



RESEARCH ARTICLE

Cycles of many lengths in Hamiltonian graphs

Matija Bucić¹, Lior Gishboliner² and Benny Sudakov³

¹Institute for Advanced Study and Princeton University, Washington Road, Fine Hall, US-08544 Princeton, USA;

E-mail: matija.bucic@ias.edu.

²Department of Mathematics, ETH, Zürich, Rämistrasse 101, Zürich, CH-8092, Switzerland;

E-mail: lior.gishboliner@math.ethz.ch.

³Department of Mathematics, ETH, Zürich, Rämistrasse 101, Zürich, CH-8092, Switzerland;

E-mail: benjamin.sudakov@math.ethz.ch.

Received: 6 May 2021; Accepted: 28 May 2022

2020 Mathematics Subject Classification: Primary – 05C35; Secondary – 05C38, 05C45, 05D40

Abstract

In 1999, Jacobson and Lehel conjectured that, for $k \geq 3$, every k -regular Hamiltonian graph has cycles of $\Theta(n)$ many different lengths. This was further strengthened by Verstraëte, who asked whether the regularity can be replaced with the weaker condition that the minimum degree is at least 3. Despite attention from various researchers, until now, the best partial result towards both of these conjectures was a \sqrt{n} lower bound on the number of cycle lengths. We resolve these conjectures asymptotically by showing that the number of cycle lengths is at least $n^{1-o(1)}$.

Contents

1	Introduction	1
2	A sketch and the main ideas	3
3	Setting up the stage	5
4	Finding cycles of many different lengths	6
4.1	Main lemmas	7
4.2	Putting it all together	13
5	Concluding remarks	19

1. Introduction

The study of cycles in graphs goes back to the early days of graph theory and has been fundamental ever since. Of particular interest are Hamilton cycles, i.e., cycles passing through all the vertices of a graph. Starting with the cornerstone theorem of Dirac [8], there are many results giving sufficient conditions for a graph to be Hamiltonian, for some other classical examples see [4, 5, 6, 12, 22]. In 1973, Bondy [3] made the ‘meta-conjecture’ that any nontrivial condition which guarantees the existence of a Hamilton cycle should also guarantee that the given graph is pancyclic, i.e., contains cycles of all possible lengths, with possibly a simple family of exceptions. This assertion turned out to be influential, and by now, there are numerous appealing results of this type. For example, Bondy himself [2] proved that Ore’s sufficient condition for Hamiltonicity (that the sum of degrees of any pair of nonadjacent vertices is at least n) implies that the graph is either pancyclic or isomorphic to the complete bipartite

© The Author(s), 2022. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives licence (<https://creativecommons.org/licenses/by-nc-nd/4.0/>), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original work is unaltered and is properly cited. The written permission of Cambridge University Press must be obtained for commercial re-use or in order to create a derivative work.

graph $K_{n/2, n/2}$. Bauer and Schmeichel [1], relying on previous results of Schmeichel and Hakimi [23], have shown that the sufficient conditions for Hamiltonicity of Bondy [4], Chvátal [5] and Fan [12] all imply pancyclicity, barring a small family of exceptions. Jackson and Ordaz [17] conjectured that any graph G whose connectivity $\kappa(G)$ is strictly larger than its independence number $\alpha(G)$ must be pancyclic. This conjecture is motivated by the classical theorem of Chvátal and Erdős [6] that a graph with $\kappa(G) \geq \alpha(G)$ must be Hamiltonian. An approximate version of the conjecture was proven by Keevash and Sudakov [18], who showed that $\kappa(G) \geq 600\alpha(G)$ is sufficient for pancyclicity.

Pancyclicity is just an instance of a wider class of problems, which study the properties of the set of cycle lengths of a graph with connection to other graph parameters. The set of cycle lengths of G is called its *cycle spectrum* and denoted $\mathcal{C}(G)$. There are by now numerous results relating properties of $\mathcal{C}(G)$ to various graph parameters. For example, Erdős [10] conjectured that a graph G with girth g and average degree d must satisfy $|\mathcal{C}(G)| \geq \Omega(d^{\lfloor \frac{g-1}{2} \rfloor})$. The case $g = 5$ was settled by Erdős, Faudree, Rousseau and Schelp [11]. Later, Sudakov and Verstraëte [24] proved the full conjecture in a strong form. Another example is a result of Gould, Haxell and Scott [15] that a graph with minimum degree cn must have a cycle of any even length between 4 and $ec(G) - K$, where $ec(G)$ is the length of a longest even cycle in G and K is a constant depending only on c . We should also mention the recent work of Gao, Huo, Liu and Ma [13], who proved several conjectures relating properties of $\mathcal{C}(G)$ to the minimum degree, connectivity or chromatic number of G .

Bondy's meta-conjecture is about conditions for Hamiltonicity which imply pancyclicity. A natural question in the opposite direction is as follows: Let us *assume* that a graph G is Hamiltonian; under which assumptions can we also guarantee that G is pancyclic? Since pancyclicity is sometimes too strong of a requirement, we can relax it and ask to find many cycle lengths. Questions of this type were first introduced by Jacobson and Lehel at the 1999 conference 'Paul Erdős and His Mathematics', where they asked for the minimum size of the cycle spectrum of a k -regular Hamiltonian graph G on n vertices. The aforementioned result of Bondy [2] implies that if $k = \lceil n/2 \rceil$, then G is pancyclic unless $G = K_{n/2, n/2}$. At the other extreme, if $k = 2$, then G clearly has just one cycle. Jacobson and Lehel conjectured that, already for $k \geq 3$, the number of cycle lengths should be linear in n . This is best possible since they also observed that one cannot expect to have pancyclicity. Indeed, assuming $2k$ divides n , take $\frac{n}{2k}$ disjoint copies of $K_{k, k}$, ordered in a cycle, remove an edge from each of them and add an edge between any two consecutive copies such that the resulting graph is k -regular. It is not hard to see that, in this construction, the possible cycle lengths are precisely all the even integers between 4 and $2k$ and between $\frac{2n}{k}$ and n . This gives in total $\frac{n}{2} \cdot \frac{k-2}{k} + k$ different lengths.

Soon after the above question was first circulated, Gould, Jacobson and Pfender proved that $|\mathcal{C}(G)| \geq \Omega(\sqrt{n})$ for every k -regular n -vertex Hamiltonian graph G (with $k \geq 3$). This bound was subsequently obtained by several other authors. Yet, prior to our work, the \sqrt{n} bound was the best known result. In particular, Girão, Kittipassorn and Narayanan [14] remarked that improving this estimate would be of considerable interest. Furthermore, the following strengthening of the above conjecture of Jacobson and Lehel, which replaces the k -regularity condition with the assumption that the minimum degree is at least 3, was proposed by Verstraëte [28].

Conjecture 1.1. An n -vertex Hamiltonian graph G with $\delta(G) \geq 3$ has $\Omega(n)$ different cycle lengths.

While the special case of this conjecture for regular graphs already seems quite challenging, it is natural to expect that the full Conjecture 1.1 is even harder. The reason for this is that often problems become more difficult when the regularity requirement is replaced by a minimum degree assumption. One well-known example is a conjecture of Thomassen [25], that a graph with a large enough minimum degree contains a subgraph of large minimum degree and large girth. This conjecture is open even for girth 7. However, this statement becomes easy if the given graph is regular; see e.g., [21]. Such situations arise also for questions related to the one studied here: A classical result of Smith (see [27] and also [26]) states that every Hamiltonian 3-regular graph G contains a second Hamilton cycle. As was shown by Entringer and Swart [9], this is no longer true if instead of 3-regularity we assume that $\delta(G) \geq 3$ (even if all degrees are equal to 3 or 4). Girão, Kittipassorn and Narayanan [14] required an involved

proof to even show that a Hamiltonian G with $\delta(G) \geq 3$ contains a second cycle of length at least $n - o(n)$. In contrast, for regular G , this proof can be simplified considerably and gives a better bound.

It is worth noting that if one replaces the minimum degree requirement $\delta(G) \geq 3$, with the requirement that the average degree is at least 3, then the aforementioned lower bound of $\Omega(\sqrt{n})$ is tight. More generally, Milans, Pfender, Rautenbach, Regen and West [20] have shown that a graph G with n vertices and m edges satisfies $|\mathcal{C}(G)| \geq (1 - o(1))\sqrt{m - n}$, and this is tight up to lower-order terms.

In this paper, we prove the following theorem, which resolves Conjecture 1.1 asymptotically:

Theorem 1.1. *An n -vertex Hamiltonian graph G with $\delta(G) \geq 3$ contains cycles of $n^{1-o(1)}$ different lengths.*

Our proof of Theorem 1.1 is constructive: It gives a polynomial-time algorithm for finding cycles of $n^{1-o(1)}$ different lengths in a Hamiltonian graph of minimum degree 3, provided a Hamilton cycle is specified.

2. A sketch and the main ideas

The overarching idea that we employ is to split the Hamilton cycle into pieces (usually paths or pairs of paths) and then find paths with lengths on a different ‘scale’ in different parts. To illustrate what we mean, let us consider the following situation. Suppose that we managed to split our Hamilton cycle into two paths P_1, P_2 such that there are many chords inside the vertex-set of each P_i (or, more precisely, that inside each P_i there is a linear number of vertices touching a chord whose other endpoint is also on P_i). Suppose that we found $k = \Omega(\sqrt{n})$ paths Q_1, \dots, Q_k between the endpoints of P_1 (which only use the vertices of P_1) such that $|Q_1|, \dots, |Q_k|$ are all different and all belong to an interval of width \sqrt{n} . Suppose further that we found $\ell = \Omega(\sqrt{n})$ paths R_1, \dots, R_ℓ between the endpoints of P_2 (which only use the vertices of P_2) such that the lengths of any two of these paths are at least \sqrt{n} apart, namely, $||R_i| - |R_j|| > \sqrt{n}$ for all $i \neq j$. In this situation, we can combine any one of the Q_i ’s with any one the R_j ’s, joining them into a cycle of length $|Q_i| + |R_j|$. The crucial point is that the $k\ell$ numbers $|Q_i| + |R_j|$ are all different. In other words, we use the ‘condensed’ lengths Q_1, \dots, Q_k to ‘fill in the gaps’ between the ‘spread-out’ lengths R_1, \dots, R_ℓ (see Lemma 3.2 for the details on this). In total, this would give us $k\ell = \Omega(n)$ different cycle lengths. Hence, achieving both above goals would establish Conjecture 1.1.

We believe that both above statements should be true, namely, that one can find both $\Omega(\sqrt{n})$ distinct path lengths all contained in an interval of width \sqrt{n} and $\Omega(\sqrt{n})$ path lengths which are \sqrt{n} apart. Observe that both of these statements are essentially implied by Conjecture 1.1 (see Section 5 for more details) and that our main result shows that both hold asymptotically (i.e., with \sqrt{n} replaced by $n^{1/2-o(1)}$). On the other hand, these statements shift the difficulty from finding many lengths (note that there have been a number of proofs that there are at least \sqrt{n} different lengths over the years) to controlling what kind of lengths we find.

Our actual strategy for tackling Conjecture 1.1 is a bit more involved. Instead of splitting our cycle into just two parts, we split it into a larger number k of parts (with k to be chosen as roughly $\sqrt{\log n}$). Here, each part will be a pair of cycle sections (subpaths of the cycle) with at least $n^{1-o(1)}$ chords between them, with different section pairs situated ‘on top of’ each other (see Figure 2). Now, with the goal of finding $n^{1-\varepsilon}$ different lengths (where ε is an appropriately chosen vanishing function of n), we shall proceed as follows. Inside the first of the k parts, we shall find $\Omega(n^\varepsilon)$ path lengths all belonging to an interval of width n^ε . Then, inside the second part, we shall find about $\Omega(n^\varepsilon)$ lengths $\ell_1 < \dots < \ell_t$ such that any two consecutive lengths are $\Theta(n^\varepsilon)$ apart, namely $\ell_{i+1} - \ell_i = \Theta(n^\varepsilon)$ for all i . Now, by combining the paths we found in these two parts, we will get $\Omega(n^{2\varepsilon})$ different path lengths, all belonging to an interval of width $O(n^{2\varepsilon})$ and only using vertices from the first two parts of the partition. Continuing in this manner, we will find inside the third part $\Omega(n^\varepsilon)$ lengths which are $\Theta(n^{2\varepsilon})$ apart, inside the fourth part $\Omega(n^\varepsilon)$ lengths which are $\Theta(n^{3\varepsilon})$ apart and so on. This will always allow us to combine the new lengths we find with the lengths found so far to get $\Omega(n^{i\varepsilon})$ different path lengths, all belonging to an interval of width $O(n^{i\varepsilon})$, only using vertices from the first i parts. Unfortunately, in each iteration, we

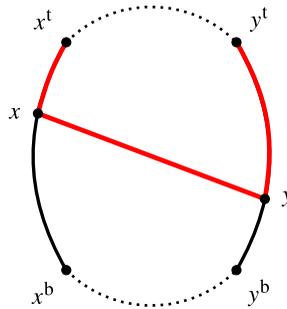


Figure 1. A section pair with a chord (x, y) and its corresponding trivial path marked in red.

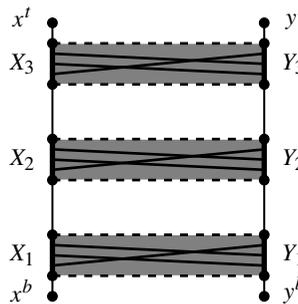


Figure 2. A parallel collection of subsection pairs.

will lose a polylogarithmic factor in the number of paths we find, which will result in the optimal number of iterations being $\sqrt{\frac{\log n}{\log \log n}}$ (this corresponds to having $\varepsilon = \sqrt{\frac{\log \log n}{\log n}}$). After this number of iterations, we will find $n^{1-o(1)}$ different lengths. This iterative process is handled in Lemma 4.8.

Let us now focus on a single iteration, and sketch the main ideas involved. For simplicity, suppose that this is the third iteration, namely, that our goal is to find (inside the third of the k parts of the partition) $\Omega(n^\varepsilon)$ path lengths $\ell_1 < \dots < \ell_t$ with $\ell_{i+1} - \ell_i = \Theta(n^{2\varepsilon})$ for all i . Up to this step, we have already found $\Theta(n^{2\varepsilon})$ path lengths in an interval of width $O(n^{2\varepsilon})$ inside the first two parts. Now, we consider a maximum collection e_1, \dots, e_m of chords inside the third part such that, for all $i \neq j$, the lengths of e_i and e_j differ by at least $n^{2\varepsilon}$. Each chord e_i gives rise to a path inside the third part (namely, the path that consists of the chord and pieces of the cycle), and the lengths of any two of these m paths differ by at least $n^{2\varepsilon}$. Now, observe that if $m \geq n^{1-3\varepsilon}$, then by combining these paths with the $\Theta(n^{2\varepsilon})$ path lengths we found in the first two parts of the partition, we obtain altogether $m \cdot \Omega(n^{2\varepsilon}) = \Omega(n^{1-\varepsilon})$ different cycle lengths and thus achieve our goal already at this stage. So we may assume that $m \leq n^{1-3\varepsilon}$. Since e_1, \dots, e_m is a maximal family, the length of any other chord must be at distance at most $n^{2\varepsilon}$ to that of one of the e_i 's. By averaging (and as each part of the partition contains $n^{1-o(1)}$ chords), we see that there is a family E of at least $n^{1-o(1)}/m \geq n^{3\varepsilon-o(1)}$ different chords, whose lengths all belong to an interval of width $n^{2\varepsilon}$. The reason such a family E is useful is as follows: Suppose we partition the two cycle sections of the third part into subpaths X_1, X_2, \dots of length $n^{2\varepsilon}$. Then, for any two such subpaths X_i, X_j of the same cycle section, which are not consecutive (and hence are at distance larger than $n^{2\varepsilon}$ on the path), any chord touching X_i must interlace (i.e., cross) any chord touching X_j . For if not, then the difference of the lengths of these two chords is larger than $n^{2\varepsilon}$, contradicting the fact that both lengths belong to an interval of width $n^{2\varepsilon}$. So we see that E decomposes into pairwise-interlacing pieces; see Figure 3 for an illustration. This structure, together with some additional arguments, then allows us to find the $\Omega(n^\varepsilon)$ desired path lengths $\ell_1 < \dots < \ell_t$. We remark that, while it is not hard to find such lengths with $\ell_{i+1} - \ell_i = \Omega(n^{2\varepsilon})$, which already allows us to find $\Omega(n^{3\varepsilon})$ lengths, it is essential

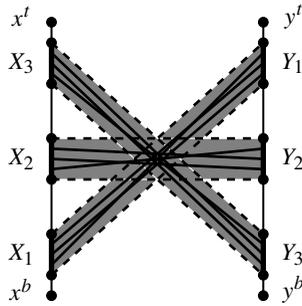


Figure 3. An interlacing collection of subsection pairs.

for the next iteration that these lengths are not too far apart, in other words ensuring in addition that $\ell_{i+1} - \ell_i \leq O(n^{2\epsilon})$ is crucial in order to be able to continue our argument.

Executing the above strategy presents significant technical difficulties, in part due to the need to join the various pieces in suitable ways. This sometimes makes various parts of the argument a bit cumbersome, and we try to alleviate this situation with a number of figures illustrating the argument.

Finally, we note that splitting the cycle into pieces (i.e., subpaths or pairs of sections) is nontrivial and requires some work. To illustrate this, note that if our graph only consisted of ‘diameter’ chords (joining vertices which are as far apart on the cycle as possible), then it is not possible to split the cycle into two paths with each of them still containing many chords. The way we go around this issue is to show that one can ‘reroute’ the original Hamilton cycle using two chords (and large parts of the Hamilton cycle), thus obtaining a new cycle which admits the desired split.

3. Setting up the stage

Let us introduce some terminology that we will use in the proof. For a path P , denote by $|P|$ the number of edges in P . Given a path or a cycle, we will call its subpaths *sections*. We will often consider chords between two sections of a cycle, and hence, it is convenient to have the following setup. A *section pair* is a pair of vertex-disjoint paths X, Y . We will always denote the endpoints of X by x^t and x^b , where x^t is called the top and x^b the bottom. Similarly, the endpoints of Y are denoted y^t, y^b . For distinct $x_1, x_2 \in X$, we say that x_1 is *above* x_2 if x_1 is closer to x^t along X than x_2 ; otherwise we say that x_1 is *below* x_2 . For sets $X_1, X_2 \subseteq X$, we say that X_1 is *above* (resp., *below*) X_2 if every $x_1 \in X_1$ is above (resp., below) every $x_2 \in X_2$. When considering a sequence X_1, \dots, X_t of disjoint subsets of X , unless otherwise specified, we assume they are labeled in such a way that X_i is below X_j for all $1 \leq i < j \leq t$. We use x_i^t (resp., x_i^b) to denote the top (resp., bottom) vertex of X_i .

For $x_1, x_2 \in X$, denote by $X[x_1, x_2]$ the subpath of X between x_1 and x_2 and by $d_X(x_1, x_2) = |X[x_1, x_2]|$ the length of this subpath, namely, the distance between x_1 and x_2 along X . Define $Y[y_1, y_2]$ and $d_Y(y_1, y_2)$ analogously for $y_1, y_2 \in Y$. A *subsection pair* of a section pair X, Y is a section pair consisting of a subpath of X and a subpath of Y .

A chord is an edge with one endpoint in X and one in Y . Let $(x_1, y_1), (x_2, y_2)$ be two chords which have no common vertices. We say that $(x_1, y_1), (x_2, y_2)$ are *parallel* if x_i is above x_{3-i} and y_i is above y_{3-i} for some $i = 1, 2$; otherwise we say that $(x_1, y_1), (x_2, y_2)$ are *interlacing*. In other words, $(x_1, y_1), (x_2, y_2)$ are interlacing if x_1 is above x_2 but y_1 is below y_2 , or vice versa.

The following statement is equivalent to (the symmetric case of) the Erdős-Szekeres lemma.

Lemma 3.1. *Let $k \geq 1$, and let X, Y be a section pair with at least $(k - 1)^2 + 1$ chords, no two of which share vertices. Then there is a set E of k chords such that either every two chords in E are parallel or every two chords in E are interlacing.*

Given a collection of disjoint subsection pairs $X_1, Y_1; \dots; X_t, Y_t$ (where as usual we will assume that X_i is below X_j for all $i < j$), we say that the collection is parallel if also Y_i is below Y_j for all $i < j$, and we say that it is interlacing if Y_i is above Y_j for all $i < j$. See Figures 2 and 3.

The length of a chord (x, y) is defined as $d_X(x^t, x) + d_Y(y^t, y)$, which is one less than the length of the path $X[x^t, x], (x, y), Y[y, y^t]$. This path will be called the *trivial path* corresponding to the chord (x, y) .

A *top-bottom path-pair* (for the section pair X, Y) is a pair of vertex-disjoint paths which are contained in $X \cup Y$, start at x^t, y^t and end at x^b, y^b . So, for example, X, Y is a (trivial) top-bottom path-pair. The length of a top-bottom path-pair $\mathcal{P} = (P_1, P_2)$ is defined as $|P_1| + |P_2|$ and denoted by $|\mathcal{P}|$.

Given $x \in X, y \in Y$, a path from x to y is called a *below-path* if it only uses vertices of X and Y which are below x and y , respectively.

At several points in the proof, we will split our cycle into disjoint parts (paths or section pairs) and concatenate paths (or path-pairs) which we find in different parts. Evidently, the length of the concatenated path is the sum of lengths of the individual paths. Hence, the set of path lengths we obtain in this way is the *sum-set* of the sets of path lengths we find in each of the different parts. Formally, given a sequence of sets L_1, \dots, L_t , we define the sum-set $L_1 + \dots + L_t := \{\ell_1 + \dots + \ell_t \mid \ell_i \in L_i\}$. Given a set L of integers, we say that two elements of L are *consecutive* (with respect to L) if there is no element of L between them. By *interval* we always mean an interval of natural numbers. We will often use the trivial fact that if x, y belong to an interval I , then $|x - y| \leq |I| - 1$. For numbers x, y , we say that x, y are *at least b apart* (or just *b apart*) if $|x - y| \geq b$ and *at most b apart* if $|x - y| \leq b$. The following auxiliary lemmas about sum-sets will come in handy later.

Lemma 3.2. *Let $L_1, \dots, L_t \subseteq \mathbb{N}$ with the property that for all $1 \leq i \leq t$, every two consecutive elements of L_i are at least a and at most b apart. Then there is a subset of $L_1 + \dots + L_t$ of size $1 + \sum_{i=1}^t (|L_i| - 1)$ in which every two consecutive elements are at least a and at most b apart.*

Proof. We will prove the lemma by induction on t . If $t = 1$, then the statement is trivial. Assume it holds for $t - 1$ so that we can find a subset S of $L_1 + \dots + L_{t-1}$ of size $1 + \sum_{i=1}^{t-1} (|L_i| - 1)$ with all consecutive elements being at least a and at most b apart. Let $x_1 < \dots < x_s$ be the elements of S , and let $y_1 < \dots < y_r$ be the elements of L_t . Then $\{x_1 + y_1 < x_2 + y_1 < \dots < x_s + y_1 < x_s + y_2 < \dots < x_s + y_r\}$ is a subset of $L_1 + \dots + L_t$ of size $|S| + |L_t| - 1 = 1 + \sum_{i=1}^t (|L_i| - 1)$ having the desired property. \square

Lemma 3.3. *Let L_1, L_2 be sets of positive integers. Suppose that each L_i is a subset of an interval of size ℓ_i and that any two elements of L_1 are at least ℓ_2 apart. Then $L_1 + L_2$ has $|L_1||L_2|$ elements, within some interval of size $\ell_1 + \ell_2$.*

Proof. Let $r_1 < \dots < r_t$ be the elements of L_1 and $s_1 < \dots < s_p$ be the elements of L_2 . We claim that all $r_i + s_j$ are distinct. To see this, assume that $r_i + s_j = r_{i'} + s_{j'}$. If $i \neq i'$, then we must have $|r_i - r_{i'}| \geq \ell_2$. However, $|s_j - s_{j'}| < \ell_2$, and so this is impossible. Hence, $i = i'$. This further implies that $j = j'$ and completes the proof. This shows that $L_1 + L_2$ indeed has $|L_1||L_2|$ elements. Furthermore, notice that all elements in $L_1 + L_2$ belong into $[r_1 + s_1, r_t + s_p]$, and since $r_t - r_1 < \ell_1$ and $s_p - s_1 < \ell_2$, this interval indeed has size at most $\ell_1 + \ell_2$. \square

All our logarithms are in base 2 unless otherwise specified. We will omit floor/ceiling signs whenever these are not crucial.

4. Finding cycles of many different lengths

In this section, we will prove our main result, Theorem 1.1. We begin, in the following subsection, by proving a number of lemmas which provide us with the main tools to attack the problem. We then proceed to put everything together in the subsequent subsection.

4.1. Main lemmas

The following lemma states that, in every section pair with many chords, one can find either a parallel or an interlacing collection of subsection pairs, each of which still has many chords.

Lemma 4.1. *Let X, Y be a section pair with m chords, and let $k \geq 1$. Suppose that no vertex of $X \cup Y$ is incident to more than $\frac{m}{10k^2}$ chords. Then there is either a parallel or an interlacing collection of subsection pairs $X_1, Y_1; \dots; X_k, Y_k$ such that $e(X_i, Y_i) \geq \Omega(\frac{m}{k^4})$ for all $1 \leq i \leq k$.*

Proof. We partition X into subpaths $X'_1, \dots, X'_s, X'_{s+1}$ such that, for each $1 \leq i \leq s$, X'_i touches at least $\frac{m}{10k^2}$ and at most $\frac{m}{5k^2}$ chords and such that X'_{s+1} touches less than $\frac{m}{10k^2}$ chords. We do this as follows: Take X'_1 to be the minimum initial segment of X which touches at least $\frac{m}{10k^2}$ chords. By minimality, the segment obtained by removing the last vertex x of X'_1 touches less than $\frac{m}{10k^2}$ chords. Also, x itself touches at most $\frac{m}{10k^2}$ chords by the assumption of the lemma. So overall, X'_1 touches at most $\frac{m}{5k^2}$ chords. We now remove X'_1 and continue in this fashion. As long as the remaining subpath touches at least $\frac{m}{10k^2}$ chords, we can extract an initial segment of it which touches at least $\frac{m}{10k^2}$ and at most $\frac{m}{5k^2}$ chords. In the end, we are left with a, possibly empty, subpath touching less than $\frac{m}{10k^2}$ chords, and we take this subpath to be X'_{s+1} .

In the same way, partition Y into subpaths $Y'_1, \dots, Y'_t, Y'_{t+1}$ with each $Y'_i, 1 \leq i \leq t$, touching at least $\frac{m}{10k^2}$ and at most $\frac{m}{5k^2}$ chords and with Y'_{t+1} touching less than $\frac{m}{10k^2}$ chords. Clearly, $4k^2 \leq (m - \frac{m}{10k^2}) / \frac{m}{5k^2} \leq s, t \leq 10k^2$. Define an auxiliary bipartite graph G with sides $[s]$ and $[t + 1]$, in which (i, j) is an edge if $e(X'_i, Y'_j) \geq \frac{m}{400k^4}$. Let $I \subseteq [s]$. We claim that $|N_G(I)| \geq |I|/4$. To see this, note that

$$\begin{aligned} |I| \cdot \frac{m}{10k^2} &\leq \sum_{i \in I} e(X'_i, Y) \leq \sum_{j \in N_G(I)} e(X, Y'_j) + (t + 1) \cdot |I| \cdot \frac{m}{400k^4} \\ &\leq |N_G(I)| \cdot \frac{m}{5k^2} + 2t \cdot |I| \cdot \frac{m}{400k^4} \leq |N_G(I)| \cdot \frac{m}{5k^2} + |I| \cdot \frac{m}{20k^2}. \end{aligned}$$

Rearranging gives $|N_G(I)| \geq |I|/4$. Now, by a well-known generalization of Hall’s theorem, there is a matching in G which saturates at least $s/4 \geq k^2$ of the elements of $[s]$. By Lemma 3.1, such a matching contains either a parallel or an interlacing family of chords of size k . It is easy to see that such a family gives sets $X_1, \dots, X_k, Y_1, \dots, Y_k$ as in the statement of the lemma. □

In a section pair X, Y with many chords, we will be able to find many paths of different lengths, for example, between x^t and y^b . However, in order to be able to join these paths into cycles of many different lengths, we need to find top-bottom path-pairs that include these paths. This is necessary in order to combine these lengths with lengths found in other section pairs above or below X, Y . In order to find such top-bottom path-pairs, it is useful to have a special chord $(x, y) \in E(X, Y)$ which interlaces all (or many) other chords in $E(X, Y)$. Indeed, observe that we can use such a chord (x, y) to walk from y^t to x^b , while using the chords which interlace (x, y) to obtain many path lengths from x^t to y^b . This way, the two paths (one from x^t to y^b and one from y^t to x^b) do not intersect each other; see the left part of Figure 10 (where the special chord (x, y) is denoted by (x_j, y_j)).

Evidently, a chord (x, y) as above does not exist if all chords in $E(X, Y)$ are pairwise-parallel. Hence, it is reasonable to try and partition X, Y into parallel subsections and try and find such a chord (x, y) in each of them. This again may fail if in a given part there is a vertex of a very high degree. As the following lemma shows, this is essentially the only obstacle. For an illustration of the two outcomes of Lemma 4.2, we refer the reader to Figures 8 and 9. As the statement of Lemma 4.2 suggests, we cannot guarantee that the special chord (x, y) interlaces a constant fraction of all other chords; we have to pay a log factor.

Lemma 4.2. *Let X, Y be a section pair with $m \geq 2$ chords. Then there is a parallel collection of subsection pairs $X_1, Y_1; \dots; X_t, Y_t$ such that $\sum_{i=1}^t e(X_i, Y_i) \geq m/24$ and one of the following holds:*

1. *For every i , there is a vertex in $X_i \cup Y_i$ which is incident to at least $\frac{e(X_i, Y_i)}{6 \log m}$ chords in $E(X_i, Y_i)$.*
2. *For every i , there is a chord in $E(X_i, Y_i)$ which interlaces at least $\frac{e(X_i, Y_i)}{6 \log m}$ of the chords in $E(X_i, Y_i)$.*

Proof. We will produce a partition $X_1, \dots, X_{t'}$ of X and a partition $Y_1, \dots, Y_{t'}$ of Y such that $\sum_{i=1}^{t'} e(X_i, Y_i) \geq m/12$, and each pair X_i, Y_i satisfies the condition of one of the items 1 or 2. Evidently, for one of the items, the pairs satisfying it would contribute at least half of all chords. Keeping only the pairs satisfying this item, we will obtain the desired collection.

We now construct such partitions $X_1, \dots, X_{t'}$ and $Y_1, \dots, Y_{t'}$ through the following process. We start with the trivial partition $\{X\}$ of X and $\{Y\}$ of Y . Note that we may assume that $m \geq 16$ or the trivial partition already satisfies the condition of item 1. Suppose that, at a given step of the process, there is a pair of sets X_i, Y_i which violates the conditions of both items 1 and 2. Partition X_i into two subsections X', X'' with X' above X'' , such that X' and X'' each touch at least $\left(\frac{1}{2} - \frac{1}{6 \log m}\right) e(X_i, Y_i)$ and at most $\left(\frac{1}{2} + \frac{1}{6 \log m}\right) e(X_i, Y_i)$ of the chords between X_i and Y_i . This is possible since every vertex of X_i is incident to less than $\frac{e(X_i, Y_i)}{6 \log m}$ of these chords (by the assumption that item 1 fails). Let x be the lowest vertex of X' which is incident to a chord, and let y be the lowest vertex of Y_i which is adjacent to x . Partition Y_i into subsections Y', Y'' such that Y' is above Y'' and y is the lowest vertex of Y' . Observe that if a chord $e \in E(X', Y'') \cup E(X'', Y')$ does not contain y , then e interlaces (x, y) . So the number of such chords is less than $\frac{e(X_i, Y_i)}{6 \log m}$. Also, $e(X_i, y) < \frac{e(X_i, Y_i)}{6 \log m}$. (Here, we used the assumption that X_i, Y_i violates both items 1 and 2). We conclude that $e(X', Y'') + e(X'', Y') < \frac{e(X_i, Y_i)}{3 \log m}$. It follows that $e(X', Y') = e(X', Y_i) - e(X', Y'') \geq \left(\frac{1}{2} - \frac{1}{2 \log m}\right) e(X_i, Y_i)$, and similarly $e(X'', Y'') \geq \left(\frac{1}{2} - \frac{1}{2 \log m}\right) e(X_i, Y_i)$. Note that we also have the upper bound $e(X', Y'), e(X'', Y'') \leq \left(\frac{1}{2} + \frac{1}{6 \log m}\right) e(X_i, Y_i)$. We now define new partitions by replacing X_i with X', X'' and Y_i with Y', Y'' . This way, we may continue the process until every pair X_i, Y_i satisfies the condition of one of the items 1 or 2. Indeed, at each step, we replace some pair X_i, Y_i with pairs having fewer edges than X_i, Y_i . If the number of edges of some pair is at most 6, then this pair trivially satisfies item 1. Hence, the process must terminate.

To each pair (X_i, Y_i) appearing in the course of the process, we assign a binary string as follows. Assign the initial pair (X, Y) the empty string. If $(X', Y'), (X'', Y'')$ are obtained by splitting (X_i, Y_i) as above and (X_i, Y_i) is assigned the string σ , then let (X', Y') be assigned the string $\sigma, 0$ and (X'', Y'') the string $\sigma, 1$. Let $h(X_i, Y_i)$ be the length of the string assigned to a pair X_i, Y_i . It is easy to show, by induction, that

$$\sum \left(\frac{1}{2}\right)^{h(X_i, Y_i)} = 1 \tag{4.1}$$

is preserved throughout the process, where the sum ranges over all pairs X_i, Y_i at a given moment. Moreover, by our upper and lower bounds on $e(X', Y'), e(X'', Y'')$, we get by induction that

$$0 < m \cdot \left(\frac{1}{2} - \frac{1}{2 \log m}\right)^{h(X_i, Y_i)} \leq e(X_i, Y_i) \leq m \cdot \left(\frac{1}{2} + \frac{1}{6 \log m}\right)^{h(X_i, Y_i)}$$

for every pair (X_i, Y_i) appearing in the process. In particular, $h(X_i, Y_i) \leq 2 \log m$ for each such pair (X_i, Y_i) . Hence,

$$e(X_i, Y_i) \geq m \cdot \left(\frac{1}{2} - \frac{1}{2 \log m}\right)^{h(X_i, Y_i)} \geq m \cdot \left(1 - \frac{1}{\log m}\right)^{2 \log m} \left(\frac{1}{2}\right)^{h(X_i, Y_i)} \geq \frac{m}{12} \cdot \left(\frac{1}{2}\right)^{h(X_i, Y_i)}.$$

Consider the partitions $X_1, \dots, X_{t'}$ and $Y_1, \dots, Y_{t'}$ at the end of the process. By equation (4.1) and the above, we have $\sum_{i=1}^{t'} e(X_i, Y_i) \geq m/12$. This completes the proof of the lemma. \square

The final, main lemma of this subsection is Lemma 4.7 below. Before stating it, we will prove some auxiliary lemmas that will feature in its proof. We start with the following lemma.

Lemma 4.3. *Let X, Y be a section pair with m chords, and suppose that each vertex of X is incident to at most one chord. Then either there is a family of $m/2$ pairwise-interlacing chords or we can find pairs of chords (e_i, e'_i) , $e_i = (x_i, y_i)$, $e'_i = (x'_i, y'_i)$, $i = 1, \dots, t$ such that the following holds:*

1. For every $1 \leq i \leq t$, e_i, e'_i are either parallel or share a vertex in Y .
2. For every $1 \leq i < j \leq t$, both e_i and e'_i interlace both e_j and e'_j .
3. $\sum_{i=1}^t d_X(x_i, x'_i) \geq m/4$.

Proof. We execute the following process. To start, set $E_1 = E(X, Y)$. For each $i \geq 1$, we do as follows. If every two chords in E_i are interlacing, then stop. Otherwise, among all pairs of chords in E_i which are parallel or share a vertex, choose one pair $e_i, e'_i \in E_i$, $e_i = (x_i, y_i)$, $e'_i = (x'_i, y'_i)$, which maximizes $d_X(x_i, x'_i)$. Without loss of generality, let us assume that x_i is above x'_i (note that x_i, x'_i are distinct by our assumption that every vertex in X is incident to at most one chord; y_i, y'_i may be equal). Since e_i, e'_i are parallel or share a vertex, it holds that either $y_i = y'_i$ or y_i is above y'_i . Let X^* be the set of vertices of X between x_i and x'_i (including x_i, x'_i), and let Y^* be the set of vertices of Y between y_i and y'_i (including y_i, y'_i). Observe that if $e = (x, y) \in E_i$ is a chord with $y \in Y^*$, then x must be in X^* . Indeed, if x is above x_i , then e, e'_i are parallel or share a vertex and $d_X(x, x'_i) > d_X(x_i, x'_i)$; similarly, if x is below x'_i , then e, e_i are parallel or share a vertex and $d_X(x, x_i) > d_X(x_i, x'_i)$. In both cases, we get a contradiction to the maximality of e_i, e'_i .

Let F be the set of chords $(x, y) \in E_i$ with $x \in X^*$. Clearly, $|F| \geq 2$ because $e_i, e'_i \in F$. Set $E_{i+1} := E_i \setminus F$. Note that $d_X(x_i, x'_i) = |X^*| - 1 \geq |F| - 1 \geq |F|/2 = (|E_i| - |E_{i+1}|)/2$, where the first inequality holds because every vertex of X^* is incident to at most one chord. Above we have shown that F contains all chords (x, y) with $y \in Y^*$. It is now easy to see that each $e = (x, y) \in E_{i+1}$ interlaces both e_i and e'_i . Indeed, let $e = (x, y) \in E_{i+1}$. Then x is either above x_i or below x'_i , and y is either above y_i or below y'_i . If x is above x_i and y is above y_i , then e, e'_i are parallel with $d_X(x, x_i) > d(x_i, x'_i)$, contradicting the maximality of e_i, e'_i . Similarly, it is impossible that x is below x'_i and y is below y'_i . In the remaining two cases, e interlaces e_i, e'_i . This ensures that item 2 will be satisfied.

Suppose that the process continues for t steps. Then in E_{t+1} , every two chords are interlacing. So if $|E_{t+1}| \geq m/2$, then we are done. Suppose then that $|E_{t+1}| < m/2$. Consider the pairs of chords $e_i = (x_i, y_i)$, $e'_i = (x'_i, y'_i)$, $1 \leq i \leq t$. We have already shown that items 1 and 2 hold. For item 3, we have

$$\sum_{i=1}^t d_X(x_i, x'_i) \geq \frac{1}{2} \sum_{i=1}^t (|E_i| - |E_{i+1}|) = \frac{1}{2} (|E_1| - |E_{t+1}|) \geq m/4,$$

as required. □

The next two lemmas are concerned with combining paths over a family of interlacing subsections.

Lemma 4.4. *Let X, Y be a section pair with $3t$ interlacing chords, whose lengths belong into an interval of size D . Then there are paths P_1, \dots, P_t between x^t and y^t such that $1 \leq |P_{i+1}| - |P_i| \leq 2D$ for every $1 \leq i \leq t - 1$. In particular, $|P_t| \geq |P_1| + t - 1$.*

Proof. Let $(x_1, y_1), \dots, (x_{3t}, y_{3t}) \in E(X, Y)$ be the given family of interlacing chords, with x_i above x_j and y_i below y_j for each $1 \leq i < j \leq 3t$. For $1 \leq i \leq 3t - 1$, put $d_i := d_Y(y_i, y_{i+1}) - d_X(x_i, x_{i+1})$. Note that d_i is precisely the difference between the lengths of (x_i, y_i) and (x_{i+1}, y_{i+1}) , so by our assumption, we have $|d_i| \leq D - 1$. We prove the lemma by induction on t . In the base case $t = 1$, the desired path P_1 can be chosen as the trivial path of the chord (x_1, y_1) .

For the induction step, we will first find two paths Q_1, Q_2 between x_1 and x_3 such that $1 \leq |Q_2| - |Q_1| \leq 2D$ and such that Q_1, Q_2 do not use any vertices of X below x_3 or any vertices of Y above y_3 . To find such Q_1, Q_2 , we proceed as follows. For $i = 1, 2$, put $a_i := d_X(x_i, x_{i+1})$ and $b_i := d_Y(y_i, y_{i+1})$ so that $d_i = b_i - a_i$. Consider the following four paths between x_1 and x_3 .

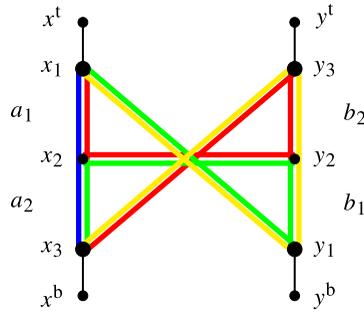


Figure 4. Different paths P'_i .

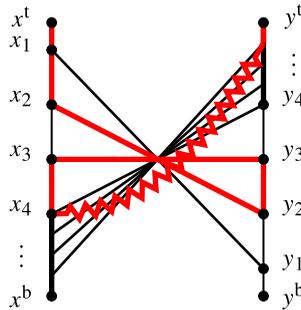


Figure 5. A combination of P'_2 and a path provided by induction (marked as zigzag in the figure).

- $P'_1 = X[x_1, x_3]$, depicted in blue in Figure 4;
- $P'_2 = X[x_1, x_2], (x_2, y_2), Y[y_2, y_3], (y_3, x_3)$, depicted in red in Figure 4;
- $P'_3 = (x_1, y_1), Y[y_1, y_2], (y_2, x_2), X[x_2, x_3]$, depicted in green in Figure 4;
- $P'_4 = (x_1, y_1), Y[y_1, y_3], (y_3, x_3)$, depicted in yellow in Figure 4.

Observe that P'_1, \dots, P'_4 do not use any vertices of X below x_3 or any vertices of Y above y_3 . We have $|P'_1| = a_1 + a_2, |P'_2| = a_1 + b_2 + 2, |P'_3| = a_2 + b_1 + 2, |P'_4| = b_1 + b_2 + 2$. Hence, $|P'_2| - |P'_1| = d_2 + 2, |P'_3| - |P'_1| = d_1 + 2, |P'_4| - |P'_1| = d_1 + d_2 + 2, |P'_3| - |P'_2| = d_1 - d_2, |P'_4| - |P'_2| = d_1$ and $|P'_4| - |P'_3| = d_2$. So we see that, for all $Q_1, Q_2 \in \{P'_1, \dots, P'_4\}$, it holds that $||Q_2| - |Q_1|| \leq |d_1| + |d_2| + 2 \leq 2D$.

Observe that the four numbers $a_1 + a_2, a_1 + b_2 + 2, a_2 + b_1 + 2, b_1 + b_2 + 2$ cannot all be equal. Indeed, if the first three are equal, then $b_1 = a_1 - 2$ and $b_2 = a_2 - 2$, which means that $b_1 + b_2 + 2 = a_1 + a_2 - 2 \neq a_1 + a_2$. It follows that there are $Q_1, Q_2 \in \{P'_1, \dots, P'_4\}$ with $|Q_1| \neq |Q_2|$, say $|Q_1| < |Q_2|$. We have already shown that $|Q_2| - |Q_1| \leq 2D$. So Q_1, Q_2 satisfy all of our requirements.

We now complete the induction step using the paths Q_1, Q_2 found above. Let $t \geq 2$, and apply the induction hypothesis with parameter $t - 1$ and with X and Y replaced by $X[x_4, x^b]$ and $Y[y^t, y_4]$, which still contain $3(t - 1)$ of our interlacing chords. This way we obtain paths R_1, \dots, R_{t-1} between x_4 and y^t such that $1 \leq |R_{i+1}| - |R_i| \leq 2D$ for every $1 \leq i \leq t - 2$. For $1 \leq i \leq t - 1$, let P_i be the concatenation of $X[x^t, x_1], Q_1, X[x_3, x_4]$ and R_i . Let P_t be the concatenation of $X[x^t, x_1], Q_2, X[x_3, x_4]$ and R_{t-1} . See Figure 5 for an illustration of the resulting paths P_i . For each $1 \leq i \leq t - 2$, we have $|P_{i+1}| - |P_i| = |R_{i+1}| - |R_i|$, and hence $1 \leq |P_{i+1}| - |P_i| \leq 2D$. Lastly, $|P_t| - |P_{t-1}| = |Q_2| - |Q_1|$, and hence $1 \leq |P_t| - |P_{t-1}| \leq 2D$ as well. This completes the proof. \square

Lemma 4.5. *Let X, Y be a section pair, let $t \geq 1$ be odd and let $X_1, Y_1; \dots; X_t, Y_t$ be an interlacing collection of subsection pairs. Let $d_1, \dots, d_t \geq 1$. Suppose that for each $1 \leq i \leq t$, there are two paths $Q_i^R, Q_i^B \subseteq X_i \cup Y_i$ with $|Q_i^R| - |Q_i^B| \geq d_i$, which go between x_i^t and y_i^t if i is odd and between x_i^b and*

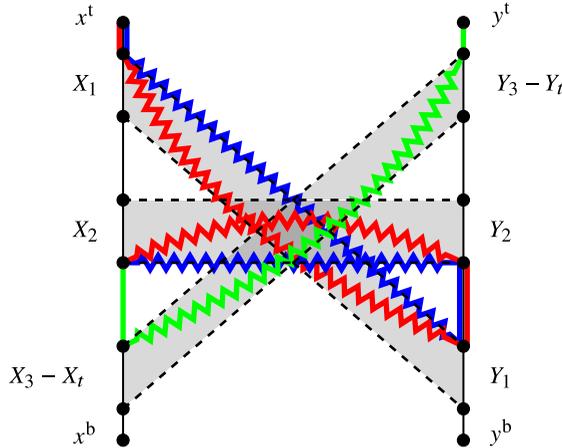


Figure 6. Q_1^R, Q_2^R are marked as red zigzag paths, Q_R^c is depicted in red, Q_1^B, Q_2^B are marked as blue zigzag paths, Q_B^c is depicted in blue and some R_i is depicted in green.

y_i^b if i is even. Set $D := \max_{i=1, \dots, t} (|X_i| + |Y_i|)$. Then there are paths $P_1, \dots, P_{(t+3)/2}$ between x^t and y^t such that $|P_{(t+3)/2}| - |P_1| \geq \sum_{i=1}^t d_i$ and such that $1 \leq |P_{i+1}| - |P_i| \leq 2D$ for every $1 \leq i \leq (t+1)/2$.

Proof. Without loss of generality, assume that X_i is above X_j for all $i < j$ (and consequently, the opposite is true for the Y_i 's). The proof is by induction on t . For the case $t = 1$, simply take P_1/P_2 to be the concatenation of $X[x^t, x_1^t]$, Q_1^B/Q_1^R and $Y[y_1^t, y^t]$. Clearly, $|P_2| - |P_1| = |Q_1^R| - |Q_1^B|$, which immediately implies that $|P_2| - |P_1| \geq d_1 \geq 1$. On the other hand, since Q_1^R, Q_1^B are both contained in $X_1 \cup Y_1$, we have $|P_2| - |P_1| = |Q_1^R| - |Q_1^B| \leq |X_1| + |Y_1| \leq D$, as required.

Suppose now that $t \geq 3$. For each $c = R, B$, define a path Q_c^c from x^t to x_2^b as follows:

$$Q_c^c = X[x^t, x_1^t], Q_1^c, Y[y_1^t, y_2^b], Q_2^c.$$

See Figure 6 for an illustration. We have $|Q_R^c| - |Q_B^c| = \sum_{i=1}^2 (|Q_i^R| - |Q_i^B|) \geq d_1 + d_2$. On the other hand, since $Q_i^R, Q_i^B \subseteq X_i \cup Y_i$ (for $i = 1, 2$), we have that $|Q_i^R| - |Q_i^B| \leq D$ and hence $|Q_R^c| - |Q_B^c| \leq 2D$. Note that Q_B^c, Q_R^c do not use any vertices of X below x_2^b or any vertices of Y above y_2^b . Now apply the induction hypothesis with parameter $t - 2$ and with X and Y replaced by $X[x_2^b, x^b]$ and $Y[y^b, y_3^b]$. This gives paths $R_1, \dots, R_{(t+1)/2}$ between x_2^b and y^t such that $|R_{(t+1)/2}| - |R_1| \geq \sum_{i=3}^t d_i$ and such that $1 \leq |R_{i+1}| - |R_i| \leq 2D$ for every $1 \leq i \leq (t-1)/2$. For $1 \leq i \leq (t+1)/2$, let P_i be the concatenation of Q_B^c and R_i . Let $P_{(t+3)/2}$ be the concatenation of Q_R^c and $R_{(t+1)/2}$. Then $|P_{(t+3)/2}| - |P_1| = |Q_R^c| - |Q_B^c| + |R_{(t+1)/2}| - |R_1| \geq \sum_{i=1}^t d_i$. Moreover, for every $1 \leq i \leq (t-1)/2$ we have $|P_{i+1}| - |P_i| = |R_{i+1}| - |R_i|$ and hence $1 \leq |P_{i+1}| - |P_i| \leq 2D$. Finally, we have $|P_{(t+3)/2}| - |P_{(t+1)/2}| = |Q_R^c| - |Q_B^c|$ and hence $1 \leq |P_{(t+3)/2}| - |P_{(t+1)/2}| \leq 2D$. This completes the proof. \square

Using Lemmas 4.3, 4.4 and 4.5, we can now prove the following lemma, which states that in a section pair with m chords one can find two paths whose lengths differ by $\Omega(m)$, provided every vertex on one side is incident to at most one chord.

Lemma 4.6. *Let X, Y be a section pair with $m \geq 12$ chords, and suppose that each vertex of X is incident to at most one chord. Then there are two paths P, P' between x^t and y^t such that $|P'| \geq |P| + \Omega(m)$. By symmetry, there are two paths Q, Q' between x^b and y^b such that $|Q'| \geq |Q| + \Omega(m)$.*

Proof. Apply Lemma 4.3. If $E(X, Y)$ contains a family of $m/2$ pairwise-interlacing chords, then we are done by Lemma 4.4. Otherwise, let $e_i = (x_i, y_i), e'_i = (x'_i, y'_i), 1 \leq i \leq t$, be as in Lemma 4.3. Without loss of generality we may assume, for each $1 \leq i \leq t$, that x_i is above x'_i , and hence either $y_i = y'_i$ or y_i

is above y'_i . For $1 \leq i \leq t$, let $X_i := X[x_i, x'_i]$, $Y_i := Y[y_i, y'_i]$; so $x_i^t = x_i$, $x_i^b = x'_i$, $y_i^t = y_i$, and $y_i^b = y'_i$. Item 2 of Lemma 4.3 guarantees that, after possibly permuting the coordinates $1, \dots, t$, we have that X_i is above X_j and Y_i is below Y_j for every $1 \leq i < j \leq t$. Set $d_i := d_X(x_i, x'_i)$. Observe that, for each $1 \leq i \leq t$, $Q_i^B := (x_i, y_i)$ and $Q_i^R := X[x_i, x'_i], (x'_i, y'_i), Y[y'_i, y_i]$ are two paths between x_i and y_i , both contained in $X_i \cup Y_i$ such that $|Q_i^R| - |Q_i^B| = d_X(x_i, x'_i) + d_Y(y_i, y'_i) \geq d_i$. Similarly, $Q_i^B := (x'_i, y'_i)$ and $Q_i^R := X[x'_i, x_i], (x_i, y_i), Y[y_i, y'_i]$ are two paths between x'_i and y'_i , both contained in $X_i \cup Y_i$ such that $|Q_i^R| - |Q_i^B| \geq d_i$. Hence, we are in the setting of Lemma 4.5. By item 3 of Lemma 4.3, we have $\sum_{i=1}^t d_i \geq m/4$. If t is even, then omit a pair (X_i, Y_i) for which $d_i = \min\{d_1, \dots, d_t\}$; this way we guarantee that t is odd (which is necessary for invoking Lemma 4.5), while making sure that the sum $\sum d_i$ over the remaining i is at least $m/8$. To complete the proof, apply Lemma 4.5 and set $P := P_1$ and $P' := P_{(t+3)/2}$. \square

Finally, we are in a position to prove Lemma 4.7. This lemma states that if a section pair has many chords whose lengths are close to each other, say all contained in some (small) interval J , then one can find a sequence of path lengths, with any two consecutive lengths differing by $\Theta(|J|)$. This lemma will play a key role in the proof of Theorem 1.1.

Lemma 4.7. *Let X, Y be a section pair, and suppose that each vertex of X is incident to at most one chord. Let $E \subseteq E(X, Y)$ be a set of chords, and let $J \subseteq \mathbb{N}$ be an interval such that, for each $e \in E$, the length of e belongs to J . Then for $k = \Omega(|E|/|J|)$, there are paths P_1, \dots, P_k between x^t and y^t such that $|P_{i+1}| - |P_i| = \Theta(|J|)$ for every $1 \leq i \leq k - 1$.*

Proof. We will assume that $|E| \gg |J|$, as otherwise, the assertion is trivial. We only consider the chords in E (removing all others). Partition X into subsections X_1, \dots, X_s such that X_1, \dots, X_{s-1} have $|J|$ vertices each, and X_s has less than $|J|$ vertices. (As usual, X_i is below X_j for $i < j$.) Note that if $i \neq j$ have the same parity, then for all $x_1 \in X_i, x_2 \in X_j$ we have $d_X(x_1, x_2) > |J|$. Without loss of generality, we shall assume that $\sum_{i \text{ even}} e(X_i) \geq \sum_{i \text{ odd}} e(X_i)$, and hence $\sum_{i \text{ even}} e(X_i) \geq |E|/2$ (the other case is symmetrical). For convenience, let us set $I_0 = \{i \in [s] : i \text{ even}\}$. For each $i \in I_0$, let Y_i be the minimal subpath of Y containing all neighbours of X_i . Note that $\sum_{i \in I_0} e(X_i, Y_i) \geq |E|/2$.

Observe that, for all $(x_1, y_1), (x_2, y_2) \in E(X_i, Y_i)$, we have $d_Y(y_1, y_2) \leq d_X(x_1, x_2) + |J| < 2|J|$, where the first inequality follows from the fact that the lengths of (x_1, y_1) and (x_2, y_2) both belong to J and hence differ by at most $|J|$. So we see that $|Y_i| \leq 2|J|$. Next, observe that, for every pair $i, j \in I_0$ with $i < j$ and for all $(x_1, y_1) \in E(X_i, Y_i), (x_2, y_2) \in E(X_j, Y_j)$, it must be the case that y_2 is below y_1 . Indeed, if not, then $(x_1, y_1), (x_2, y_2)$ either are parallel or share a vertex, which means that the difference between the lengths of (x_1, y_1) and (x_2, y_2) is at least $d_X(x_1, x_2) > |J|$, contradicting that both of these lengths are in J . So we see that Y_j is below Y_i . In other words, every chord in $E(X_i, Y_i)$ interlaces every chord in $E(X_j, Y_j)$.

Let I' (resp., I'') be the set of all $i \in I_0$ with $0 < e(X_i, Y_i) < 12$ (resp., $e(X_i, Y_i) \geq 12$). Suppose first that $\sum_{i \in I'} e(X_i, Y_i) \geq |E|/4$. Then by picking one chord from $E(X_i, Y_i)$ for each $i \in I'$, we obtain a family of pairwise-interlacing chords of size $\Omega(|E|)$. Let $(x_1, y_1), \dots, (x_{3t}, y_{3t})$ be such a family with $t = \Omega(|E|)$. By Lemma 4.4, there are paths P_1, \dots, P_t between x^t and y^t such that $1 \leq |P_{i+1}| - |P_i| \leq 2|J|$ for every $1 \leq i \leq t - 1$. It is easy to see that, under these conditions, we can find indices $1 \leq i_1 < \dots < i_k$ with $k = \Omega(t/|J|) = \Omega(|E|/|J|)$ such that $|P_{i_{j+1}}| - |P_{i_j}| = \Theta(|J|)$ for every $1 \leq j \leq k - 1$. Thus, the assertion of the lemma holds in this case.

Suppose now that $\sum_{i \in I''} e(X_i, Y_i) \geq |E|/4$. Write $I'' = \{i_1, \dots, i_t\}$, where $i_1 < \dots < i_t$. If t is even, then we only consider i_1, \dots, i_{t-1} , thus making sure that the number of indices in consideration is odd. Observe that $e(X_{i_t}, Y_{i_t}) \leq |X_{i_t}| \leq |J|$ (since every vertex of X is incident to at most one chord), and hence $\sum_{j=1}^{t-1} e(X_{i_j}, Y_{i_j}) \geq |E|/4 - |J| \geq |E|/8$. So with a slight abuse of notation, we will assume from now on that t is odd and $\sum_{j=1}^t e(X_{i_j}, Y_{i_j}) \geq |E|/8$.

Fix any $1 \leq j \leq t$. Applying Lemma 4.6 to X_{i_j} and Y_{i_j} gives paths $P, P' \subseteq X_{i_j} \cup Y_{i_j}$ between $x_{i_j}^t$ and $y_{i_j}^t$ with $|P'| \geq |P| + \Omega(e(X_{i_j}, Y_{i_j}))$, as well as paths $Q, Q' \subseteq X_{i_j} \cup Y_{i_j}$ between $x_{i_j}^b$ and $y_{i_j}^b$ with $|Q'| \geq |Q| + \Omega(e(X_{i_j}, Y_{i_j}))$. Hence, we are in the setting of Lemma 4.5 with $d_j = \Omega(e(X_{i_j}, Y_{i_j}))$.

Moreover, $D := \max_{j=1, \dots, t} (|X_{i_j}| + |Y_{i_j}|) \leq 3|J|$. By Lemma 4.5, there are paths $P_1, \dots, P_{(t+3)/2}$ between x^t and y^t such that $|P_{(t+3)/2}| - |P_1| \geq \sum_{j=1}^t d_j = \sum_{j=1}^t \Omega(e(X_{i_j}, Y_{i_j})) = \Omega(|E|)$ and such that $1 \leq |P_{i+1}| - |P_i| \leq 2D = O(|J|)$ for every $1 \leq i \leq (t+1)/2$. Again, it is easy to see that under these conditions, we can find indices $1 \leq i_1 < \dots < i_k$ with $k = \Omega(|E|/|J|)$ such that $|P_{i_{j+1}}| - |P_{i_j}| = \Theta(|J|)$ for every $1 \leq j \leq k-1$. So we have established the assertion of the lemma in this case as well. This completes the proof. \square

4.2. Putting it all together

The following is the main lemma in the proof of Theorem 1.1.

Lemma 4.8. *Let X, Y be a section pair, and suppose that each vertex of X is incident to at most one chord. Let $X_1, Y_1; \dots; X_t, Y_t$ be a parallel collection of subsection pairs such that for every i , there are m chords between X_i and Y_i . Provided $t \geq \sqrt{\log m}$, there are at least $\Omega(m/2^{4\sqrt{\log m \log \log m}})$ different lengths of paths from x^t to y^t .*

Proof. We may and will assume throughout that m is sufficiently large. Let us for convenience define $\varepsilon := \sqrt{\frac{\log \log m}{\log m}}$. Observe that $m^\varepsilon = 2^{\sqrt{\log m \log \log m}}$, so our goal is to find $\Omega(m^{1-4\varepsilon})$ different lengths of paths from x^t to y^t . Let us assume towards a contradiction that there are less than $\Omega(m^{1-4\varepsilon})$ different lengths of paths from x^t to y^t . Recall that x_i^t/x_i^b denote the top/bottom vertices of X_i , and similarly y_i^t/y_i^b denote the top/bottom vertices of Y_i . We will prove the following claim by induction on i .

Induction statement. For a large enough constant $C > 0$ and for every $i \leq 1/\varepsilon$, there are below-paths from x_i^t to y_i^t of $\frac{m^{i\varepsilon}}{6(C \log m)^{2(i-1)}}$ different lengths, all of which belong to an interval of size $m^{i\varepsilon}$.

Before turning to the proof of this statement, let us show that it indeed implies the lemma. By choosing $i = \lfloor 1/\varepsilon \rfloor$, we have $m^{i\varepsilon} \geq m^{1-\varepsilon}$ and

$$(\log m)^{1/\varepsilon} = 2^{\sqrt{\log m \log \log m}} = m^\varepsilon. \tag{4.2}$$

So, using, e.g., that $C^2 \leq \log m$, we get

$$\frac{m^{i\varepsilon}}{(C \log m)^{2(i-1)}} \geq \frac{m^{1-\varepsilon}}{(\log m)^{3i}} \geq \frac{m^{1-\varepsilon}}{(\log m)^{3/\varepsilon}} \geq m^{1-4\varepsilon}.$$

This implies we find at least $\Omega(m^{1-4\varepsilon})$ path lengths, as desired.

From now on, our goal is to prove the above induction statement. For the base case of $i = 1$, note that, since for any chord (x, y) , the trivial path $X[x^t, x], (x, y), Y[y, y^t]$ has length exactly the length of (x, y) plus one, there can be at most $m^{1-4\varepsilon} \leq m^{1-\varepsilon}$ different chord lengths. Since there are m chords between X_1 and Y_1 , there must exist a length which appears at least m^ε many times. Apply Lemma 4.4 with $t = \lfloor m^\varepsilon/3 \rfloor$ to the family of chords having this length, noting that chords of the same length must be interlacing and that for such chords the parameter D in Lemma 4.4 equals 1. Lemma 4.4 gives us a collection of $t \geq m^\varepsilon/6$ lengths of paths between x_1^t and y_1^t with all consecutive differences being either 1 or 2 apart so in total belonging into an interval of size at most $2t + 1 \leq m^\varepsilon$, as desired.

Let us now move on to the induction step. So assume that for some $1 \leq i \leq 1/\varepsilon - 1$, there are below-paths going from x_i^t to y_i^t of $\frac{m^{i\varepsilon}}{6(C \log m)^{2(i-1)}}$ different lengths, all of which belong to an interval of size $m^{i\varepsilon}$. Note that we can extend all of these paths to below-paths from x_{i+1}^b to y_{i+1}^b by using $X[x_i^t, x_{i+1}^b]$ and $Y[y_i^t, y_{i+1}^b]$ and still keep the same property concerning their lengths, as each length is increased by the same number.

Our goal now is to find in X_{i+1}, Y_{i+1} top-bottom path-pairs with many (roughly $m^\varepsilon/\text{poly log } m$) different lengths, among which any two consecutive ones are $\Theta(m^{i\varepsilon})$ apart. We can use any such top-bottom path-pair to extend any of our below-paths from x_{i+1}^b to y_{i+1}^b into a below-path from x_{i+1}^t to y_{i+1}^t with the length being the sum of the lengths of our original path and that of the top-bottom path-pair

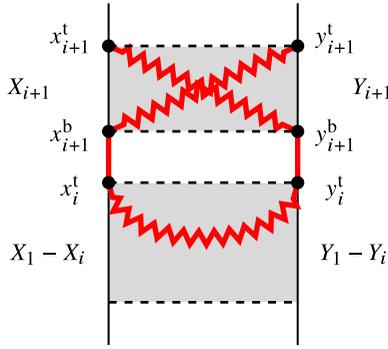


Figure 7. The lower part of the path is provided by the induction and can be chosen among $\tilde{\Omega}(m^{i\varepsilon})$ paths with lengths in an interval of size $m^{i\varepsilon}$. The upper top-bottom path-pair for X_{i+1}, Y_{i+1} can be chosen among a collection of $\tilde{\Omega}(m^\varepsilon)$ of them with lengths $m^{i\varepsilon}$ apart.

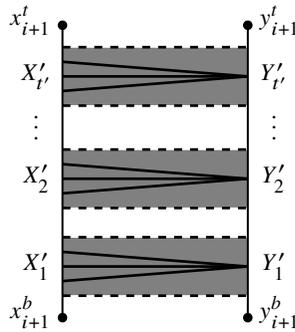


Figure 8. Case 1 of Lemma 4.2. Note that trivial paths using any two distinct edges of a star always have different lengths.

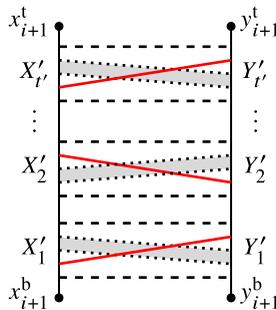


Figure 9. Case 2 of Lemma 4.2. Edges (x_j, y_j) are depicted in red. Shaded regions have many chords and will become X''_j, Y''_j .

we used. Lemma 3.3 will then allow us to ‘fill in the gaps’ of size $\Theta(m^{i\varepsilon})$ between the lengths of the top-bottom path-pairs we found, by using the path lengths given by the inductive assumption. This will complete the induction step. See Figure 7 for an illustration.

Let us apply Lemma 4.2 to the section pair X_{i+1}, Y_{i+1} to obtain a parallel collection of subsection pairs $X'_1, Y'_1; \dots; X'_{i+1}, Y'_{i+1}$ such that $\sum_{j=1}^{i+1} e(X'_j, Y'_j) \geq m/24$ and either for every j we have a vertex in $X'_j \cup Y'_j$ incident to $e(X'_j, Y'_j)/(6 \log m)$ chords, or for every j there is a chord (x_j, y_j) in $E(X'_j, Y'_j)$ interlacing at least $e(X'_j, Y'_j)/(6 \log m)$ chords in $E(X'_j, Y'_j)$. See Figures 8 and 9. In the former case,

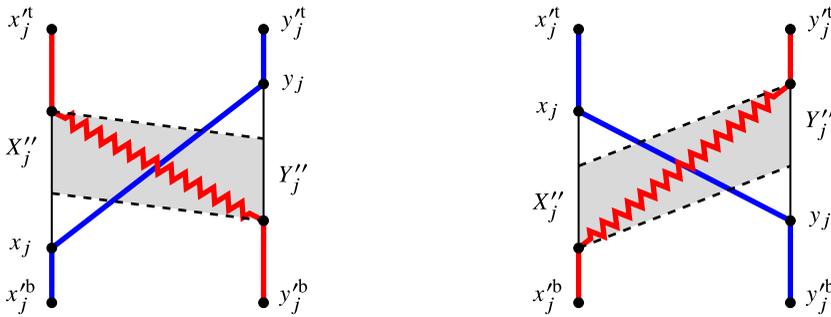


Figure 10. Two options for how (x_j, y_j) can interlace X''_j, Y''_j and the corresponding top-bottom path-pairs we use.

we get $\Omega(m/\log m) \geq \Omega(m^{1-4\epsilon})$ chords, any two of which are either parallel or share a vertex, which means that all of these chords have different lengths. This in turn means that we have $\Omega(m^{1-4\epsilon})$ different lengths of (trivial) paths from x^t to y^t , giving us a contradiction. So we may assume that we are in the latter case.

For each j , consider all chords in $E(X'_j, Y'_j)$ which interlace (x_j, y_j) . Observe that they can be of two types: Either their X -vertex is above x_j and their Y -vertex is below y_j , or vice versa. Consider only the interlacing chords of the more frequent type, and define X''_j, Y''_j to be the smallest subsections of X'_j, Y'_j which contain all these chords. In particular, $x_j \notin X''_j, y_j \notin Y''_j$, and (X''_j, Y''_j) is interlaced by (x_j, y_j) as a section pair. Moreover, we still have $\sum_{j=1}^{t'} e(X''_j, Y''_j) \geq \Omega(m/\log m)$ chords altogether. Consider the intervals of the form $[k, 2k]$ with k a power of 2 and $k \leq m/2$. Each $e(X''_j, Y''_j)$ belongs to such an interval, of which there are $O(\log m)$. So by averaging, there exists $1 \leq k \leq m/2$ and a subset $S \subseteq [t']$ such that for any $j \in S$ we have $e(X''_j, Y''_j) \in [k, 2k]$, and $\sum_{j \in S} e(X''_j, Y''_j) \geq \Omega(m/\log^2 m)$. Furthermore, by deleting $e(X''_j, Y''_j) - k \leq \frac{1}{2}e(X''_j, Y''_j)$ chords from each pair X''_j, Y''_j with $j \in S$ (so in total at most a half of all such chords), we may assume for convenience that $e(X''_j, Y''_j) = k$ for all $j \in S$. Observe in particular that

$$|S|k \geq \Omega(m/\log^2 m). \tag{4.3}$$

We now construct a collection of top-bottom path-pairs for X'_j, Y'_j for each $j \in S$. Depending on which of the two types above was more frequent, we have two essentially symmetric cases. In the first one, X''_j is above x_j and Y''_j is below y_j . Here, we will take our path pairs to all contain the path which goes along X'_j from the bottom to x_j , uses the chord (x_j, y_j) and then goes along Y'_j from y_j to the top of Y'_j . The second path in each pair is going to follow X'_j from the top until the top of X''_j , then follow a path inside $X''_j \cup Y''_j$ from the top of X''_j to the bottom of Y''_j , and then follow along Y'_j to its bottom. Symmetrically, in the second case, our path pairs will all use the path through (x_j, y_j) from the top of X'_j to the bottom of Y'_j , with the other path (from the bottom of X'_j to the top of Y'_j) passing through $X''_j \cup Y''_j$. See Figure 10 for these two symmetric cases. Since the argument is analogous in both cases, from now on, let us assume we are in the first case.

Observe that if we find a collection of path lengths inside $X''_j \cup Y''_j$ going from the top of X''_j to the bottom of Y''_j , then we have the same collection of lengths of top-bottom path-pairs for X'_j, Y'_j up to a possible translation by a fixed constant. This is because the top-bottom path-pairs we consider only differ in which path they use between the top of X''_j and the bottom of Y''_j .

It will be convenient to consider the ‘flipped’ section pair X''_j, \bar{Y}''_j , which we obtain by reversing the path Y''_j (and denote it as \bar{Y}''_j). In particular, the top of \bar{Y}''_j is the bottom of Y''_j and any top-to-top path in X''_j, \bar{Y}''_j is just a path between the top of X''_j and the bottom of Y''_j .

Claim 1. *For each $j \in S$, one of the following holds.*

1. *There are at least $2k^{1-(i+1)\varepsilon}$ lengths, any two at least $k^{i\varepsilon}$ apart, of top-to-top paths in X''_j, \bar{Y}''_j .*
2. *There is an interval $I_j \subseteq \mathbb{N}$ of size $k^{i\varepsilon}$ such that the length with respect to X''_j, \bar{Y}''_j of at least $k^{(i+1)\varepsilon}/4$ chords belongs to I_j .*

Proof. All our chord lengths in this proof will be with respect to X''_j, \bar{Y}''_j . Take a minimal collection of disjoint intervals of size $k^{i\varepsilon}$ each, which cover all chord lengths (i.e., the length with respect to X''_j, \bar{Y}''_j of every chord in $E(X''_j, \bar{Y}''_j)$ belongs to one of these intervals). For each such interval I , either I can be taken as the desired I_j , or there must be less than $k^{(i+1)\varepsilon}/4$ chords with length in I . As there are k chords in $E(X''_j, \bar{Y}''_j)$, this means that we needed to use at least $4k^{1-(i+1)\varepsilon}$ intervals to cover all chord lengths. Since we took a minimal collection, each interval is nonempty. By picking one length from every second interval, we find $2k^{1-(i+1)\varepsilon}$ chord lengths, any two separated by one of our intervals (since we took a length from every second interval). Hence, any two such lengths are at least $k^{i\varepsilon}$ apart. For each such length, take a chord of this length and take the trivial path between the top of X''_j and the top of \bar{Y}''_j corresponding to this chord. This gives the $2k^{1-(i+1)\varepsilon}$ path lengths required by item 1. \square

Let S_1 be the subset of S consisting of j for which case 1 of the above claim occurred. Let us first assume this was the more common case, namely that $|S_1| \geq |S|/2$.

Claim 2. *If $|S_1| \geq |S|/2$, then there are $\Omega(m^{1-(i+1)\varepsilon}/\log^2 m)$ lengths of top-bottom path-pairs for X_{i+1}, Y_{i+1} which are all at least $m^{i\varepsilon}$ apart.*

Proof. For each $j \in S_1$, consider the $2k^{1-(i+1)\varepsilon}$ top-to-top paths for X''_j, \bar{Y}''_j given by item 1 of Claim 1. Each of these paths is a path from the top of X''_j to the bottom of \bar{Y}''_j . Hence, as explained above, this gives rise to top-bottom path-pairs for X'_j, Y'_j with at least $2k^{1-(i+1)\varepsilon} \geq 2$ (recall that $i \leq 1/\varepsilon - 1$) lengths, any two of which are at least $k^{i\varepsilon}$ apart. Denote the set of these lengths by L_j . Now observe that by choosing a path-pair for X'_j, Y'_j of length $\ell_j \in L_j$ for each $j \in S_1$ and joining these paths with subpaths of X, Y , we obtain a top-bottom path-pair for X_{i+1}, Y_{i+1} of length $\sum_{j \in S_1} \ell_j + C_0$, where C_0 is the sum of the lengths of the subpaths we used to join our top-bottom path-pairs into a top-bottom path-pair for X_{i+1}, Y_{i+1} ; so C_0 is independent of our choices for ℓ_j . Lemma 3.2 now tells us that we can find $|S_1|k^{1-(i+1)\varepsilon}$ lengths of top-bottom path-pairs for X_{i+1}, Y_{i+1} which are at least $k^{i\varepsilon}$ apart. By taking every $m^{i\varepsilon}/k^{i\varepsilon}$ -th such length, we get at least $\lfloor |S_1|k^{1-(i+1)\varepsilon}/(m^{i\varepsilon}/k^{i\varepsilon}) \rfloor \geq \frac{1}{2}|S_1|k^{1-\varepsilon}/m^{i\varepsilon} \geq \frac{1}{4}|S|k^{1-\varepsilon}/m^{i\varepsilon} \geq \frac{1}{4}|S|k/m^{(i+1)\varepsilon} \geq \Omega(m^{1-(i+1)\varepsilon}/\log^2 m)$ lengths which are at least $m^{i\varepsilon}$ apart, where the last inequality uses (4.3). This proves the claim. \square

Let us now use Claim 2 to complete the proof in the case that $|S_1| \geq |S|/2$. Observe that we may combine our top-bottom path-pairs for X_{i+1}, Y_{i+1} (provided by Claim 2) with top-to-top below-paths from x_{i+1}^t to y_{i+1}^t , provided by the inductive assumption, to obtain below-paths from x_{i+1}^t to y_{i+1}^t , which can then be extended along X, Y into top-to-top paths for X, Y . Each such path length is of the form $\ell_1 + \ell_2 + C_0$, where ℓ_1 may be chosen among our $\Omega(m^{1-(i+1)\varepsilon}/\log^2 m)$ lengths of top-bottom path-pairs for X_{i+1}, Y_{i+1} , which are at least $m^{i\varepsilon}$ apart; ℓ_2 may be chosen among $\Omega(m^{i\varepsilon}/(C \log m)^{(2i-2)})$ lengths which belong in an interval of size $m^{i\varepsilon}$ (as guaranteed by the induction hypothesis), and C_0 is independent of the choice of ℓ_1, ℓ_2 . Therefore, by Lemma 3.3 we get at least $\Omega(m^{1-\varepsilon}/(C \log m)^{2i}) \geq \Omega(m^{1-4\varepsilon})$ (see equation (4.2)) different lengths of top-to-top paths for X, Y , contradicting our assumption that the number of such lengths is smaller than $\Omega(m^{1-4\varepsilon})$. This completes the proof in the case $|S_1| \geq |S|/2$.

So we must have the second case in Claim 1 occurring more often, namely that $S_2 := S \setminus S_1$ has size at least $|S|/2$. By definition, for any $j \in S_2$ there is an interval I_j of size $k^{i\varepsilon}$ such that at least $k^{(i+1)\varepsilon}/4$ chords in $E(X''_j, \bar{Y}''_j)$ have their length with respect to the subsection pair X''_j, \bar{Y}''_j belong in I_j . Applying Lemma 4.7 (with $J = I_j$), we find $\Omega(k^\varepsilon)$ top-to-top paths in X''_j, \bar{Y}''_j whose lengths are $\Theta(k^{i\varepsilon})$ apart. As before, this gives us top-bottom path-pairs for X'_j, Y'_j of lengths which are $\Theta(k^{i\varepsilon})$ apart. Once again, we can choose one such length for each X'_j, Y'_j and combine the corresponding top-bottom path-pairs into

a top-bottom path-pair for X_{i+1}, Y_{i+1} , with the resulting length being the sum of the lengths of the top-bottom path-pairs we chose plus a fixed number (which is independent of our choices). Lemma 3.2 then tells us that we can find top-bottom path-pairs for X_{i+1}, Y_{i+1} of $\Omega(|S_2|k^\varepsilon) \geq \Omega(|S|k^\varepsilon)$ lengths which are $\Theta(k^{i\varepsilon})$ apart, say between $c_1k^{i\varepsilon}$ and $c_2k^{i\varepsilon}$. Once again, we only consider every $m^{i\varepsilon}/c_1k^{i\varepsilon}$ -th length, thus guaranteeing that every two such lengths are at least $m^{i\varepsilon}$ and at most $\frac{c_2}{c_1}m^{i\varepsilon} = O(m^{i\varepsilon})$ apart. The number of these lengths is at least $c_1k^{i\varepsilon}/m^{i\varepsilon} \cdot \Omega(|S|k^\varepsilon) \geq \Omega(k^{(i+1)\varepsilon-1}m^{1-i\varepsilon}/\log^2 m) \geq \Omega(m^\varepsilon/\log^2 m)$, with the first inequality using equation (4.3). Take the first $\Theta(m^\varepsilon/\log^2 m)$ of them. Since these lengths are at most $O(m^{i\varepsilon})$ apart, they are all contained in an interval of size $O(m^{(i+1)\varepsilon})$. Now similarly as in the previous case, we can combine any of these top-bottom path-pairs with our below paths from x_i^t to y_i^t , which are provided by the induction hypothesis. Lemma 3.3 ensures that, by doing so, we get at least $\Omega(m^\varepsilon/\log^2 m) \cdot \frac{m^{i\varepsilon}}{(C \log m)^{2(i-1)}} = \Omega\left(\frac{m^{(i+1)\varepsilon}}{C^{2i-2}(\log m)^{2i}}\right)$ lengths which all belong to an interval of size $O(m^{(i+1)\varepsilon})$. By potentially losing a constant proportion of the lengths, we can guarantee they all lie in an interval of size at most $m^{(i+1)\varepsilon}$, and by choosing C to be large enough, we will have at least $\frac{m^{(i+1)\varepsilon}}{(C \log m)^{2i}}$ of them, completing the induction step and hence the proof. \square

We are now in a position to prove Theorem 1.1, which we rephrase in the following quantitative form:

Theorem 4.9. *Every n -vertex Hamiltonian graph with minimum degree 3 has cycle spectrum of size at least $\Omega\left(n/2^{6\sqrt{\log n \log \log n}}\right)$.*

Proof. Let us fix a Hamilton cycle H in our graph and a direction for this cycle. We define the length of a chord $\{x, y\}$ to be the length of the shorter of the two paths along the cycle between x and y ; in particular, the length is always at most $n/2$. For every vertex v , we fix a single chord $c(v)$ incident to v . By the pigeonhole principle, there must exist an $\ell \leq n/4$ such that at least $n/\log n$ vertices have the length of their chosen chord in $[\ell, 2\ell]$. Now let us partition H into consecutive sections B_1, \dots, B_s, R (appearing in this order along H), where $|B_s| = 4\ell$ and $s = \lfloor \frac{n}{4\ell} \rfloor \geq 1$, and the starting vertex for B_1 is chosen uniformly at random among the vertices of the cycle. We will talk of the first half of B_j (which is just the set of the first 2ℓ vertices of B_j), the first quarter of B_j (the set of the first ℓ vertices), etc.

A vertex v is said to be *good* if $c(v)$ is contained in some B_j and if the two endpoints of $c(v)$ belong to different halves of B_j . We claim that the probability that a given v with $c(v) \in [\ell, 2\ell]$ is good is at least $\frac{1}{8}$. To see this, write $c(v) = \{v, w\}$, and consider the shorter path (along the cycle) between v and w ; recall that the length of $c(v)$ is defined as the length of this path. There are two cases: If when walking on this path from v to w , we walk in the direction of the cycle (which was fixed at the beginning), then v is guaranteed to be good if it belongs to the second quarter of some B_j . Otherwise, i.e., if we walk against the direction of the cycle, then v is guaranteed to be good if it belongs to the third quarter of some B_j . In any case, the probability that v is good is at least $\frac{1}{4} \cdot \frac{|B_1| + \dots + |B_s|}{n}$. Since $|B_1| + \dots + |B_s| \geq n/2$, this probability is at least $\frac{1}{8}$, as claimed. By linearity of expectation, the expected number of good vertices is at least $\frac{n}{8 \log n}$. Let us now fix an outcome B_1, \dots, B_s, R for which we have at least this many good vertices. Let I be the set of $i \in [s]$ such that B_i contains at least $\frac{\ell}{4 \log n}$ good vertices. Observe that the sections $B_i, i \notin I$, contribute in total at most $s \cdot \frac{\ell}{4 \log n} \leq \frac{n}{16 \log n}$ to the overall number of good vertices. Therefore, at least $\frac{n}{16 \log n}$ good vertices belong to sections B_i with $i \in I$. In particular, $|I| \geq \frac{n}{64\ell \log n}$ since each B_i has 4ℓ vertices.

For each $i \in I$, we will later find inside B_i paths of at least $1 + \Omega(\ell/2^5 \sqrt{\log n \log \log n})$ different lengths joining the endpoints of B_i . Let us first complete the proof using this. To do so, choose for each $i \in I$, one of these paths joining the endpoints of B_i , and combine these paths into a cycle using H . The length of the resulting cycle is equal to the sum of the lengths of the paths we chose plus a fixed number, independent of our choices (i.e., the total length of the pieces of H we used to join the paths). So by Lemma 3.2, these cycles take at least $\Omega\left(\frac{n}{\ell \log n} \cdot \ell/2^5 \sqrt{\log n \log \log n}\right) \geq \Omega\left(n/2^{6\sqrt{\log n \log \log n}}\right)$ different lengths, as desired.

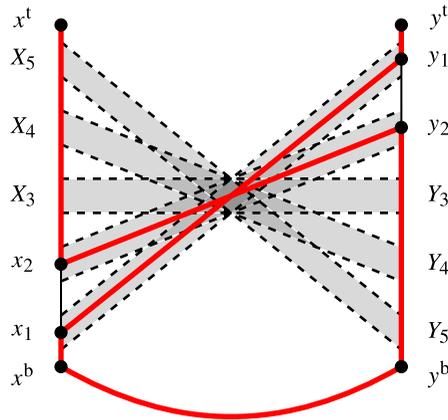


Figure 11. *Rerouting through a path B'_i marked in red.*

What remains to be proved is that, for $i \in I$, B_i contains paths of $1 + \Omega(\ell/2^5\sqrt{\log n \log \log n})$ different lengths joining its endpoints. By definition, B_i has at least $\frac{\ell}{4\log n}$ good vertices. Since B_i contains at least one chord, we know it has at least two path lengths (one coming from the length of B_i itself and the other from the path only using a single chord). In particular, we may assume that $\ell \gg 2^5\sqrt{\log n \log \log n}$ or we are done. Let X denote the first half of B_i and Y its second half. We think of X, Y as a section pair (omitting the middle edge of B_i) and consider the endpoints of B_i to be the top vertices of X and Y . By definition, every chord $c(v)$ which corresponds to a good vertex v has one endpoint in X and one in Y . Let us assume without loss of generality that X contains at least half (so at least $\frac{\ell}{8\log n}$) of all good vertices in B_i . Delete any chord which was not chosen by one of the good vertices in X . This way, we make sure that every vertex in X is incident to at most one chord. Also, at least $\frac{\ell}{8\log n}$ chords remain.

Let us now apply Lemma 4.1 to this section pair X, Y with parameter $m = \frac{\ell}{8\log n}$ and $k = \sqrt{\log n}$ and thus obtain either a vertex $v \in Y$ incident to at least $\Omega(\ell/\log^2 n)$ chords or an interlacing or parallel collection of k subsection pairs $X_1, Y_1; \dots; X_k, Y_k$, each containing at least $\Omega(\ell/\log^3 n)$ chords.

Suppose first that there is a vertex $v \in Y$ incident to at least $\Omega(\ell/\log^2 n)$ chords $(v, x), x \in X$. Observe that all these chords have different lengths. Now, for each such chord, consider the path between the endpoints of B_i , which uses only this chord and no others. These are paths of $\Omega(\ell/\log^2 n)$ different lengths between the endpoints of B_i , as required.

Suppose now that we are in the second case, namely that there exists an interlacing or parallel collection $X_1, Y_1; \dots; X_k, Y_k$ of subsection pairs with $e(X_i, Y_i) \geq \Omega(\ell/\log^3 n)$ for each $i = 1, \dots, k$. If these subsection pairs are parallel, then we may immediately apply Lemma 4.8 with $t = k$ and $m = \Omega(\ell/\log^3 n)$ to get the desired number $\Omega(\frac{\ell}{\log^3 n}/2^4\sqrt{\log n \log \log n}) \geq \Omega(\ell/2^5\sqrt{\log n \log \log n})$ of different path lengths between the endpoints of B_i . In the case that $X_1, Y_1; \dots; X_k, Y_k$ are interlacing, we need an extra step, as follows. Suppose without loss of generality that X_i is below X_j (and hence Y_i is above Y_j) for each $1 \leq i < j \leq k$. Take $(x_1, y_1) \in E(X_1, Y_1)$ and $(x_2, y_2) \in E(X_2, Y_2)$. Let us now ‘reroute’ B_i along the interlacing chords $(x_1, y_1), (x_2, y_2)$; in other words, we consider a new path B'_i , obtained by following X from the top until x_2 , jumping along (x_2, y_2) , following B_i (now in the other direction) until x_1 , jumping along (x_1, y_1) and finally following Y until its top. See Figure 11 for a picture. Note that B'_i splits into the section pair $X' := X[x^t, x_2], Y' := B_i[y_2, x_1], (x_1, y_1), Y[y_1, y^t]$, with X' having the same top as X and Y' the same top as Y . Observe that X_k, \dots, X_3 still appear in this order along X' , but, crucially, Y_3, \dots, Y_k now appear in the reverse order; see Figure 12. In other words, the subsection pairs $X_3, Y_3; \dots; X_k, Y_k$ are parallel in X', Y' . So we may now apply Lemma 4.8 with $t = k - 2$ and

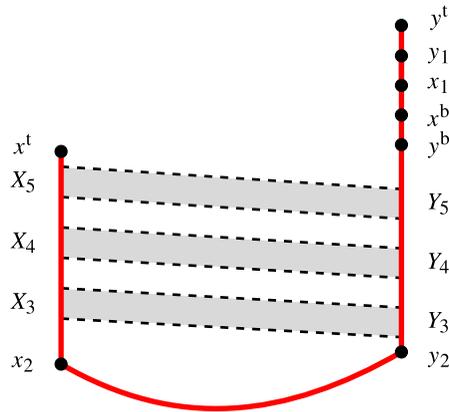


Figure 12. Rerouted path B'_i with new relative positions of the section pairs.

$m = \Omega(\ell/\log^3 n)$ to get the desired number of different path lengths between the endpoints of B_i . This completes the proof. \square

5. Concluding remarks

In this paper, we proved that an n -vertex Hamiltonian graph of minimum degree 3 has cycles of $n^{1-o(1)}$ different lengths, which shows that Conjecture 1.1 holds asymptotically. Moreover, for the original question of Jacobson and Lehel (dealing with graphs of bounded degree), we can use our ideas to get a better quantitative bound of $\frac{n}{\text{polylog}(n)}$. Still, it would be very interesting to prove a linear bound on the number of cycles, even in the 3-regular case. Towards this, we propose the following natural intermediate steps:

Conjecture 5.1. In an n -vertex Hamiltonian graph with minimum degree 3, there are:

1. $\Omega(\sqrt{n})$ cycle lengths all belonging to an interval of size $O(\sqrt{n})$.
2. $\Omega(\sqrt{n})$ cycle lengths any two of which are at least $\Omega(\sqrt{n})$ apart.

Observe first that Conjecture 1.1 immediately implies Conjecture 5.1. On the other hand, a slight strengthening of Conjecture 5.1 already implies Conjecture 1.1. Indeed, suppose that in Conjecture 5.1, we replace the assumption of minimum degree 3 with the assumption that linearly many vertices have degree at least 3. Furthermore, suppose that instead of starting with a Hamilton cycle we start with a Hamilton path, and instead of cycle lengths we consider path lengths of paths going between the endpoints of the given Hamilton path. Then by splitting our Hamilton cycle (or a ‘rerouted’ cycle) into two paths (similarly to what is done in the proof of Theorem 4.9), we can apply this strengthened version of Conjecture 5.1 and obtain linearly many cycle lengths (see Section 2 for some details).

We note that ideas similar to the general approach described above, namely of controlling the large- and small-scale behavior separately in a way which allows us to combine the gains, have been very useful in a number of situations. This idea has been used in additive combinatorics for a long time; see, for example, Lemma 1 in a 1964 paper of Graham [16] and a more recent example of its use by Conlon, Fox and Pham in the context of subset sums, to settle a number of conjectures in the area [7]. Another example is the recent remarkable work of Liu and Montgomery [19], resolving the Erdős-Hajnal odd cycle problem. In that paper, in order to find a path of some specific length ℓ between two given vertices, the authors first find a path of length somewhat close to ℓ , and then ‘fill in the gap’ by repeatedly changing the path length by 2, using certain gadgets (see Lemma 4.8 in [19]).

Finally, let us also discuss the following related question, which was already mentioned in passing in the introduction. What is the minimum C such that every n -vertex Hamiltonian graph with minimum

degree 3 contains a second cycle of length at least $n - C$? Girão, Kittipassorn and Narayanan [14] show that $C = O(n^{4/5}) = o(n)$, and conjecture that actually $C = O(1)$. It is also interesting to ask whether the answer changes if we want in addition to have $C > 0$ (i.e., if we forbid this second cycle from being a Hamilton cycle). Such a result might be useful to completely settle the conjecture of Jacobson and Lehel.

Acknowledgements. We are grateful to the anonymous referees for their careful reading of the paper and several suggestions which improved the presentation.

Funding statement. Research supported in part by a grant from the SNSF(200021_196965) and by a grant from NSF (CCF-1900460).

Conflicts of Interest. None.

References

- [1] D. Bauer and E. Schmeichel, ‘Hamiltonian degree conditions which imply a graph is pancyclic’, *J. Combin. Theory Ser. B* **48**(1) (1990), 111–116.
- [2] J. A. Bondy, ‘Pancyclic graphs I’, *J. Combin. Theory Ser. B* **11**(1) (1971), 80–84.
- [3] J. A. Bondy, ‘Pancyclic graphs: recent results, infinite and finite sets’, *Colloq. Math. Soc. János Bolyai*; dedicated to P. Erdős on his 60th birthday, vol. **10**, 1973, pp. 181–187.
- [4] J. A. Bondy, ‘*Longest paths and cycles in graphs of high degree*’, Department of Combinatorics and Optimization, University of Waterloo, 1980.
- [5] V. Chvátal, ‘On Hamilton’s ideals’, *J. Combin. Theory Ser. B* **12**(2) (1972), 163–168.
- [6] V. Chvátal and P. Erdős, ‘A note on Hamiltonian circuits’, *Discrete Math.* **2**(2) (1972), 111–113.
- [7] D. Conlon, J. Fox and H. T. Pham, ‘Subset sums, completeness and colorings’, Preprint, 2021, [arXiv:2104.14766](https://arxiv.org/abs/2104.14766).
- [8] G. A. Dirac, ‘Some theorems on abstract graphs’, *Proc. Lond. Math. Soc.* **3**(1) (1952), 69–81.
- [9] R. C. Entringer and H. Swart, ‘Spanning cycles of nearly cubic graphs’, *J. Combin. Theory Ser. B* **29**(3) (1980), 303–309.
- [10] P. Erdős, ‘Some of my favourite problems in various branches of combinatorics’, *Le Matematiche* **47**(2) (1992), 231–240.
- [11] P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, ‘The number of cycle lengths in graphs of given minimum degree and girth’, *Discrete Math.* **200**(1–3) (1999), 55–60.
- [12] G.-H. Fan, ‘New sufficient conditions for cycles in graphs’, *J. Combin. Theory Ser. B* **37**(3) (1984), 221–227.
- [13] J. Gao, Q. Huo, C.-H. Liu and J. Ma, ‘A unified proof of conjectures on cycle lengths in graphs’, *Int. Math. Res. Not. IMRN* **10** (2022), 7615–7653.
- [14] A. Girão, T. Kittipassorn and B. Narayanan, ‘Long cycles in Hamiltonian graphs’, *Israel J. Math.* **229**(1) (2019), 269–285.
- [15] R. J. Gould, P. E. Haxell and A. D. Scott, ‘A note on cycle lengths in graphs’, *Graphs Combin.* **18**(3) (2002), 491–498.
- [16] R. L. Graham, ‘Complete sequences of polynomial values’, *Duke Math. J.* **31** (1964), 275–285.
- [17] B. Jackson and O. Ordaz, ‘Chvátal–Erdős conditions for paths and cycles in graphs and digraphs. a survey’, *Discrete Math.* **84**(3) (1990), 241–254.
- [18] P. Keevash and B. Sudakov, ‘Pancyclicity of Hamiltonian and highly connected graphs’, *J. Combin. Theory Ser. B* **100**(5) (2010), 456–467.
- [19] H. Liu and R. Montgomery, ‘A solution to Erdős and Hajnal’s odd cycle problem’, Preprint, 2020, [arXiv:2010.15802](https://arxiv.org/abs/2010.15802).
- [20] K. G. Milans, F. Pfender, D. Rautenbach, F. Regen and D. B. West, ‘Cycle spectra of Hamiltonian graphs’, *J. Combin. Theory Ser. B* **102**(4) (2012), 869–874.
- [21] R. Montgomery, A. Pokrovskiy and B. Sudakov, ‘ C_4 -free subgraphs with large average degree’, *Israel J. Math.* **246** (2021), 55–71.
- [22] O. Ore, ‘Note on Hamilton circuits’, *Amer. Math. Monthly* **67** (1960), 55.
- [23] E. F. Schmeichel and S. L. Hakimi, ‘A cycle structure theorem for hamiltonian graphs’, *J. Combin. Theory Ser. B* **45**(1) (1988), 99–107.
- [24] B. Sudakov and J. Verstraëte, ‘Cycle lengths in sparse graphs’, *Combinatorica* **28**(2) (2008), 357–372.
- [25] C. Thomassen, ‘Girth in graphs’, *J. Combin. Theory Ser. B* **35**(2) (1983), 129–141.
- [26] C. Thomassen, ‘Independent dominating sets and a second Hamiltonian cycle in regular graphs’, *J. Combin. Theory Ser. B* **72**(1) (1998), 104–109.
- [27] W. T. Tutte, ‘On Hamiltonian circuits’, *J. Lond. Math. Soc.* **1**(2) (1946), 98–101.
- [28] J. Verstraëte, ‘Extremal Problems for Cycles in Graphs’, in *Recent Trends in Combinatorics* vol. **159** (Springer, Cham, 2016) 83–116. [doi:10.1007/978-3-319-24298-9_4](https://doi.org/10.1007/978-3-319-24298-9_4).