Glasgow Math. J. **42** (2000) 139–156. © Glasgow Mathematical Journal Trust 2000. Printed in the United Kingdom

COMPLETE MAXIMAL SPACELIKE SURFACES IN AN ANTI-DE SITTER SPACE $\mathbf{H}_2^4(c)$*

QING-MING CHENG

Department of Mathematics, Faculty of Science, Josai University, Sakado, Saitama 350-0295, Japan
e-mail: cheng@math.josai.ac.jp

(Received 14 August, 1998)

Abstract. In this paper, we prove that if M^2 is a complete maximal spacelike surface of an anti-de Sitter space $\mathbf{H}_2^4(c)$ with constant scalar curvature, then $S = 0$, $S = \frac{-10c}{11}$, $S = \frac{-4c}{3}$ or $S = -2c$, where S is the squared norm of the second fundamental form of M^2. Also

(1) $S = 0$ if and only if M^2 is the totally geodesic surface $\mathbf{H}^2(c)$;
(2) $S = \frac{-4c}{3}$ if and only if M^2 is the hyperbolic Veronese surface;
(3) $S = -2c$ if and only if M^2 is the hyperbolic cylinder of the totally geodesic surface $\mathbf{H}_1^3(c)$ of $\mathbf{H}_2^4(c)$.

1991 *Mathematics Subject Classification*. 53C40, 53C42

1. Introduction. Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p and of constant curvature c, which is called as an indefinite space form of index p. The standard models of indefinite space forms are given as follows. In an $(n+p)$-dimensional real vector space $\mathbf{R}^{n+p}$ with the standard basis, the scalar product $\langle , \rangle$ is given by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i - \sum_{j=n+1}^{n+p} x_j y_j,$$

where $x = (x_1, x_2, \cdots, x_{n+p})$ and $y = (y_1, y_2, \cdots, y_{n+p})$. Then $(\mathbf{R}^{n+p}, \langle , \rangle)$ is an indefinite Euclidean space, which is denoted by $\mathbf{R}_p^{n+p}$.

Let $S_p^{n+p}(c)$ for $c > 0$ be the hypersurface in $\mathbf{R}_p^{n+p+1}$ given as

$$\langle x, x\rangle = \frac{1}{c} =: r_0^2.$$

Then we know that the $S_p^{n+p}(c)$ inherits an indefinite Riemannian metric induced through $\mathbf{R}_p^{n+p+1}$ and has constant curvature c. This is called a de Sitter space of constant curvature c with index p.

On the other hand, let $\mathbf{H}_p^{n+p}(c)$ for $c < 0$ be the hypersurface in $\mathbf{R}_{p+1}^{n+p+1}$ given as

$$\langle x, x\rangle = \frac{1}{c} =: -r_0^2.$$

*Research partially supported by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture and by a Grant-in-Aid for Scientific Research from Josai University.

https://doi.org/10.1017/S0017089500010168 Published online by Cambridge University Press

Then we also know that the $\mathbf{H}_p^{n+p}(c)$ inherits an indefinite Riemannian metric induced through $\mathbf{R}_{p+1}^{n+p+1}$ and has constant curvature c. This is called an anti-de Sitter space of constant curvature c with index p.

Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. A submanifold M^n of $M_p^{n+p}(c)$ is said to be spacelike if the induced metric on M^n from that of the ambient space is positive definite.

E. Calabi [**1**] first studied the Bernstein problem for a maximal spacelike entire graph in the Minkowski space $\mathbf{R}_1^{n+1}$ and proved that it has to be hyperplane, when $n \leq 4$. S. Y. Cheng and S. T. Yau [**6**] proved that the conclusion remains true for all n. As a generalization of the Bernstein type problem, T. Ishihara [**8**] proved that a complete spacelike maximal submanifold M^n of $M_p^{n+p}(c)$ $(c \geq 0)$ is totally geodesic.

On the other hand, there exist many examples of complete maximal spacelike submanifolds in the anti-de Sitter space $\mathbf{H}_p^{n+p}(c)$, which are not totally geodesic. For examples, we consider the following examples.

Example 1. We consider the mapping defined by

$$u_1 = \frac{1}{\sqrt{-3c}} yz,\ u_2 = \frac{1}{\sqrt{-3c}} zx,\ u_3 = \frac{1}{\sqrt{-3c}} xy,$$

$$u_4 = \frac{1}{2\sqrt{-3c}}(x^2 - y^2),\ u_5 = \frac{1}{6\sqrt{-c}}(x^2 + y^2 + 2z^2),$$

where (x, y, z) is the natural coordinate system in $\mathbf{R}_1^3$ and $(u_1, u_2, u_3, u_4, u_5)$ is the natural coordinate system $\mathbf{R}_3^5$. This defines a complete maximal spacelike isometric immersion of $\mathbf{H}^2(\frac{c}{3})$ into $\mathbf{H}_2^4(c)$, where $\mathbf{H}^{n_i}(c_i)$ is an n_i-dimensional hyperbolic space of constant curvature c_i, which is called the hyperbolic Veronese surface.

Example 2. Let $n_1, \cdots, n_{p+1}$ be positive integers and $n = n_1 + \cdots + n_{p+1}$. Let x_i be a point of $\mathbf{H}^{n_i}(\frac{nc}{n_i})$. Then $x = (x_1, \cdots, x_{p+1})$ is a vector in $\mathbf{R}_{p+1}^{n+p+1}$ with $\langle x, x\rangle = \frac{1}{c}$. This also defines a complete maximal spacelike isometric immersion of $\mathbf{H}^{n_1}(\frac{nc}{n_1}) \times \cdots \times \mathbf{H}^{n_{p+1}}(\frac{nc}{n_{p+1}})$ into $\mathbf{H}_p^{n+p}(c)$.

Hence this case of complete maximal spacelike submanifolds in the anti-de Sitter $\mathbf{H}_p^{n+p}(c)$ is very different from the ones in the indefinite Euclidean space $\mathbf{R}_p^{n+p}$ and the de Sitter space $S_p^{n+p}(c)$. Hence, the investigation of complete maximal spacelike submanifolds in $\mathbf{H}_p^{n+p}(c)$ would be very interesting.

T. Ishihara [**8**] characterized the complete maximal spacelike submanifolds $\mathbf{H}^{n_1}(\frac{nc}{n_1}) \times \cdots \times \mathbf{H}^{n_{p+1}}(\frac{nc}{n_{p+1}})$ of $\mathbf{H}_p^{n+p}(c)$, that is, he proved that let M^n be an n-dimensional complete maximal spacelike submanifold in $\mathbf{H}_p^{n+p}(c)$, then $S \leq -npc$ and $S = -npc$ if and only if $M^n = \mathbf{H}^{n_1}(\frac{nc}{n_1}) \times \cdots \times \mathbf{H}^{n_{p+1}}(\frac{nc}{n_{p+1}})$, where S is the squared norm of the second fundamental form of M^n. When $p = 1$, the Bernstein type properties of complete maximal spacelike hypersurfaces in $\mathbf{H}_1^{n+1}(c)$ are also studied in [**3**] and [**4**].

In particular, if $n = 2$, we know that the well known examples of complete maximal spacelike surfaces in the anti-de Sitter space $\mathbf{H}_2^4(c)$ are the totally geodesic surface $H^2(c)$ with $S = 0$ and the hyperbolic Veronese surface with $S = \frac{-4c}{3}$. Therefore, it is natural to ask whether there exist the other complete maximal spacelike surfaces with $S =$ constant in $\mathbf{H}_2^4(c)$, which are different from the above ones. If there exist such surfaces, can we determine all of the value of S? In this paper we shall answer these problems.

MAIN THEOREM. *Let M^2 be a complete maximal spacelike surface of an anti-de Sitter space $\mathbf{H}_2^4(c)$ with constant scalar curvature, then $S=0$, $S=\frac{-10c}{11}$, $S=\frac{-4c}{3}$ or $S=-2c$, where S is the squared norm of the second fundamental form of M^2. And*

(1) $S=0$ *if and only if M^2 is the totally geodesic surface $\mathbf{H}^2(c)$;*
(2) $S=\frac{-4c}{3}$ *if and only if M^2 is the hyperbolic Veronese surface;*
(3) $S=-2c$ *if and only if M^2 is the hyperbolic cylinder of the totally geodesic surface $\mathbf{H}_1^3(c)$ of $\mathbf{H}_2^4(c)$.*

REMARK 1. It is still open for the author whether there exist complete maximal spacelike surfaces of the anti-de Sitter space $\mathbf{H}_2^4(c)$ with $S=\frac{-10c}{11}$.

2. Preliminaries. Let M^n be an n-dimensional spacelike submanifold of an anti-de Sitter space $\mathbf{H}_p^{n+p}(c)$ of dimension $n+p$ and with index p. We choose a local orthonormal frame field $e_1,\cdots,e_n,e_{n+1},\cdots,e_{n+p}$ in $\mathbf{H}_p^{n+p}(c)$, restricted to M^n, so that $e_1,\cdots,e_n$ are tangent to M^n. With respect to the above frame field of $\mathbf{H}_p^{n+p}(c)$, let $\omega_1,\cdots,\omega_{n+p}$ denote the dual coframe field. Then

$$\omega_\alpha=0 \quad \text{for any} \quad \alpha=n+1,\cdots,n+p. \tag{2.1}$$

It follows from Cartan's Lemma that

$$\omega_{\alpha i}=\sum_j h_{ij}^\alpha\omega_j, \quad h_{ij}^\alpha=h_{ji}^\alpha. \tag{2.2}$$

The structure equations of M^n are given by

$$\begin{cases} d\omega_i+\sum_j\omega_{ij}\wedge\omega_j=0, \quad w_{ij}+\omega_{ji}=0,\\ d\omega_{ij}+\sum_k\omega_{ik}\wedge\omega_{kj}=\Omega_{ij},\\ \Omega_{ij}=\dfrac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l, \end{cases} \tag{2.3}$$

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})-\sum_\beta(h_{ik}^\beta h_{jl}^\beta-h_{il}^\beta h_{jk}^\beta), \tag{2.4}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor) of M^n.

We have also the structure equations of the normal bundle of M^n.

$$\begin{cases} d\omega_{\alpha\beta}+\sum_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=\Omega_{\alpha\beta},\\ \Omega_{\alpha\beta}=\dfrac{1}{2}\sum_{k,l}R_{\alpha\beta kl}\omega_k\wedge\omega_l, \end{cases} \tag{2.5}$$

$$R_{\alpha\beta kl} = -\sum_t (h^\alpha_{tk} h^\beta_{tl} - h^\alpha_{tl} h^\beta_{tk}). \tag{2.6}$$

The second fundamental form $\mathbf{h}$ of M^n is given by

$$\mathbf{h} = \sum_{i,j,\alpha} h^\alpha_{ij} w_i w_j e_\alpha$$

We recall $\frac{1}{n}\sum_\alpha(\sum_i h^\alpha_{ii})e_\alpha$ the mean curvature vector. If $\sum_i h^\alpha_{ii} = 0$ for all α, then M^n is said to be maximal. The Codazzi equation and Ricci formulas for the second fundamental form and its covariant derivatives are given by

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}, \tag{2.7}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im} R_{mjkl} + \sum_m h^\alpha_{mj} R_{mikl} - \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}, \tag{2.8}$$

$$h^\alpha_{ijklm} - h^\alpha_{ijkml} = \sum_r h^\alpha_{rjk} R_{rilm} + \sum_r h^\alpha_{irk} R_{rjlm} + \sum_r h^\alpha_{ijr} R_{rklm} - \sum_\beta h^\beta_{ijk} R_{\alpha\beta lm}, \tag{2.9}$$

where h^α_{ijk}, h^α_{ijkl} and h^α_{ijklm} are the coefficients of the first, the second and the third covariant derivatives of the second fundamental form of M^n, respectively. If M^n is maximal, the scalar curvature is given by

$$R = n(n-1)c + \sum_{i,j,\alpha} (h^\alpha_{ij})^2. \tag{2.10}$$

Hence the scalar curvature is constant if and only if $S = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$ is constant.

The following Generalized Maximum Principle due to Omori [**9**] and Yau [**12**] will be used in this paper.

Generalized Maximum Principle (cf. Omori [**9**] and Yau [**12**]). *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M^n, then there exists a sequence $\{p_m\}$ of points in M^n such that*

$$\lim_{m\to\infty} F(p_m) = \sup F, \quad \lim_{m\to\infty} |\nabla F|(p_m) = 0, \quad \lim_{m\to\infty} \sup \Delta F(p_m) \leq 0.$$

3. Proof of main theorem. In this section, we assume $n = p = 2$. We first compute some local formulas in order to prove Main Theorem. Let $S_3 := \sum_{ij}(h^3_{ij})^2$ and $S_4 := \sum_{ij}(h^4_{ij})^2$. We know that $S_3 S_4$ is a function defined globally on M^2. For arbitrary fixed point p in M^2 we can choose e_1 and e_2 such that

$$h^3_{ij} = \lambda_i \delta_{ij}. \tag{3.1}$$

Since M is maximal we get $\lambda_1 = -\lambda_2 =: \lambda$. Let

$$S_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij} h^\beta_{ij}.$$

We know that the (2×2)-matrix $(S_{\alpha\beta})$ is symmetric. Hence we can assume that it is diagonal for a suitable choice of e_3 and e_4. Thus setting $\mu := h^4_{11} = -h^4_{22}$ and $\mu_1 = h^4_{12}$, we have

$$\sum_{i,j} h^3_{ij} h^4_{ij} = 2\lambda\mu = 0. \tag{3.2}$$

THEOREM 3.1. *For $\alpha = 3, 4$, we have*

$$\Delta h^\alpha_{ij} = (S + 2c)h^\alpha_{ij} - 2 \sum_{l,t,\beta\neq\alpha} h^\alpha_{lt} h^\beta_{tj} h^\beta_{il} + \sum_{l,t,\beta\neq\alpha} h^\alpha_{tl} h^\beta_{tl} h^\beta_{ij}, \tag{3.3}$$

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + (S + 2c)S + 2S_3S_4, \tag{3.4}$$

$$\frac{1}{2}\Delta S_3 = \sum_{i,j,k} (h^3_{ijk})^2 + (S + 2c)S_3 + S_3S_4, \tag{3.5}$$

where $S = S_3 + S_4 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$ is the squared norm of the second fundamental form of M^2 and $S_3 = \sum_{i,j}(h^3_{ij})^2$ and $S_4 = \sum_{i,j}(h^4_{ij})^2$.

Proof. For any α,

$$\begin{aligned}
\Delta h^\alpha_{ij} &= \sum_l h^\alpha_{ijll} = \sum_l h^\alpha_{lijl} \\
&= \sum_l h^\alpha_{lilj} + \sum_{l,t} h^\alpha_{ti} R_{tljl} + \sum_{l,t} h^\alpha_{lt} R_{tijl} - \sum_{l,\beta} h^\beta_{li} R_{\alpha\beta jl} \\
&= \sum_{l,t} h^\alpha_{ti}[c(\delta_{tj}\delta_{ll} - \delta_{tl}\delta_{lj}) - \sum_\beta (h^\beta_{tj}h^\beta_{ll} - h^\beta_{tl}h^\beta_{lj})] \\
&+ \sum_{l,t} h^\alpha_{tl}[c(\delta_{tj}\delta_{il} - \delta_{tl}\delta_{ij}) - \sum_\beta (h^\beta_{tj}h^\beta_{il} - h^\beta_{tl}h^\beta_{ij})] \\
&+ \sum_{t,l,\beta} h^\beta_{li}(h^\alpha_{tj}h^\beta_{tl} - h^\alpha_{tl}h^\beta_{tj}) \\
&= (2c + S)h^\alpha_{ij} - 2 \sum_{l,t,\beta\neq\alpha} h^\alpha_{lt} h^\beta_{tj} h^\beta_{il} + \sum_{l,t,\beta\neq\alpha} h^\alpha_{tl} h^\beta_{tl} h^\beta_{ij},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} \\
&= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + (2c + S)S + 2S_3S_4,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\Delta S_3 &= \sum_{i,j,k} (h^3_{ijk})^2 + \sum_{i,j} h^3_{ij}\Delta h^3_{ij} \\
&= \sum_{i,j,k} (h^3_{ijk})^2 + (2c + S)S_3 + S_3S_4.
\end{aligned}$$

This finishes the Proof of Theorem 3.1.

THEOREM 3.2.

$$\frac{1}{2}\Delta\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2$$
$$=\sum_{i,j,k,l,\alpha}(h^{\alpha}_{ijkl})^2+(\frac{9}{2}S+7c)\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2 \tag{3.6}$$
$$+3|\nabla S|^2-5\sum_{\alpha}S_{\alpha}\sum_{i,j,k}(h^{\alpha}_{ijk})^2.$$

Since M^2 is maximal, for any α, we have

$$h^{\alpha}_{11}+h^{\alpha}_{22}=0.$$

Hence

$$h^{\alpha}_{11l}=-h^{\alpha}_{22l},\quad h^{\alpha}_{11lk}=-h^{\alpha}_{22lk}\quad\text{for any}\quad l,k. \tag{3.7}$$

In the sequel, we will often use the formula (3.7).

Proof.

$$\sum_{i,j,k,\alpha}h^{\alpha}_{ijk}\Delta h^{\alpha}_{ijk}=\sum_{i,j,k,l,\alpha}h^{\alpha}_{ijk}h^{\alpha}_{ijkll}$$
$$=\sum_{i,j,k,l,\alpha}h^{\alpha}_{ijk}[h^{\alpha}_{ijlkl}+\nabla_l(\sum_t h^{\alpha}_{tj}R_{tikl}$$
$$+\sum_t h^{\alpha}_{ti}R_{tjkl}-\sum_{\beta}h^{\beta}_{ij}R_{\alpha\beta kl})]$$
$$=\sum_{i,j,k,l,\alpha}h^{\alpha}_{ijk}[h^{\alpha}_{ijllk}+\sum_t h^{\alpha}_{tjl}R_{tikl}$$
$$+\sum_t h^{\alpha}_{til}R_{tjkl}+\sum_t h^{\alpha}_{ijt}R_{tlkl}-\sum_{\beta}h^{\beta}_{ijl}R_{\alpha\beta kl}]$$
$$+\sum_{i,j,k,l,\alpha}h^{\alpha}_{ijk}[\sum_t h^{\alpha}_{tjl}R_{tikl}+\sum_t h^{\alpha}_{til}R_{tjkl}-\sum_{\beta}h^{\beta}_{ijl}R_{\alpha\beta kl}]$$
$$+\sum_{i,j,k,l,\alpha}h^{\alpha}_{ijk}[\sum_t h^{\alpha}_{tj}\nabla_l R_{tikl}+\sum_t h^{\alpha}_{ti}\nabla_l R_{tjkl}-\sum_{\beta}h^{\beta}_{ij}\nabla_l R_{\alpha\beta kl}] \tag{3.8}$$
$$=\sum_{i,j,k,\alpha}h^{\alpha}_{ijk}\{\nabla_k[(S+2c)h^{\alpha}_{ij}$$
$$-2\sum_{l,t,\beta\neq\alpha}h^{\alpha}_{lt}h^{\beta}_{tj}h^{\beta}_{il}+\sum_{l,t,\beta\neq\alpha}h^{\alpha}_{tl}h^{\beta}_{tl}h^{\beta}_{ij}]\}$$
$$+2\sum_{i,j,k,l,t,\alpha}h^{\alpha}_{ijk}h^{\alpha}_{tjl}R_{tikl}+2\sum_{i,j,k,l,t,\alpha}h^{\alpha}_{ijk}h^{\alpha}_{til}R_{tjkl}$$
$$+\sum_{i,j,k,l,t,\alpha}h^{\alpha}_{ijk}h^{\alpha}_{ijt}R_{tlkl}-2\sum_{i,j,k,l,\alpha,\beta}h^{\alpha}_{ijk}h^{\beta}_{ijl}R_{\alpha\beta kl}$$
$$+\sum_{i,j,k,l,\alpha}h^{\alpha}_{ijk}[\sum_t h^{\alpha}_{tj}\nabla_l R_{tikl}+\sum_t h^{\alpha}_{ti}\nabla_l R_{tjkl}-\sum_{\beta}h^{\beta}_{ij}\nabla_l R_{\alpha\beta kl}].$$

$$\begin{aligned}
&\sum_{i,j,k,\alpha} h^\alpha_{ijk}\{\nabla_k[(S+2c)h^\alpha_{ij} - 2\sum_{l,t,\beta\neq\alpha} h^\alpha_{lt}h^\beta_{tj}h^\beta_{il} + \sum_{l,t,\beta\neq\alpha} h^\alpha_{tl}h^\beta_{tl}h^\beta_{ij}]\} \\
&= (S+2c)\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{ijk}\nabla_k S \\
&- 2\sum_{i,j,k,t,l,\alpha,\beta\neq\alpha} h^\alpha_{ijk}h^\alpha_{tlk}h^\beta_{tj}h^\beta_{il} - 2\sum_{i,j,k,t,l,\alpha,\beta\neq\alpha} h^\alpha_{ijk}h^\alpha_{tl}h^\beta_{tjk}h^\beta_{il} \\
&- 2\sum_{i,j,k,t,l,\alpha,\beta\neq\alpha} h^\alpha_{ijk}h^\alpha_{tl}h^\beta_{tj}h^\beta_{ilk} + \sum_{i,j,k,t,l,\alpha,\beta\neq\alpha} h^\alpha_{ijk}h^\alpha_{tlk}h^\beta_{tl}h^\beta_{ij} \\
&+ \sum_{i,j,k,t,l,\alpha,\beta\neq\alpha} h^\alpha_{ijk}h^\alpha_{tl}h^\beta_{tlk}h^\beta_{ij} + \sum_{i,j,k,t,l,\alpha,\beta\neq\alpha} h^\alpha_{ijk}h^\beta_{ijk}h^\alpha_{tl}h^\beta_{tl} \\
&= (S+2c)\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \frac{1}{2}|\nabla S|^2 - 2\sum_{i,j,k,l,t} h^3_{ijk}h^3_{tlk}h^4_{tj}h^4_{il} \\
&- 4\sum_{i,j,k,t} h^4_{ijk}h^3_{tjk}h^4_{ti}h^3_{ii} - 4\sum_{i,j,k,t} h^4_{ijk}h^3_{tik}h^4_{tj}h^3_{jj} + \sum_{i,j,k,l,t} h^3_{ijk}h^3_{tlk}h^4_{tl}h^4_{ij} \\
&+ \frac{S_3}{2}\sum_{i,j,k}(h^4_{ijk})^2 + 4\lambda\sum_{i,j,k} h^4_{11k}h^4_{ij}h^3_{ijk} \quad \text{(by (3.2))} \\
&= (S+2c)\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \frac{1}{2}|\nabla S|^2 + \frac{S_3}{2}\sum_{i,j,k}(h^4_{ijk})^2 \\
&- 2\sum_{i,j,k,l,t} h^3_{ijk}h^3_{tlk}h^4_{tj}h^4_{il} - 8\sum_{i,j,k,t} h^4_{ijk}h^3_{tjk}h^4_{ti}h^3_{ii} \\
&+ \sum_k(\sum_{i,j} h^3_{ijk}h^4_{ij})^2 + 4\lambda\sum_{i,j,k} h^4_{11k}h^4_{ij}h^3_{ijk}.
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\sum_{i,j,k,l,t,\alpha} h^\alpha_{ijk}h^\alpha_{ijt}R_{tlkl} \\
&= \sum_{i,j,k,l,t,\alpha} h^\alpha_{ijk}h^\alpha_{ijt}[c(\delta_{tk}\delta_{ll} - \delta_{tl}\delta_{lk}) - \sum_\beta(h^\beta_{tk}h^\beta_{ll} - h^\beta_{tl}h^\beta_{lk})] \\
&= (\frac{S}{2}c)\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2.
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&-2\sum_{i,j,k,l,\alpha,\beta} h^\alpha_{ijk}h^\beta_{ijl}R_{\alpha\beta kl} \\
&= 2\sum_{i,j,k,l,\alpha,\beta} h^\alpha_{ijk}h^\beta_{ijl}\sum_t(h^\alpha_{tk}h^\beta_{tl} - h^\alpha_{tl}h^\beta_{tk}) \\
&= -4\sum_{i,j,k,l} h^3_{ijk}h^4_{ijl}h^4_{lk}(h^3_{ll} - h^3_{kk}).
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&2\sum_{i,j,k,l,t,\alpha} h^\alpha_{ijk}h^\alpha_{tjl}R_{tikl} + 2\sum_{i,j,k,l,t,\alpha} h^\alpha_{ijk}h^\alpha_{til}R_{tjkl}\\
&= 2\sum_{i,j,k,l,t,\alpha} h^\alpha_{ijk}h^\alpha_{tjl}[c(\delta_{tk}\delta_{il}-\delta_{tl}\delta_{ik}) - \sum_\beta(h^\beta_{tk}h^\beta_{il}-h^\beta_{tl}h^\beta_{ik})]\\
&+ 2\sum_{i,j,k,l,t,\alpha} h^\alpha_{ijk}h^\alpha_{itl}[c(\delta_{tk}\delta_{jl}-\delta_{tl}\delta_{jk}) - \sum_\beta(h^\beta_{tk}h^\beta_{jl}-h^\beta_{tl}h^\beta_{jk})]\\
&= 4c\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 - 4\sum_{i,j,k,l,t,\alpha,\beta} h^\alpha_{ijk}h^\alpha_{tjl}h^\beta_{tk}h^\beta_{il}\\
&+ 4\sum_{i,j,k,l,t,\alpha,\beta} h^\alpha_{ijk}h^\alpha_{tjl}h^\beta_{tl}h^\beta_{ik}\\
&= 4c\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + 2S_3\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2\\
&- 4\sum_{i,j,k,t,l} h^3_{ijk}h^3_{tjl}(h^4_{tk}h^4_{il}-h^4_{tl}h^4_{ik})\\
&- 4\sum_{i,j,k,t,l} h^4_{ijk}h^4_{tjl}(h^4_{tk}h^4_{il}-h^4_{tl}h^4_{ik}).
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\sum_{i,j,k,l,\alpha} h^\alpha_{ijk}[\sum_t h^\alpha_{tj}\nabla_l R_{tikl} + \sum_t h^\alpha_{ti}\nabla_l R_{tjkl}\\
&- \sum_\beta h^\beta_{ij}\nabla_l R_{\alpha\beta kl}]\\
&= -\sum_{i,j,k,t,l,\alpha,\beta} h^\alpha_{ijk}h^\alpha_{tj}\nabla_l(h^\beta_{tk}h^\beta_{il}-h^\beta_{tl}h^\beta_{ik})\\
&- \sum_{i,j,k,t,l,\alpha,\beta} h^\alpha_{ijk}h^\alpha_{ti}\nabla_l(h^\beta_{tk}h^\beta_{jl}-h^\beta_{tl}h^\beta_{jk})\\
&+ \sum_{i,j,k,t,l,\alpha,\beta} h^\alpha_{ijk}h^\beta_{ij}\nabla_l(h^\alpha_{tk}h^\beta_{tl}-h^\alpha_{tl}h^\beta_{tk})\\
&= -2\sum_{i,j,k,t,l,\alpha,\beta} h^\alpha_{ijk}h^\alpha_{tj}(h^\beta_{tkl}h^\beta_{il}-h^\beta_{tl}h^\beta_{ikl})\\
&+ \sum_{i,j,k,t,l,\alpha,\beta} h^\alpha_{ijk}h^\beta_{ij}(h^\alpha_{tkl}h^\beta_{tl}-h^\alpha_{tl}h^\beta_{tkl})\\
&= S_3\sum_{i,j,k}(h^3_{ijk})^2 + \frac{S_3}{2}\sum_{i,j,k}(h^4_{ijk})^2\\
&+ \sum_k(\sum_{i,j} h^4_{ij}h^3_{ijk})^2 - 4\lambda\sum_{i,j,k} h^4_{11k}h^4_{ij}h^3_{ijk}\\
&- 2\sum_{i,j,k,t,l} h^4_{ijk}h^4_{tkl}h^4_{il}h^4_{tj} + 2\sum_{i,j,k,t,l} h^4_{ijk}h^4_{ikl}h^4_{tl}h^4_{tj}\\
&- 4\sum_{i,j,k,t,l} h^3_{ijk}h^4_{ikl}h^4_{jl}(h^3_{ii}-h^3_{jj}).
\end{aligned} \tag{3.13}$$

Hence, (3.8)∼(3.13) yield

https://doi.org/10.1017/S0017089500010168 Published online by Cambridge University Press

$$
\begin{aligned}
&\sum_{i,j,k,\alpha} h^\alpha_{ijk}\Delta h^\alpha_{ijk}\\
&=(\frac{3}{2}S+7c)\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2+\frac{1}{2}|\nabla S|^2+3S_3\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2\\
&-4\sum_{i,j,k,l} h^3_{ijk}h^4_{ijl}h^4_{lk}(h^3_{ll}-h^3_{kk})\\
&-4\sum_{i,j,k,l} h^3_{ijk}h^4_{ikl}h^4_{jl}(h^3_{ii}-h^3_{jj})\\
&-2\sum_{i,j,k,l,t} h^3_{ijk}h^3_{tlk}h^4_{tj}h^4_{il}-8\sum_{i,j,k,t} h^4_{ijk}h^3_{tjk}h^4_{ti}h^3_{ii}\\
&-4\sum_{i,j,k,t,l} h^3_{ijk}h^3_{tjl}(h^4_{tk}h^4_{il}-h^4_{tl}h^4_{ik})\\
&-4\sum_{i,j,k,t,l} h^4_{ijk}h^4_{tjl}(h^4_{tk}h^4_{il}-h^4_{tl}h^4_{ik})\\
&-2\sum_{i,j,k,t,l} h^4_{ijk}h^4_{tkl}h^4_{il}h^4_{tj}+2\sum_{i,j,k,t,l} h^4_{ijk}h^4_{ikl}h^4_{tl}h^4_{tj}\\
&+2\sum_k(\sum_{i,j} h^3_{ijk}h^4_{ij})^2.
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&8\sum_{i,j,k,t} h^4_{ijk}h^3_{tjk}h^4_{ti}h^3_{ii}\\
&=8\lambda\sum_{j,k,t}(h^4_{1jk}h^3_{tjk}h^4_{t1}-h^4_{2jk}h^3_{tjk}h^4_{t2})\\
&=8\lambda\sum_{j,k}(h^3_{1jk}h^4_{1jk}\mu+h^3_{2jk}h^4_{1jk}\mu_1\\
&-h^4_{2jk}h^3_{1jk}\mu_1+h^4_{2jk}h^3_{2jk}\mu)\\
&=8\lambda\mu_1(h^4_{1jk}h^3_{2jk}-h^4_{2jk}h^3_{1jk})\quad \text{(by (3.2))}\\
&=32\lambda\mu_1(h^4_{222}h^3_{111}-h^4_{111}h^3_{222}).
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&4\sum_{i,j,k,l} h^3_{ijk}h^4_{ijl}h^4_{lk}(h^3_{ll}-h^3_{kk})\\
&=4\sum_{i,j} h^3_{ij1}h^4_{ij2}h^4_{12}(h^3_{22}-h^3_{11})\\
&+4\sum_{i,j} h^3_{ij2}h^4_{ij1}h^4_{12}(h^3_{11}-h^3_{22})\\
&=-8\lambda\mu_1\sum_{ij}(h^3_{1ij}h^4_{2ij}-h^4_{1ij}h^3_{2ij})\quad \text{(by (3.2))}\\
&=32\lambda\mu_1(h^4_{222}h^3_{111}-h^4_{111}h^3_{222}).
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
&4\sum_{i,j,k,l} h^3_{ijk}h^4_{ikl}h^4_{lj}(h^3_{ii}-h^3_{jj})\\
&=4\sum_{i,j,} h^3_{12k}h^4_{1kl}h^4_{l2}(h^3_{11}-h^3_{22})\\
&+4\sum_{i,j,} h^3_{21k}h^4_{2kl}h^4_{1l}(h^3_{22}-h^3_{11})\\
&=8\lambda\sum_{k,l}(h^3_{12k}h^4_{1kl}h^4_{l2}-h^3_{12k}h^4_{2kl}h^4_{1l})\\
&=16\lambda\mu_1(h^4_{222}h^3_{111}-h^4_{111}h^3_{222})\quad\text{(by (3.2)).}
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
&4\sum_{i,j,k,t,l,\alpha} h^\alpha_{ijk}h^\alpha_{tjl}(h^4_{tk}h^4_{il}-h^4_{tl}h^4_{ik})\\
&=4\sum_{i,j,k,t,\alpha} h^\alpha_{ijk}h^\alpha_{tj1}(h^4_{tk}h^4_{i1}-h^4_{t1}h^4_{ik})\\
&+4\sum_{i,j,k,t,\alpha} h^\alpha_{ijk}h^\alpha_{tj2}(h^4_{tk}h^4_{i2}-h^4_{t2}h^4_{ik})\\
&=4\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{1j1}(h^4_{1k}h^4_{i1}-h^4_{11}h^4_{ik})\\
&+4\sum_{i,j,k,t,\alpha} h^\alpha_{ijk}h^\alpha_{1j2}(h^4_{1k}h^4_{i2}-h^4_{12}h^4_{ik})\\
&+4\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{2j1}(h^4_{2k}h^4_{i1}-h^4_{21}h^4_{ik})\\
&+4\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{2j2}(h^4_{2k}h^4_{i2}-h^4_{22}h^4_{ik})\\
&=-8\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{11j}h^4_{ik}h^4_{11}\\
&+4\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{1j1}(h^4_{1k}h^4_{i1}-h^4_{2k}h^4_{2i})\\
&-8\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{12j}h^4_{ik}h^4_{12}+8\sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{12j}h^4_{1i}h^4_{2k}\\
&=-2S_4\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2.
\end{aligned}
\tag{3.18}
$$

For any α,

$$
\begin{aligned}
&2\sum_{i,j,k,l,t} h^\alpha_{ijk}h^\alpha_{tlk}h^4_{tj}h^4_{il}\\
&=2\sum_{i,j,k,l} h^\alpha_{ijk}h^\alpha_{1lk}h^4_{1j}h^4_{il}+2\sum_{i,j,k,l,t} h^\alpha_{ijk}h^\alpha_{2lk}h^4_{2j}h^4_{il}\\
&=2\mu\sum_{i,k,l}(h^\alpha_{1ik}h^\alpha_{1lk}-h^\alpha_{2ik}h^\alpha_{2lk})h^4_{il}+4\mu_1\sum_{i,k,l} h^\alpha_{1ik}h^\alpha_{2lk}h^4_{il}\\
&=0.
\end{aligned}
\tag{3.19}
$$

https://doi.org/10.1017/S0017089500010168 Published online by Cambridge University Press

$$
\begin{aligned}
&2\sum(\sum_{i,j} h^3_{ijk}h^4_{ij})^2 \\
&= 2\sum_k(\sum_{ij} h^4_{ij}h^3_{ijk})^2 = S_4\sum_{i,j,k}(h^3_{ijk})^2.
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
&2\sum_{i,j,k,t,l} h^4_{ijk}h^4_{ikl}h^4_{tl}h^4_{tj} \\
&= 2\sum_{i,j,k,t} h^4_{ijk}h^4_{ik1}h^4_{t1}h^4_{tj} + 2\sum_{i,j,k,t} h^4_{ijk}h^4_{ik2}h^4_{t2}h^4_{tj} \\
&= S_4\sum_{i,j,k}(h^4_{ijk})^2.
\end{aligned} \tag{3.21}
$$

According to (3.14) $\sim$ (3.21), we get

$$
\begin{aligned}
&\sum_{i,j,k,\alpha} h^\alpha_{ijk}\Delta h^\alpha_{ijk} \\
&= (\frac{9}{2}S + 7c)\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \frac{1}{2}|\nabla S|^2 \\
&- 80\lambda\mu_1(h^3_{111}h^4_{222} - h^4_{111}h^3_{222}).
\end{aligned} \tag{3.22}
$$

Since, for any α,

$$
\sum_{i,j,k}(h^\alpha_{ijk})^2 = 4\{(h^\alpha_{111})^2 + (h^\alpha_{222})^2\}
$$

and

$$
\begin{aligned}
|\nabla S|^2 &= \sum_l(\nabla_l S)^2 \\
&= \sum_l(2\sum_{i,j,\alpha} h^\alpha_{ij}h^\alpha_{ijl})^2 \\
&= 16\sum_l(\lambda h^3_{11l} + \mu h^4_{11l} + \mu_1 h^4_{12l})^2 \\
&= 2\sum_{i,j,k,\alpha} S_\alpha(h^\alpha_{ijk})^2 - 32\lambda\mu_1(h^3_{111}h^4_{222} - h^4_{111}h^3_{222}),
\end{aligned}
$$

we obtain

$$
-8\lambda\mu_1(h^3_{111}h^4_{222} - h^4_{111}h^3_{222}) = \frac{1}{4}|\nabla S|^2 - \frac{1}{2}\sum_{i,j,k,\alpha} S_\alpha(h^\alpha_{ijk})^2. \tag{3.23}
$$

From

$$
\frac{1}{2}\Delta\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 = \sum_{i,j,k,\alpha} h^\alpha_{ijk}\Delta h^\alpha_{ijk} + \sum_{i,j,k,l,\alpha}(h^\alpha_{ijkl})^2,
$$

(3.22) and (3.23) yields the Theorem 3.2.

https://doi.org/10.1017/S0017089500010168 Published online by Cambridge University Press

LEMMA 1.

$$\sum_{i,j,k,l,\alpha}(h^{\alpha}_{ijkl})^2=\sum_{i\neq j,\alpha}(h^{\alpha}_{iijj}-h^{\alpha}_{jjii})^2+\sum_{i\neq j,\alpha}(h^{\alpha}_{iijj}+h^{\alpha}_{jjii})^2$$
$$+\sum_{i\neq j,\alpha}(h^{\alpha}_{iiij}-h^{\alpha}_{iiji})^2+\sum_{i\neq j,\alpha}(h^{\alpha}_{iiij}+h^{\alpha}_{iiji})^2.$$

Proof.

$$\sum_{i,j,k,l,\alpha}(h^{\alpha}_{ijkl})^2=\sum_{i,\alpha}(h^{\alpha}_{iiii})^2+3\sum_{i\neq j,\alpha}(h^{\alpha}_{iijj})^2+\sum_{i\neq j,\alpha}(h^{\alpha}_{iiij})^2+3\sum_{i\neq j,\alpha}(h^{\alpha}_{iiji})^2.$$

Since

$$h^{\alpha}_{iiii}=-h^{\alpha}_{jjii}\quad\text{for any}\quad\alpha\quad\text{and}\quad j\neq i$$

and

$$h^{\alpha}_{iiij}=-h^{\alpha}_{jjij}\quad\text{for any}\quad\alpha\quad\text{and}\quad j\neq i,$$

we know that Lemma 1 holds.

LEMMA 2.

$$h^3_{1122}-h^3_{2211}=\lambda(2c+S+S_4),$$

$$h^3_{1112}-h^3_{1121}=0,$$

$$h^4_{1122}-h^4_{2211}=(2c+S)\mu,$$

$$h^4_{1112}-h^4_{1121}=-(2c+S+S_3)\mu_1.$$

Proof. From the Ricci formula (2.8), we have

$$\begin{aligned}h^{\alpha}_{iijj}-h^{\alpha}_{jjii}&=h^{\alpha}_{ijij}-h^{\alpha}_{ijji}\\&=\sum_t h^{\alpha}_{tj}R_{tiij}+\sum_t h^{\alpha}_{it}R_{tjij}-\sum_t h^{\beta}_{ij}R_{\alpha\beta ij}\\&=(h^{\alpha}_{ii}-h^{\alpha}_{jj})c-\frac{S}{2}(h^{\alpha}_{jj}-h^{\alpha}_{ii})\\&-\sum_{t,\beta\neq\alpha}h^{\alpha}_{tj}h^{\beta}_{ti}h^{\beta}_{ij}+\sum_{t,\beta\neq\alpha}h^{\alpha}_{it}h^{\beta}_{tj}h^{\beta}_{ij}.\end{aligned}$$

Hence

$$h^3_{1122}-h^3_{2211}=\lambda(2c+S+S_4),$$

$$h^4_{1122}-h^4_{2211}=(2c+S)\mu.$$

By the same proof, we can obtain

$$h^3_{1112} - h^3_{1121} = 2\lambda\mu\mu_1 = 0 \quad \text{(by (3.2))},$$
$$h^4_{1112} - h^4_{1121} = -(2c + S + S_3)\mu_1.$$

Thus we complete the proof of Lemma 2.

Next, we shall prove the Main Theorem. Since the scalar curvature is constant if and only if S is constant, in the sequel, we assume that S is constant.

Proof of Main Theorem. If $S = 0$, then M is totally geodesic because S is constant. Next we assume $S \neq 0$. Since S is constant, we have

$$\nabla_l S = 0, \quad \text{for} \quad l = 1, 2,$$

namely,

$$\begin{cases} 2\lambda h^3_{11l} + 2\mu h^4_{11l} + 2\mu_1 h^4_{12l} = 0. \\ 2\lambda h^3_{111} + 2\mu h^4_{111} - 2\mu_1 h^4_{222} = 0, \\ 2\lambda h^3_{222} + 2\mu h^4_{222} + 2\mu_1 h^4_{111} = 0. \end{cases} \tag{3.24}$$

Hence we obtain

$$S_3 \sum_{i,j,k} (h^3_{ijk})^2 = S_4 \sum_{i,j,k} (h^4_{ijk})^2, \tag{3.25}$$

from

$$\sum_{i,j,k} (h^\alpha_{ijk})^2 = 4[(h^\alpha_{111})^2 + (h^\alpha_{222})^2]. \tag{3.26}$$

We know that $S_3 S_4$ are function defined globally on M. Because S is constant, from Gauss equation, we infer that the sectional curvature is bounded from below and that function $S_3 S_4$ is bounded because $0 \leq S_3 S_4 \leq S^2$. Since M is complete, from the Generalized Maximum Principle due to Omori and Yau, we know that there exists a sequence $\{p_m\} \subset M^2$ of points such that

$$\lim_{m\to\infty} (S_3 S_4)(p_m) = \inf(S_3 S_4), \tag{3.27}$$

$$\lim_{m\to\infty} |\nabla(S_3 S_4)|(p_m) = 0, \tag{3.28}$$

$$\lim_{m\to\infty} \inf \Delta(S_3 S_4)(p_m) \geq 0. \tag{3.29}$$

From Theorem 3.1, we have

$$\sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 = -S(S + 2c) - 2S_3 S_4. \tag{3.30}$$

because S is constant. Hence $\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2$ is bounded and

$$\lim_{m\to\infty}\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2(p_m)=\sup\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2, \tag{3.31}$$

$$\lim_{m\to\infty}|\nabla\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2|(p_m)=-2\lim_{m\to\infty}|\nabla(S_3S_4)|(p_m)=0. \tag{3.32}$$

From

$$\Delta\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2=-2S_3\Delta S_4-4\nabla S_3\cdot\nabla S_4-2S_4\Delta S_3 \tag{3.33}$$

and Theorem 3.1 and Theorem 3.2, we obtain that $\sum_{i,j,k,l,\alpha}(h^{\alpha}_{ijkl})^2$ is bounded. Thus, we can assume $\lim_{m\to\infty}S_3(p_m)=\tilde{S}_3$, $\lim_{m\to\infty}S_4(p_m)=\tilde{S}_4$, $\lim_{m\to\infty}\lambda(p_m)=\tilde{\lambda}$, $\lim_{m\to\infty}\mu(p_m)=\tilde{\mu}$, $\lim_{m\to\infty}\mu_1(p_m)=\tilde{\mu}_1$, $\lim_{m\to\infty}h^{\alpha}_{ijk}(p_m)=\tilde{h}^{\alpha}_{ijk}$ and $\lim_{m\to\infty}h^{\alpha}_{ijkl}(p_m)=\tilde{h}^{\alpha}_{ijkl}$, by taking a subsequence if necessary. Since

$$\lim_{m\to\infty}|\nabla(S_3S_4)|(p_m)=0$$

and

$$\nabla_l(S_3S_4)=S_3\nabla_lS_4+S_4\nabla_lS_3,$$

we have

$$\lim_{m\to\infty}(S_3\nabla_lS_4+S_4\nabla_lS_3)(p_m)=0.$$

From $S=S_3+S_4$, we get

$$\nabla_lS_3=-\nabla_lS_4.$$

Therefore,

$$\lim_{m\to\infty}(S_3-S_4)(\nabla_lS_4)(p_m)=\lim_{m\to\infty}(S_4-S_3)(\nabla_lS_3)(p_m)=0. \tag{3.34}$$

Hence,

$$\tilde{S}_3=\tilde{S}_4 \quad\text{or}\quad \lim_{m\to\infty}(\nabla_lS_4)(p_m)=\lim_{m\to\infty}(\nabla_lS_3)(p_m)=0. \tag{3.35}$$

(1). In the case where $\tilde{S}_3=\tilde{S}_4$. From Theorem 3.1, we have

$$0=\lim_{m\to\infty}\{\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2+(S+2c)S+2S_3S_4\}(p_m)=\sup\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2+(\frac{3}{2}S+2c)S.$$

Hence,

$$\sup\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2=-(\frac{3}{2}S+2c)S.$$

https://doi.org/10.1017/S0017089500010168 Published online by Cambridge University Press

From Theorem 3.1, we have

$$\begin{aligned}\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 &= -(S+2c)S - 2S_3S_4\\ &= -(\frac{3}{2}S+2c)S + \frac{1}{2}(S_3 - S_4)^2\\ &\geq -(\frac{3}{2}S+2c)S.\end{aligned}$$

Hence, we have

$$\inf \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 \geq -(\frac{3}{2}S+2c)S = \sup \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2,$$

that is,

$$\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 \equiv -(\frac{3}{2}S+2c)S \tag{3.36}$$

is constant. Therefore, $S_3 \equiv S_4$ on M^2 and they are constant. Hence, on M^2,

$$\nabla_l S_3 = \nabla_l S_4 = 0.$$

From (3.25), we have $\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 = 0$ on M^2. According to (3.36), we have $S = \frac{-4c}{3}$.

(2). In the case where $\tilde{S}_3 \neq \tilde{S}_4$. We have, for $l = 1, 2$,

$$\lim_{m\to\infty} \nabla_l S_3(p_m) = \lim_{m\to\infty} \nabla_l S_4(p_m) = 0.$$

From $|\nabla S_\alpha|^2 = 4S_\alpha \sum_{i,j,k}(h^\alpha_{ijk})^2$, we have

$$\lim_{m\to\infty} S_3 \sum_{i,j,k}(h^3_{ijk})^2(p_m) = \lim_{m\to\infty} S_4 \sum_{i,j,k}(h^4_{ijk})^2(p_m) = 0,$$

that is

$$\tilde{S}_3 \lim_{m\to\infty} \sum_{i,j,k}(h^3_{ijk})^2(p_m) = \tilde{S}_4 \lim_{m\to\infty} \sum_{i,j,k}(h^4_{ijk})^2(p_m) = 0. \tag{3.37}$$

If

$$\lim_{m\to\infty} \sum_{i,j,k}(h^3_{ijk})^2(p_m) = \lim_{m\to\infty} \sum_{i,j,k}(h^4_{ijk})^2(p_m) = 0,$$

we have

$$\sup \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 = 0.$$

Hence, from Theorem 3.1, we have

$$0 = S(S+2c) + 2S_3S_4 = S(\frac{3}{2}S + 2c) - \frac{1}{2}(S_3 - S_4)^2.$$

Hence $S > \frac{-4c}{3}$ and S_3S_4 is constant. Hence S_3 and S_4 are constant because $S = S_3 + S_4$ and S_3S_4 are constant. Since $S > 0$, we can assume $S_3 > 0$. From the proof of Theorem 3.1, we have, for $\alpha = 3, 4$,

$$0 = \frac{1}{2}\Delta S_\alpha = \sum_{i,j,k}(h^\alpha_{ijk})^2 + (S+2c)S_\alpha + S_3S_4.$$

Hence,

$$(S+2c)S_\alpha + S_3S_4 = 0.$$

Therefore, $S = -2c$ and M^2 is the hyperbolic cylinder of the totally geodesic hypersurface $\mathbf{H}^3_1(c)$ from the Theorem due to Ishihara [**8**].

Next we can assume $\lim_{m\to\infty}\sum_{i,j,k}(h^4_{ijk})^2(p_m) \neq 0$ without loss of the generality. We have $\tilde{S}_4 = 0$. Because $S = S_3 + S_4 > 0$ is constant, we have $\tilde{S}_3 \neq 0$. Hence,

$$\lim_{m\to\infty}\sum_{i,j,k}(h^3_{ijk})^2(p_m) = 0.$$

We shall prove $S = \frac{-10c}{11}$ in this case. Since $\tilde{S}_4 = 0$, we have

$$\lim_{m\to\infty}\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2(p_m) + (2c+S)S = 0.$$

Hence,

$$\sup\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 = -(S+2c)S.$$

From (3.31) and (3.32), we know that $\lim_{m\to\infty}|\nabla\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2| = 0$. Hence,

$$\lim_{m\to\infty}\sum_{i,j,k}h^4_{ijk}h^4_{ijkl}(p_m) = 0 \quad \text{for} \quad l = 1, 2,$$

because of $\lim_{m\to\infty}\sum_{i,j,k}(h^3_{ijk})^2(p_m) = 0$. Thus we conclude

$$\tilde{h}^4_{111}\tilde{h}^4_{2211} = -\tilde{h}^4_{222}\tilde{h}^4_{1121}, \quad \tilde{h}^4_{111}\tilde{h}^4_{1112} = \tilde{h}^4_{222}\tilde{h}^4_{1122}.$$

According to Lemma 2 we have

$$\tilde{h}^4_{1122} = \tilde{h}^4_{2211}, \quad \tilde{h}^4_{1112} = \tilde{h}^4_{1121}.$$

Hence

$$(\tilde{h}^4_{111})^2\tilde{h}^4_{2211} + (\tilde{h}^4_{222})^2\tilde{h}^4_{2211} = 0.$$

https://doi.org/10.1017/S0017089500010168 Published online by Cambridge University Press

Since $\sum_{i,j,k}(\tilde{h}^4_{ijk})^2 \neq 0$, then $\tilde{h}^4_{2211} = \tilde{h}^4_{1122} = \tilde{h}^4_{1121} = \tilde{h}^4_{1112} = 0$. On the other hand, since S is constant we have

$$\sum_{i,j,\alpha} h^\alpha_{ij} h^\alpha_{ijlk} + \sum_{i,j,\alpha} h^\alpha_{ijk} h^\alpha_{ijl} = 0 \quad \text{for any} \quad l, k.$$

Hence

$$\begin{aligned}
2\tilde{\lambda}\tilde{h}^3_{1112} &= 0, \\
2\tilde{\lambda}\tilde{h}^3_{1121} &= 0, \\
2\tilde{\lambda}\tilde{h}^3_{1122} &= -\frac{1}{2}\sum_{i,j,k,\alpha}(\tilde{h}^\alpha_{ijk})^2, \\
2\tilde{\lambda}\tilde{h}^3_{2211} &= \frac{1}{2}\sum_{i,j,k,\alpha}(\tilde{h}^\alpha_{ijk})^2.
\end{aligned}$$

We infer

$$(\tilde{h}^3_{1122} + \tilde{h}^3_{2211})^2 + (\tilde{h}^3_{1112} + \tilde{h}^3_{1121})^2 = 0.$$

Therefore, from Lemma 2 we have

$$\sum_{i,j,k,l,\alpha} (\tilde{h}^\alpha_{ijkl})^2 = (S + 2c)^2 S. \tag{3.38}$$

From Theorem 3.1, we have

$$\begin{aligned}
&\frac{1}{2}\Delta \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 \\
&= -\Delta(S_3 S_4) \\
&= -S_3 \Delta S_4 - 2\nabla S_3 \cdot \nabla S_4 - S_4 \Delta S_3.
\end{aligned}$$

Hence

$$\lim_{m\to\infty} \frac{1}{2}\Delta \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2(p_m) = -\lim_{m\to\infty}(S_3 \Delta S_4)(p_m) = 2S(S + 2c)S \tag{3.39}$$

because of $\tilde{S}_4 = 0$ and $\lim_{m\to\infty}\sum_{i,j,k}(h^3_{ijk})^2(p_m) = 0$. From (3.37), (3.38), (3.39) and Theorem 3.2, we have

$$2(S + 2c)S^2 = \frac{1}{2}\lim_{m\to\infty} \Delta \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2(p_m) = (S + 2c)^2 S - (\frac{9}{2}S + 7c)(S + 2c)S.$$

Thus $S = -\frac{10c}{11}$. From the above proof, we know that $S = \frac{-4c}{3}$ if and only if $S_3 = S_4$ is constant on M^2 and $\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 \equiv 0$. By making use of the similar method to one which was used in **[7]** by Chern, do Carmo and Kobayashi, we can prove that M^2 is isometric to the hyperbolic Veronese surface. We complete the proof of Main Theorem.

REFERENCES

1. E. Calabi, Examples of Bernstein problems for nonlinear equations, *Proc. Symp. Pure and Appl. Math.* **15** (1970), 223–230.

2. Q. M. Cheng, Complete spacelike submanifolds in a de Sitter space with parallel mean curvature vector, *Math. Z.* **206** (1991), 333–339.

3. Q. M. Cheng, Complete maximal spacelike hypersurfaces of $\mathbf{H}_1^4(c)$, *Manuscripta Math.* **82** (1994), 149–160.

4. Q. M. Cheng, Spacelike surfaces in an anti-de Sitter space, *Colloquium Math.* **46** (1994), 201–208.

5. Q. M. Cheng and H. Nakagawa, Totally umbilical hypersurfaces, *Hiroshima Math. J.* **20** (1990), 1–10.

6. S. Y. Cheng and S. T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski space, *Ann. of Math.* **104** (1976), 407–419.

7. S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, in *Functional analysis and related fields* (Springer-Verlag, 1970) 59–75.

8. T. Ishihara, Maximal spacelike submanifolds of a pseudoriemannian space form of constant curvature, *Michigan Math. J.* **35** (1988), 345–352.

9. H. Omori, Isometric immersion of Riemannian manifolds, *J. Math. Soc. Japan* **19** (1967), 205–214.

10. A. E. Treibergs, Entire hypersurfaces of constant mean curvature in Minkowski 3-space, *Invent. Math.* **66** (1982), 39–56.

11. J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.* **88** (1968), 62–105.

12. S. T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure and Appl. Math.* **28** (1975), 201–228.