)L_{-15}(s) \tag{3.2}$$

$$\sum_{m,k}{}' \Big(\frac{1}{(m^2+mk+4k^2)^s} - \frac{1}{2m^2+mk+2k^2)^s}\Big) = 2L_{-3}(s)L_5(s) \tag{3.3}$$

$$\sum_{m,k}{}' \Big(\frac{1}{(m^2+15k^2)^s} - \frac{1}{(3m^2+5k^2)^s}\Big) = 2\Big(1+\frac{1}{2^{2s-1}} + \frac{1}{2^{s-1}}\Big)L_{-3}(s)L_5(s) \tag{3.4}$$

Proof Assertion (3.1) follows from the result [18]

$$\sum{}' \left(\frac{1}{(2m^2+mk+k^2)^s} + \frac{1}{m^2+mk+4k^2)^s}\right) = \zeta_{\mathbb{Q}(\sqrt{-15})}(s)$$

and the formula $\zeta_{\mathbb{Q}(\sqrt{-15})}(s) = \zeta(s)L_{-15}(s)$. Assertion (3.2) follows from results of K. Williams [16] and Zucker [19]. Using Williams's notation, we set $\phi(q) := \sum_{-\infty}^{+\infty} q^{n^2}$ and get

$$\phi(q)\phi(q^{15}) + \phi(q^3)\phi(q^5) = 2 + \sum_{n\geq 1} a_n(-60)\frac{q^n}{1-q^n},$$

where

$$a_n(-60) = \begin{cases} 0 & \text{if } n \equiv 0,3,5,6,9,10 \pmod{60}, \\ 2 & \text{if } n \equiv 1,4,8,14,16,17,19,22,23,26,31,32,47,49,53,58 \pmod{60}, \\ -2 & \text{if } n \equiv 2,7,11,13,28,29,34,37,38,41,43,44,46,52,56,59 \pmod{60}. \end{cases}$$

As explained in [19], often we may get

$$Q(a,b,c;s) = \sum{}' \frac{1}{(am^2+bmn+cn^2)^s}$$

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in terms of $L_{\pm h}$ when expressing them as Mellin transforms of products of various Jacobi functions $\theta_3(q)$ for different arguments. More precisely,

$$Q(1,0,\lambda;s)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\sum{}' e^{-(m^2t+\lambda n^2t)}\,dt=\frac{1}{\Gamma(s)}\int_0^{\infty}(\theta_3(q)\theta_3(q^{\lambda})-1)\,dt,$$

where $e^{-t}=q$ and $\theta_3(q)=1+2q^2+2q^4+2q^9+\cdots$; thus writing $\theta_3(q)\theta_3(q^{\lambda})-1$ as a Lambert series $\sum_{n\geq 1}a_n\frac{q^n}{1-q^n}$, very often the integral is given in terms of L-series. So we get

$$\begin{aligned}Q(1,0,15;s)+Q(3,0,5;s)&=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}(\theta_3(q)\theta_3(q^{15})+\theta_3(q^3)\theta_3(q^5)-2)\,dt\\&=\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\Big(\sum_{n\geq 1}a_n(-60)\frac{e^{-tn}}{1-e^{-tn}}\Big)\,dt.\end{aligned}$$

Since

$$\Gamma(s)=\int_0^{+\infty}e^{-y}y^{s-1}\,dy$$

making the change variable $nt=y$, it follows that

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\frac{e^{-tn}}{1-e^{-tn}}\,dt&=\int_0^{+\infty}\Big(\frac{y}{n}\Big)^{s-1}\frac{e^{-y}}{1-e^{-y}}\frac{dy}{n}\\&=\frac{1}{\Gamma(s)}\frac{1}{n^s}\int_0^{+\infty}\frac{y^{s-1}}{e^y-1}dy=\frac{1}{n^s}\zeta(s).\end{aligned}$$

Thus

$$Q(1,0,15;s)+Q(3,0,5;s)=\zeta(s)\sum_{n\geq 1}a_n(-60)\frac{1}{n^s}.$$

But

$$\begin{aligned}L_{-60}(s)=\frac{1}{1^s}-\frac{1}{7^s}-\frac{1}{11^s}-\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{31^s}\\-\frac{1}{37^s}-\frac{1}{41^s}-\frac{1}{43^s}+\frac{1}{47^s}+\frac{1}{49^s}+\frac{1}{53^s}-\frac{1}{59^s}+\cdots\pmod{60}\end{aligned}$$

and

$$\begin{aligned}L_{-15}(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{7^s}+\frac{1}{8^s}-\frac{1}{11^s}-\frac{1}{13^s}\\-\frac{1}{14^s}+\frac{1}{16^s}+\frac{1}{17^s}+\frac{1}{19^s}-\frac{1}{22^s}+\cdots\pmod{15}.\end{aligned}$$

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So,

$$\frac{1}{2}\sum_{n\geq 1} a_n(-60)\frac{1}{n^s} = L_{-60}(s) + \frac{1}{2^s}(-1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{11^s} + \frac{1}{13^s} - \frac{1}{14^s} + \frac{1}{16^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{22^s} - \frac{1}{23^s} - \frac{1}{26^s} - \frac{1}{28^s} + \frac{1}{29^s} + \cdots) \pmod{30}.$$

Let us define

$$L_{-15}(s) := \sum_{n\geq 1}\frac{\chi_{-15}(n)}{n^s} = L_+(s) + L_-(s),$$

where

$$L_+(s) = \sum_{\substack{n\geq 1\\ n \text{ pair}}}\frac{\chi_{-15}(n)}{n^s} \qquad L_-(s) = \sum_{\substack{n\geq 1\\ n \text{ impair}}}\frac{\chi_{-15}(n)}{n^s}.$$

Obviously,

$$L_+(s) = \frac{1}{2^s}L_{-15}(s), \quad L_{-60}(s) = L_-(s), \quad L_{-15}(s) = L_-(s) + \frac{1}{2^s}L_{-15}(s).$$

Thus,

$$\frac{1}{2}\sum_{n\geq 1}\frac{a_n(-60)}{n^s} = L_-(s) + \frac{1}{2^s}(L_+(s) - L_-(s)) = \left(1 + \frac{1}{2^{2s-1}} - \frac{1}{2^{s-1}}\right)L_{-15}(s).$$

From this last equality we deduce formula (3.2). From [20] we get

$$Q(1,1,4;s) = \zeta(s)L_{-15}(s) + L_{-3}(s)L_5(s).$$

So from formula (3.1) we obtain formula (3.3). Equality (3.4) derives from a formula by Zucker and Robertson [20] giving

$$Q(1,0,15;s) = \left(1 - \frac{1}{2^{s-1}} + \frac{1}{2^{2s-1}}\right)\zeta(s)L_{-15}(s) + \left(1 + \frac{1}{2^{s-1}} + \frac{1}{2^{2s-1}}\right)L_{-3}(s)L_5(s).$$

So, thanks to formula (3.2)

$$\begin{aligned} Q(1,0,15;s) - Q(3,0,5;s) &= 2Q(1,0,15;s) - (Q(1,0,15;s) + Q(3,0,5;s)) \\ &= 2(1 + \frac{1}{2^{s-1}} + \frac{1}{2^{2s-1}})L_{-3}(s)L_5(s). \end{aligned}$$

■

By subtracting (3.3) from (3.4) for $s = 2$ and using [19]

$$L_5 = \frac{4\pi^2}{25\sqrt{5}},$$

we get the proposition. ■

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The proof of Theorem 1.1 is just a combination of the three propositions.

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