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A THEORY OF POLYTOPES

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The basic properties of polytopes and their faces are derived from a set of axioms which are satisfied, in particular, by polytopes in Euclidean, hyperbolic or (hemi-)spherical space. The underlying space is not assumed to be either dense or unbounded.

1. INTRODUCTION

A *polytope* is the convex hull of a finite set of points. This definition makes sense in any set on which there is defined a hull operator, or closure operator in the terminology of Cohn [**3**]. However, polytopes in n-dimensional Euclidean space have a richer theory, relating to their *faces* and thus involving linearity. It is of interest to study also the facial properties of polytopes in n-dimensional hyperbolic space or in an open hemisphere of n-dimensional spherical space. What, then, is the natural framework for a theory of polytopes?

The polytopes studied here include the examples just mentioned, but also include examples with only finitely many points. Using axioms taken from our previous paper [**4**], we establish the basic facial properties of polytopes. It is noteworthy that we do not assume either *density* ($a \neq b$ implies $c \in [a, b]$ for some $c \neq a, b$) or *unboundedness* ($a \neq b$ implies $a \in [b, c]$ for some $c \neq a, b$), although these properties are liberally used in the standard treatments of polytopes, such as Brøndsted [**2**], Grünbaum [**5**] or Prenowitz and Jantosciak [**7**].

2. CONVEXITY

Suppose that with any unordered pair $\{a, b\}$ of elements of a set X there is associated a *segment*, that is a subset $[a, b]$ of X containing a and b. We shall say that a *linear geometry* is defined on X if the following six axioms are satisfied:

(C) if $c \in [a, b_1]$ and $d \in [c, b_2]$, then $d \in [a, b]$ for some $b \in [b_1, b_2]$,
(P) if $c_1 \in [a, b_1]$ and $c_2 \in [a, b_2]$, then $[b_1, c_2] \cap [b_2, c_1] \neq \emptyset$,
(L1) $[a, a] = \{a\}$,
(L2) if $b \in [a, c]$, $c \in [b, d]$ and $b \neq c$, then $b \in [a, d]$,
(L3) if $c \notin [a, b]$ and $b \notin [a, c]$, then $[a, b] \cap [a, c] = \{a\}$,
(L4) if $c \in [a, b]$, and $[a, b] = [a, c] \cup [c, b]$.

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The axioms **(C)** and **(P)** were first used by Peano in 1889 in an analysis of Pasch's axiom for plane geometry. The axioms **(L1)**–**(L4)**, as we shall see, enable the points of a line to be totally ordered. Simple examples, with $|X| \leqslant 5$, show that each of the six axioms is independent of the others. An example of a linear geometry with only finitely many points is the set $X = \{a, a', b, b', c\}$ with $[a, a'] = \{a, a', c\}$, $[b, b'] = \{b, b', c\}$ and $[x, y] = \{x, y\}$ otherwise.

We do not intend to study here all the properties of linear geometries, but shall concentrate attention on those which are needed for our ultimate goal. Some properties will not require all six axioms, but we prefer not to clutter the development by specifying the axioms actually required for each result.

PROPOSITION 1. *If $c_1, c_2 \in [a, b]$, then $[c_1, c_2] \subseteq [a, b]$.*

PROOF: Let $c \in [c_1, c_2]$. From $c_1 \in [a, b]$ and $c \in [c_1, c_2]$ we obtain, by **(C)**, $c \in [a, b']$ for some $b' \in [b, c_2]$. From $c_2 \in [a, b]$ and $b' \in [b, c_2]$ we obtain, by **(C)** and **(L1)**, $b' \in [a, b]$. From $b' \in [a, b]$ and $c \in [a, b']$ we obtain, by **(C)** and **(L1)** again, $c \in [a, b]$. ∎

PROPOSITION 2. *If $c_1 \in [a, b_1]$, $c_2 \in [a, b_2]$ and $c \in [c_1, c_2]$, then $c \in [a, b]$ for some $b \in [b_1, b_2]$.*

PROOF: Applying **(C)** twice, we obtain $c \in [a, b']$ for some $b' \in [b_1, c_2]$ and $b' \in [a, b]$ for some $b \in [b_1, b_2]$. From $c \in [a, b']$ and $a, b' \in [a, b]$ we now obtain $c \in [a, b]$, by Proposition 1. ∎

We define a subset C of X to be *convex* if $x, y \in C$ implies $[x, y] \subseteq C$. For example, the segment $[x, y]$ is itself convex, by Proposition 1.

PROPOSITION 3. *For any convex set C and any point $a \notin C$, the set*

$$D = \bigcup_{b \in C} [a, b]$$

is convex, and is contained in every convex set which contains both C and a.

PROOF: Suppose $x_1 \in [a, b_1]$ and $x_2 \in [a, b_2]$, where $b_1, b_2 \in C$. If $x \in [x_1, x_2]$ then, by Proposition 2, $x \in [a, b]$ for some $b \in [b_1, b_2] \subseteq C$. Thus $x \in D$, and hence D is convex. The second statement of the proposition is obvious. ∎

It follows at once from the definition that convex sets have the following properties:

(i) the empty set $\emptyset$ and the whole space X are convex sets,
(ii) the intersection of any family of convex sets is again a convex set,
(iii) the union of any family of convex sets which is totally ordered by inclusion is again a convex set.

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Consequently the collection of all convex sets, partially ordered by inclusion, is a *complete lattice*, since any family $\{C_\alpha\}$ of convex sets has an infimum, namely $\bigcap C_\alpha$, and a supremum, namely the intersection of all convex sets which contain $\bigcup C_\alpha$.

For any set $S \subseteq X$, we define its *convex hull* $[S]$ to be the intersection of all convex sets which contain S. Thus $[S]$ is itself a convex set. Moreover our notations are consistent, since the convex hull of the set $\{x, y\}$ is the segment $[x, y]$.

PROPOSITION 4. *Convex hulls have the following properties:*

(o) $[\emptyset] = \emptyset$,
(i) $S \subseteq [S]$,
(ii) $S \subseteq T$ *implies* $[S] \subseteq [T]$,
(iii) $[[S]] = [S]$,
(iv) $x \in [S]$ *implies* $x \in [F]$ *for some finite set* $F \subseteq S$.

PROOF: The only property whose derivation is not immediate is (iv). To prove (iv), call a set $S \subseteq X$ *admissible* if $[S]$ is the union of all sets $[F]$, where F runs through the finite subsets of S. Thus any finite set is admissible. We show first that if $\mathcal{T}$ is a family of admissible sets which is totally ordered by inclusion, then $S = \bigcup_{T \in \mathcal{T}} T$ is also admissible.

Since the family $\{[T] : T \in \mathcal{T}\}$ is also totally ordered, the set $C = \bigcup_{T \in \mathcal{T}} [T]$ is convex. Moreover $S \subseteq C$, since $T \subseteq [T] \subseteq C$ for every $T \in \mathcal{T}$. Since $[C] = C$, it follows that $[S] \subseteq C$. On the other hand, $[T] \subseteq [S]$ for every $T \in \mathcal{T}$, and hence $C \subseteq [S]$. Thus $C = [S]$. Evidently

$$\bigcup_{F \subseteq S,\, |F| < \infty} [F] \subseteq [S].$$

On the other hand, for each $T \in \mathcal{T}$,

$$[T] = \bigcup_{F \subseteq T,\, |F| < \infty} [F] \subseteq \bigcup_{F \subseteq S,\, |F| < \infty} [F]$$

and hence

$$[S] = C = \bigcup_{T \in \mathcal{T}} [T] \subseteq \bigcup_{F \subseteq S,\, |F| < \infty} [F].$$

Thus $[S] = \bigcup_{F \subseteq S,\, |F| < \infty} [F]$, as we wished to show.

Now let S be an arbitrary nonempty subset of X and consider the family $\mathcal{F}$ of all admissible subsets of S. The family $\mathcal{F}$ is not empty, since it contains every finite subset of S. If we regard $\mathcal{F}$ as partially ordered by inclusion then, by Hausdorff's maximality theorem, $\mathcal{F}$ contains a *maximal* totally ordered subfamily $\mathcal{T}$. The union R of all sets in $\mathcal{T}$ is an admissible subset of S, by what we have just proved. Moreover $R \in \mathcal{T}$, since $\mathcal{T}$ is maximal. We wish to show that $R = S$.

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Assume on the contrary that there exists a point $x \in S \setminus R$. By Proposition 3, the set $D = \bigcup_{y \in [R]} [x, y]$ is convex and in fact $D = [x \cup R]$. Since R is admissible, it follows that $x \cup R$ is also admissible. This is the required contradiction. ∎

Since $[x \cup [S]] = [x \cup S]$, it follows from Proposition 3 that

$$[x \cup S] = \bigcup_{y \in [S]} [x, y].$$

In particular, *if* $x \in [a, b, c]$, *then* $x \in [a, d]$ *for some* $d \in [b, c]$. However, much more is true:

PROPOSITION 5. *For any sets* $S, T \subseteq X$,

$$[S \cup T] = \bigcup_{x \in [S],\, y \in [T]} [x, y].$$

PROOF: Put

$$R = \bigcup_{x \in [S],\, y \in [T]} [x, y].$$

Since $S \cup T \subseteq R \subseteq [S \cup T]$, we need only prove that R is convex. Thus we wish to show that if $z' \in [x', y']$ and $z'' \in [x'', y'']$, where $x', x'' \in [S]$ and $y', y'' \in [T]$, and if $z \in [z', z'']$, then $z \in [x, y]$ for some $x \in [S]$ and $y \in [T]$.

Since $z \in [x', y', z''] \subseteq [x', x'', y', y'']$, we have $z \in [x', w']$ for some $w \in [x'', y', y'']$. Then $w \in [x'', y]$ for some $y \in [y', y''] \subseteq [T]$. Thus $z \in [x', x'', y]$, and hence $z \in [x, y]$ for some $x \in [x', x''] \subseteq [S]$. ∎

The next result is a counterpart to Proposition 2:

PROPOSITION 6. *If* $c_1 \in [a, b_1]$, $c_2 \in [a, b_2]$ *and* $b \in [b_1, b_2]$, *then there exists a point* $c \in [a, b] \cap [c_1, c_2]$.

PROOF: By **(P)**, there exists a point $c' \in [a, b] \cap [c_1, b_2]$ and a point $c \in [a, c'] \cap [c_1, c_2]$. Since $[a, c'] \subseteq [a, b]$, the result follows. ∎

For convenience of reference, we now bring together several elementary properties:

PROPOSITION 7. *For any* $a, b, c, d \in X$,

(i) *if* $c \in [a, b]$ *and* $d \in [a, c]$, *then* $c \in [b, d]$;
(ii) *if* $c \in [a, b]$ *and* $b \in [a, c]$, *then* $b = c$;
(iii) *if* $c \in [a, b]$, *then* $[a, c] \cap [b, c] = \{c\}$.

PROOF: (i) There exists a point $e \in [b, d] \cap [c, c]$, by **(P)**, and $e = c$, by **(L1)**.
(ii) Take $d = b$ in (i).

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(iii) If $d \in [a, c] \cap [b, c]$, then $c \in [b, d]$, by (i), and hence $d = c$, by (ii). ∎

It is worth remarking that Proposition 7 can also be deduced from the axioms **(L1)–(L4)**.

Proposition 8. *If $x \in [a, a'] \cap [b, b']$ and $y \in [a, b]$, then $x \in [y, y']$ for some $y' \in [a', b']$.*

Proof: Since $x \in [a, a']$ and $y \in [a, b]$ there exists, by **(P)**, a point $z \in [a', y] \cap [b, x]$. From $x \in [b, b']$ and $z \in [b, x]$ we obtain, by Proposition 7(i), $x \in [b', z] \subseteq [y, a', b']$. Hence $x \in [y, y']$ for some $y' \in [a', b']$. ∎

Proposition 9. *For any set $S \subseteq X$ and any $c \in [S]$,*

$$[S] = \bigcup_{s \in S} [c \cup (S \setminus s)].$$

Proof: It is sufficient to prove the result for finite sets S, by Proposition 4(iv). Suppose $S = \{s_1, \ldots, s_n\}$. The result is trivial for $n = 1$ and it holds for $n = 2$, by **(L4)**. We use induction and assume that the result holds for all finite sets containing at most n elements, where $n \geqslant 2$. Let $T = s_0 \cup S$ and suppose $d \in [T]$. Then, by Proposition 5, $d \in [s_0, c]$ for some $c \in [S]$. Hence, by Proposition 5 and the induction hypothesis,

$$[T] = \bigcup_{z \in [S]} [s_0, z] = \bigcup_{i=1}^{n} [c \cup (T \setminus s_i)].$$

But, by Proposition 5, $[c \cup (T \setminus s_i)]$ is the union of all segments $[x, y]$, with $x \in [s_0, c]$ and $y \in [S \setminus s_i]$. Since $[s_0, c] = [s_0, d] \cup [d, c]$, it follows from Proposition 5 again that

$$[c \cup (T \setminus s_i)] = [d \cup (T \setminus s_i)] \cup [c \cup d \cup (S \setminus s_i)] \quad (i = 1, \ldots, n).$$

Hence

$$\begin{aligned} [T] &= \bigcup_{i=1}^{n} [d \cup (T \setminus s_i)] \cup \bigcup_{i=1}^{n} [c \cup d \cup (S \setminus s_i)] \\ &= \bigcup_{i=1}^{n} [d \cup (T \setminus s_i)] \cup [d \cup S]. \end{aligned}$$

Thus the result holds also for all finite sets containing $n + 1$ elements. ∎

Proposition 10. *Let $S = \{s_1, \ldots, s_n\}$ be a finite set and let d, e be distinct elements of $[S]$. Then $[d, e] \subseteq [p, q]$, where $p, q \in \bigcup_{i=1}^{n} [S \setminus s_i]$.*

Proof: By Proposition 9,

$$[S] = \bigcup_{i=1}^{n} [d \cup (S \setminus s_i)] = \bigcup_{i=1}^{n} [e \cup (S \setminus s_i)].$$

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Thus $e \in [d \cup (S \setminus s_j)]$ and $d \in [e \cup (S \setminus s_k)]$ for some $j, k \in \{1, \ldots, n\}$. It follows from Proposition 5 that $e \in [d, p]$ for some $p \in [S \setminus s_j]$ and $d \in [e, q]$ for some $q \in [S \setminus s_k]$. Hence $d, e \in [p, q]$, by **(L2)**, and $[d, e] \subseteq [p, q]$, by Proposition 1. ∎

In agreement with the usual notation for intervals in $\mathbb{R}$ we set, for any $a, b \in X$,

$$[a, b) = [a, b] \setminus b, \quad (a, b] = [a, b] \setminus a, \quad (a, b) = [a, b] \setminus \{a, b\}.$$

It is easily shown (see [4]) that $(a, b]$ and (a, b) are convex, for any $a, b \in X$.

3. Linearity

If a and b are distinct points, we define the *line* $\langle a, b\rangle$ to be the set of all points c such that either $c \in [a, b]$ or $a \in [b, c]$ or $b \in [c, a]$. If $a = b$ we set $\langle a, b\rangle = \{a\}$.

Clearly $\langle a, b\rangle = \langle b, a\rangle$ and $[a, b] \subseteq \langle a, b\rangle$. In particular, $a, b \in \langle a, b\rangle$. Furthermore, if a, b, c are distinct points such that $c \in \langle a, b\rangle$, then also $a \in \langle b, c\rangle$ and $b \in \langle c, a\rangle$.

Proposition 11. *If $c, d \in \langle a, b\rangle$ and $c \neq d$, then $\langle c, d\rangle = \langle a, b\rangle$.*

Proof: Clearly we must have $a \neq b$. It is sufficient to show that if $c \neq a, b$ then $\langle a, c\rangle = \langle a, b\rangle$. In fact, by symmetry we need only show that $\langle a, c\rangle \subseteq \langle a, b\rangle$. The following proof is arranged so as to appeal to Proposition 7(i) rather than **(L4)**, as far as possible.

Let $x \in \langle a, c\rangle$, so that either $x \in [a, c]$ or $a \in [c, x]$ or $c \in [a, x]$. We wish to show that either $x \in [a, b]$ or $a \in [b, x]$ or $b \in [a, x]$. Evidently we may assume that $x \neq a, b, c$.

Suppose first that $c \in [a, b]$. If $a \in [c, x]$ then $a \in [b, x]$, by **(L2)**. If $c \in [a, x]$ then $x \in [a, b]$ or $b \in [a, x]$, by **(L3)**. If $x \in [a, c]$ then $c \in [b, x]$, by Proposition 7(i), and hence $x \in [a, b]$, by **(L2)**.

Suppose next that $a \in [b, c]$. If $x \in [a, c]$ then $a \in [b, x]$, by Proposition 7(i). If $a \in [c, x]$ then $b \in [c, x]$ or $x \in [b, c]$, by **(L3)**. Moreover, by Proposition 7(i), $b \in [c, x]$ implies $b \in [a, x]$ and $x \in [b, c]$ implies $x \in [a, b]$. If $c \in [a, x]$ then $a \in [b, x]$, by **(L2)**.

Suppose finally that $b \in [a, c]$. If $a \in [c, x]$, then $a \in [b, x]$ by Proposition 7(i). If $c \in [a, x]$ then $c \in [b, x]$, by Proposition 7(i), and hence $b \in [a, x]$, by **(L2)**. If $x \in [a, c]$ then $x \in [a, b]$ or $x \in [b, c]$, by **(L4)**. Moreover, by Proposition 7(i), $x \in [b, c]$ implies $b \in [a, x]$. ∎

We show next how the points of each line may be totally ordered:

Proposition 12. *Given any two distinct points $x, y \in X$, the points of the line $\ell = \langle x, y\rangle$ may be totally ordered so that $x \leqslant y$ and so that, for any points $a, b \in \ell$*

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with $a \leqslant b$, *the segment* $[a, b]$ *consists of all* $c \in \ell$ *such that* $a \leqslant c \leqslant b$. *Moreover, this total ordering is unique.*

PROOF: For any $a, b \in \ell$, we write $a \leqslant b$ if either of the following conditions is satisfied:

(i) $a \in [x, b]$ and either $y \in [x, b]$ or $b \in [x, y]$,
(ii) $x \in [a, y]$ and either $b \in [a, y]$ or $y \in [a, b]$.

It should be noted that (i) and (ii) say the same thing if $a = x$ and that they cannot both hold if $a \neq x$.

The definition obviously implies $x \leqslant y$. Also, from the definition of a line it is clear that for any $a \in \ell$ we have $a \leqslant a$. Let a, b be points of ℓ such that both $a \leqslant b$ and $b \leqslant a$. We wish to show that this implies $b = a$.

Assume first that $a \in [x, b]$ and either $y \in [x, b]$ or $b \in [x, y]$. Since $x \neq y$, this implies $x \notin [b, y]$. Since $b \leqslant a$, it follows that $b \in [x, a]$. Hence $b = a$, by Proposition 7(ii). The case $b \in [x, a]$ and either $y \in [x, a]$ or $a \in [x, y]$ may be discussed similarly. Suppose finally that $x \in [a, y]$ and $x \in [b, y]$. Then either $a \in [b, y]$ or $b \in [a, y]$, by **(L3)**. We cannot have also $y \in [a, b]$, since this would imply $y = a$, respectively $y = b$, and hence $x = y$. Consequently we must have $b \in [a, y]$ and $a \in [b, y]$, which implies $b = a$.

Suppose next that a, b, c are distinct points of ℓ such that $a \leqslant b$ and $b \leqslant c$. We wish to show that this implies $a \leqslant c$.

Assume first that $b \in [x, c]$ and either $y \in [x, c]$ or $c \in [x, y]$. If $a \in [x, b]$ then $a \in [x, c]$ and hence $a \leqslant c$. If $x \in [a, y]$ then $y \in [x, c]$ implies $y \in [a, c]$, by **(L2)**, and $c \in [x, y]$ implies $c \in [a, y]$, by **(L4)**. In both cases $a \leqslant c$. Assume next that $x \in [b, y]$ and either $c \in [b, y]$ or $y \in [b, c]$. If $x \in [a, y]$ and $b \in [a, y]$ then $c \in [a, y]$, respectively $y \in [a, c]$. Thus again $a \leqslant c$. All remaining cases may be discussed similarly.

It remains to show that if a, b are distinct points of ℓ then either $a \leqslant b$ or $b \leqslant a$. Assume that neither relation holds. Then if $a \in [x, b]$ we must have $x \in [b, y]$, and if $x \in [b, y]$, and if $x \in [b, y]$ we must have $b \in [a, y]$. Hence $b \in [x, y]$, by **(L2)**, and $b = x$, by Proposition 7(ii). But this implies $b = a$, which contradicts our assumption. Therefore $a \notin [x, b]$. Similarly we can show that $b \notin [a, x]$. Consequently $x \in [a, b]$.

If $x \in [a, y]$ we must have $a \in [b, y]$ and hence $x \in [b, y]$, and if $x \in [b, y]$ we must have $b \in [a, y]$. Therefore $x \notin [a, y]$, and similarly $x \notin [b, y]$. Thus either $a \in [x, y]$ or $y \in [a, x]$. But $a \in [x, y]$ would imply $x \in [b, y]$, by **(L2)**, and $y \in [a, x]$ would imply $x \in [b, y]$, by Proposition 7(i). Thus we cannot escape a contradiction.

This proves that the relation $\leqslant$ is a total ordering of the line ℓ such that $x \leqslant y$. Suppose now that $a \leqslant b$ and $c \in [a, b]$. If (i) holds, then $c \in [x, b]$ and hence $c \leqslant b$.

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Moreover $a \in [x, c]$ and either $y \in [x, c]$ or $c \in [x, y]$, so that $a \leqslant c$. Similarly it may be seen that $a \leqslant c \leqslant b$ if (ii) holds.

Suppose on the other hand that $c \in \ell$ and $a \leqslant c \leqslant b$. We shall show that $c \in [a, b]$. Indeed, if $a \in [c, b]$ then, by what we have just proved, $c \leqslant a \leqslant b$ and hence $c = a$. Similarly if $b \in [a, c]$, then $c = b$.

Finally suppose that $\leqslant$ is any total ordering of ℓ with the properties in the statement of the proposition. Let a, b be any distinct elements of ℓ and assume first that (i) holds. If $b \leqslant x$, then $b \leqslant a \leqslant x$ and $x \leqslant b \leqslant y$, since $y \notin [x, b]$. Hence $b = a$, which is a contradiction. Consequently $x \leqslant b$ and $x \leqslant a \leqslant b$. Assume next that (ii) holds. Then $a \leqslant x \leqslant y$ and either $a \leqslant b \leqslant y$ or $a \leqslant y \leqslant b$, since $y \neq a$. In any event $a \leqslant b$, and so the total ordering is the one originally defined. $\square$

Points which lie on the same line will be said to be *collinear*. The concept of collinearity makes it possible to obtain sharper forms for some of our earlier results. For example, the following result may be used in conjunction with the axiom $(\mathbf{P})$, and will then be called $(\mathbf{P})'$:

(a) *Let a, b_1, b_2 be non-collinear points and let $c_1 \in (a, b_1)$, $c_2 \in (a, b_2)$. If $d \in [b_1, c_2] \cap [b_2, c_1]$, then actually $d \in (b_1, c_2) \cap (b_2, c_1)$.*

PROOF: It is sufficient to show that $d \neq b_1, c_1$. By Proposition 11, $d = b_1$ implies $b_2 \in \langle b_1, c_1 \rangle = \langle a, b_1 \rangle$, which is a contradiction. Similarly $d = c_1$ implies $c_2 \in \langle a, b_1 \rangle$ and again $b_2 \in \langle a, b_1 \rangle$. $\square$

Similarly we can prove the following results, which may be used in combination with Propositions 2, 6 and 8 respectively, and will then be called Propositions $2'$, $6'$, $8'$:

(b) *Let a, b_1, b_2 be non-collinear points and let $c_1 \in (a, b_1)$, $c_2 \in (a, b_2]$, $c \in (c_1, c_2)$. If $c \in [a, b]$ for some $b \in [b_1, b_2]$, then actually $c \in (a, b)$ and $b \in (b_1, b_2)$.*

(c) *Let a, b_1, b_2 be non-collinear points and let $c_1 \in (a, b_1)$, $c_2 \in (a, b_2]$, $b \in (b_1, b_2)$. If there exists a point $c \in [a, b] \cap [c_1, c_2]$, then actually $c \in (a, b) \cap (c_1, c_2)$.*

(d) *Let a, a', b, b' be non-collinear points and let $x \in (a, a') \cap (b, b')$. If $x \in [y, y']$ for some $y \in (a, b)$ and $y' \in [a', b']$, then actually $x \in (y, y')$ and $y' \in (a', b')$.*

We define a subset A of X to be *affine* if $x, y \in A$ implies $\langle x, y \rangle \subseteq A$. For example, the line $\langle x, y \rangle$ is itself affine, by Proposition 11. The next result is of fundamental importance:

PROPOSITION 13. *If C is a convex set, then the set*

$$A = \bigcup_{x, x' \in C} \langle x, x' \rangle$$

is affine, and is contained in every affine set which contains C.

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PROOF: Only the first statement of the proposition requires proof. Obviously we may assume that C contains more than one point. Thus we wish to show that if $b \in \langle x_1, x_2 \rangle$, $c \in \langle x_3, x_4 \rangle$, and $a \in \langle b, c \rangle$, where $x_1, \ldots, x_4 \in C$ and $x_1 \neq x_2$, $x_3 \neq x_4$, $b \neq c$, then there exist $x_5, x_6 \in C$ with $x_5 \neq x_6$ such that $a \in \langle x_5, x_6 \rangle$. This is evident if $c \in \langle x_1, x_2 \rangle$, since then also $a \in \langle x_1, x_2 \rangle$. Consequently we assume $c \notin \langle x_1, x_2 \rangle$, and similarly $b \notin \langle x_3, x_4 \rangle$.

By symmetry, and the convexity of C, we need only consider the following five cases:

(A) $x_2 \in [b, x_1]$, $x_4 \in [c, x_3]$, $c \in [b, a]$;
(B) $x_2 \in [b, x_1]$, $x_4 \in [c, x_3]$, $a \in [b, c]$;
(C) $x_2 \in [b, x_1]$, $c \in [x_3, x_4]$, $a \in [b, c]$;
(D) $x_2 \in [b, x_1]$, $c \in [x_3, x_4]$, $c \in [a, b]$;
(E) $x_2 \in [b, x_1]$, $c \in [x_3, x_4]$, $b \in [a, c]$.

Consider first case (A). By **(P)** there exists a point $y \in [a, x_2] \cap [c, x_1]$. Since $c \notin \langle x_1, x_2 \rangle$, we have $y \neq x_2$ and hence $a \in \langle x_2, y \rangle$. Consequently we may assume $y \notin C$. By **(P)** also there exists a point $x_5 \in [y, x_3] \cap [x_1, x_4]$. Then $x_5 \in C$ and $x_5 \in [a, x_2, x_3]$. Hence $x_5 \in [a, x_6]$ for some $x_6 \in [x_2, x_3]$. Thus $x_6 \in C$ and $a \in \langle x_5, x_6 \rangle$ if $x_5 \neq x_6$. We can now assume $x_5 = x_6 \in [x_2, x_3]$. If $x_3 \neq x_5$ then $x_2 \in \langle x_3, x_5 \rangle$, $y \in \langle x_3, x_5 \rangle$, and hence $a \in \langle x_3, x_5 \rangle$. Consequently we can now assume $x_3 = x_5 \in [x_1, x_4]$. Then by **(L2)**, $x_4 \in [c, x_1]$. Since $y \notin C$ it follows from **(L4)** that $y \in [c, x_4]$ and then from Proposition 7(i) that $x_4 \in [x_1, y] \subseteq [a, x_1, x_2]$. Hence $x_4 \in [a, x_7]$ for some $x_7 \in [x_1, x_2]$. Then $x_7 \in C$ and $a \in \langle x_4, x_7 \rangle$ if $x_4 \neq x_7$. In fact we cannot have $x_4 = x_7$, since this would imply $x_4 \in [x_1, x_2]$, $x_3 \in [x_1, x_2]$ and hence

$$c \in \langle x_3, x_4 \rangle \subseteq \langle x_1, x_2 \rangle.$$

Similarly in case **(B)** there exist points $y \in [b, x_4] \cap [a, x_3]$ and $x_5 \in [y, x_1] \cap [x_2, x_4]$. The argument can now be completed as in the previous case.

In case **(C)** there exists a point $x_5 \in [c, x_2] \cap [a, x_1]$. Moreover $x_5 \in C$, since $c \in C$, and $x_5 \neq x_1$, since $c \notin \langle x_1, x_2 \rangle$. Hence $a \in \langle x_1, x_5 \rangle$.

Similarly in case **(D)** there exists a point $x_5 \in [a, x_2] \cap [c, x_1]$ and the argument can be completed as in the previous case.

Finally in case **(E)** we have $c \in C$ and $x_2 \in [a, c, x_1]$. Hence $x_2 \in [a, x_5]$ for some $x_5 \in [c, x_1]$. Then $x_5 \in C$, $x_5 \neq x_2$ and $a \in \langle x_2, x_5 \rangle$. $\square$

It follows at once from the definition that affine sets have the following properties:

(i) the empty set $\emptyset$ and the whole space X are affine sets,
(ii) the intersection of any family of affine sets is again an affine set,
(iii) the union of any family of affine sets which is totally ordered by inclusion is again an affine set.

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For any set $S \subseteq X$, we define its *affine hull* $\langle S\rangle$ to be the intersection of all affine sets which contain S. Thus $\langle S\rangle$ is itself an affine set. Moreover our notations are consistent, since the affine hull of the set $\{x, y\}$ is the line $\langle x, y\rangle$. It follows immediately from the definition that the properties (o)–(iii) in Proposition 4 continue to hold with '[ ]' replaced by '$\langle\ \rangle$'. We are going to show that this is true also for the property (iv).

For any set $S \subseteq X$ we have $S \subseteq [S] \subseteq \langle S\rangle$ and hence

$$\langle S\rangle = \langle [S]\rangle.$$

Consequently, by Proposition 13,

$$\langle S\rangle = \bigcup_{x, x' \in [S]} \langle x, x'\rangle.$$

But for given $x, x' \in [S]$ there exists a finite set $F \subseteq S$ such that $x, x' \in [F]$, and so $\langle x, x'\rangle \subseteq \langle F\rangle$. Hence

$$\langle S\rangle = \bigcup_{F \subseteq S,\, |F| < \infty} \langle F\rangle.$$

From Proposition 13 we can derive the *exchange property* of affine sets, which was first stated by Grassmann in 1862:

PROPOSITION 14. *For an arbitrary set $S \subseteq X$, if $y \in \langle S \cup x\rangle$ but $y \notin \langle S\rangle$, then $x \in \langle S \cup y\rangle$.*

PROOF: Obviously we may assume that $x \notin [S \cup y]$. By Proposition 13, $y \in \langle z_1, z_2\rangle$, where $z_1, z_2 \in [S \cup x]$. Therefore, by Proposition 5, $z_1 \in [x, w_1]$ and $z_2 \in [x, w_2]$ for some $w_1, w_2 \in [S]$.

If $y \in [z_1, z_2]$, then $y \in [x, w_1, w_2]$ and hence $y \in [x, w]$ for some $w \in [w_1, w_2] \subseteq [S]$. Thus $y \neq w$ and $x \in \langle y, w\rangle \subseteq \langle S \cup y\rangle$.

By symmetry it only remains to consider the case $z_2 \in (y, z_1)$. We shall assume $x \notin \langle y, w_1, w_2\rangle$ and derive a contradiction.

Since $x \notin \langle y, w_1, w_2\rangle$, we must have $z_1 \neq x, w_1$. Thus $z_1 \in (x, w_1)$ and hence, by Proposition 2$'$, $z_2 \in (x, w)$ for some $w \in (y, w_1)$. Hence $w \in \langle x, z_2\rangle = \langle x, w_2\rangle$. Since $x \notin \langle y, w_1, w_2\rangle$, we must have $w = w_2$. Since $y \in \langle w, w_1\rangle$, this is a contradiction. ∎

We say that a set $S \subseteq X$ is *affine independent* if, for every $x \in S$, $x \notin \langle S \setminus x\rangle$. A subset T of a set S is an *affine generator* of S if $\langle T\rangle = \langle S\rangle$ and an *affine basis* of S if, in addition, T is affine independent.

It is easily seen that an infinite set is affine independent if every finite subset is affine independent, and any affine independent set is contained in a maximal affine independent set. From the exchange property we can deduce in the usual way that

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T is an affine basis of S if and only if it is a maximal affine independent subset of S. Furthermore, if T and T' are subsets of S such that T is affine independent and $\langle T' \rangle = \langle S \rangle$, then there exists a subset T'' of T', with $T \cap T'' = \emptyset$, such that $T \cup T''$ is affine independent and $\langle T \cup T'' \rangle = \langle S \rangle$. It follows that if a set S has a finite affine basis, then all affine bases of S are finite and have the same cardinality.

We shall say that a set S has *dimension* d if all affine bases of S are finite and have cardinality $d+1$, and that it is *infinite-dimensional* if its affine bases are all infinite. In particular, the empty set $\emptyset$ has dimension -1, a point has dimension 0 and a line has dimension 1. An affine set of dimension 2 will be called a *plane*.

An affine set H is said to be a *hyperplane* of X if $H \neq X$ and if X is the only affine set which properly contains H. Equivalently, an affine set $H \subset X$ is a hyperplane if $\langle x \cup H \rangle = X$ for every $x \in X \setminus H$. It is easily shown that any affine set $A \subset X$ is contained in a hyperplane.

When X is finite-dimensional, an affine set $H \subseteq X$ is a hyperplane if and only if $\dim H = \dim X - 1$. However, the familiar formula

$$\dim A + \dim B = \dim\langle A \cup B \rangle + \dim(A \cap B)$$

need not hold for affine sets A, B with $A \cap B \neq \emptyset$.

4. Half-spaces

Pasch, in 1882, pointed out the incompleteness of Euclid's axioms for geometry and introduced the following additional axiom: "If a line in the plane of a triangle does not pass through any of its vertices but intersects one of its sides, then it also intersects another of its sides". It will now be shown that Pasch's axiom holds in any linear geometry.

Lemma 15. *If a, b, c, d are points such that*

$$[a,\, b] \cap \langle c,\, d \rangle = \emptyset, \quad [a,\, c] \cap \langle b,\, d \rangle = \emptyset, \quad [b,\, c] \cap \langle a,\, d \rangle = \emptyset,$$

then $d \notin \langle a, b, c \rangle$.

Proof: The hypotheses evidently imply that a, b, c are not collinear and $d \notin [a, b] \cup [a, c] \cup [b, c]$. We shall assume $d \in \langle a, b, c \rangle$ and derive a contradiction.

It follows from Proposition 10 that if $d \in [a, b, c]$, then $d \in [a, e]$ for some $e \in [b, c]$, which contradicts $[b, c] \cap \langle a, d \rangle = \emptyset$. Thus we now suppose $d \notin [a, b, c]$.

By Proposition 13, $d \in \langle p, q \rangle$, where $p, q \in [a, b, c]$ and $p \neq q$. Moreover we may choose the notation so that $q \in (p, d)$. Furthermore, by Proposition 10, we may take $p, q \in [a, b] \cup [a, c] \cup [b, c]$.

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If $p, q \in [a, b]$, then $d \in \langle a, b\rangle$ and either $a \in (b, d)$ or $b \in (a, d)$. In either case $\langle a, d\rangle \cap [b, c] \neq \emptyset$. It is obvious also that $\langle a, d\rangle \cap [b, c] \neq \emptyset$ if $q = a$ and $p \in [b, c]$ or if $p = a$ and $q \in [b, c]$.

By symmetry it only remains to consider the case $p \in (a, b)$ and $q \in (b, c)$. Then $q \in (b, r)$ for some $r \in (a, d)$. Since $q \in (b, c)$ and $r \neq c$, either $r \in (b, c)$ or $c \in (b, r)$. If $r \in (b, c)$, then $(a, d) \cap (b, c) \neq \emptyset$. If $c \in (b, r)$, then $c \in (q, r)$ and hence $c \in (d, e)$ for some $e \in (a, p) \subseteq (a, b)$. Thus $\langle c, d\rangle \cap (a, b) \neq \emptyset$. ∎

PROPOSITION 16. *Let a, b, c be non-collinear points and let ℓ be a line in the plane $\langle a, b, c\rangle$ such that $a, b, c \notin \ell$. If ℓ intersects (a, b), then ℓ also intersects either (a, c) or (b, c), but not both.*

PROOF: We show first that ℓ cannot intersect (a, b), (a, c) and (b, c). By symmetry it is sufficient to show that if $d \in (a, b)$, $e \in (a, c)$ and $f \in (b, c)$, then $f \notin (d, e)$. But if $f \in (d, e)$ then, by Proposition 2′, $f \in (a, g)$ for some $g \in (b, c)$. Hence $a \in \langle b, c\rangle$, which is a contradiction.

It remains to show that ℓ intersects two 'sides' of the 'triangle' $[a, b, c]$. Since $a, b, c \notin \ell$, this follows at once from Proposition 10 if ℓ contains two distinct points of $[a, b, c]$. Thus we now assume that ℓ contains a point $e \notin [a, b, c]$. Since $e \in \langle a, b, c\rangle$, it follows from Lemma 15 that the points a, b, c may be named so that there exists a point $p \in \langle a, e\rangle \cap [b, c]$. Thus the hypothesis is now that ℓ contains a point $d \in (a, b) \cap (a, c) \cap (b, c)$.

Suppose first that $d \in (a, b)$, which implies $p \neq b$. If $a \in (e, p)$, then $d \in (e, f)$ for some $f \in (b, p)$. If $p \in (a, e)$, then there exists a point $f \in (d, e) \cap (b, p)$. In both cases $f \in \langle d, e\rangle \cap (b, c)$.

Suppose next that $d \in (b, c)$. We may assume that $p \neq d$ and, without loss of generality, that $d \in (b, p)$. If $a \in (e, p)$, then there exists a point $f \in (d, e) \cap (a, b)$. If $p \in (a, e)$, then $d \in (e, f)$ for some $f \in (a, b)$. In both cases $f \in \langle d, e\rangle \cap (a, b)$.

The case $d \in (a, c)$ is reduced to the case $d \in (a, b)$ by interchanging b and c. ∎

Although Proposition 16 may seem rather special, we shall show that it has some important general consequences. The statement of the following lemma is taken from Lenz [**6**], but is proved here under weaker hypotheses:

LEMMA 17. *Let H be a hyperplane. If x_1, x_2, x_3, x_4 are points of $X \setminus H$ such that*

$$(x_1, x_2) \cap H \neq \emptyset, \quad (x_2, x_3) \cap H \neq \emptyset, \quad (x_3, x_4) \cap H \neq \emptyset,$$

then also $(x_4, x_1) \cap H \neq \emptyset$.

PROOF: Obviously we can sssume $x_3 \neq x_1$ and $x_4 \neq x_2$. The lines $\langle x_1, x_2\rangle$, $\langle x_2, x_3\rangle$ and $\langle x_3, x_4\rangle$ intersect H in unique points h_1, h_2 and h_3. Moreover $h_1 \in (x_1, x_2)$, $h_2 \in (x_2, x_3)$ and $h_3 \in (x_3, x_4)$.

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Suppose first that x_1, x_2, x_3 are collinear. Then $h_1 = h_2$ and either $x_1 \in (h_2, x_3)$ or $x_3 \in (h_2, x_1)$. If $x_1 \in (h_2, x_3)$ then, by **(P)**, there exists a point $h \in [h_2, h_3] \cap (x_1, x_4)$. If $x_3 \in (h_2, x_1)$ then, by **(C)**, there exists a point $h \in [x_1, x_4]$ such that $h_3 \in [h_2, h]$. If $h_3 = h_2$, then $h_2 \in (x_1, x_4)$. If $h_3 \neq h_2$, then $h \in H \cap (x_1, x_4)$.

Thus we may assume that x_1, x_2, x_3 are not collinear, and hence $h_1 \neq h_2$. Suppose now that x_1, x_3, x_4 are collinear, so that $h_3 \in \langle x_1, x_3 \rangle$. Since $\langle h_1, h_2, h_3 \rangle \neq \langle x_1, x_2, x_3 \rangle$, we must have $h_3 \in \langle h_1, h_2 \rangle$. By Proposition 16, $h_3 \notin (x_1, x_3)$. Since x_1, x_3, x_4 are collinear, it follows that $h_3 \in (x_1, x_4)$.

Thus we may assume that x_1, x_3, x_4 are not collinear and indeed, by similar arguments, we may assume that no three of the points x_1, x_2, x_3, x_4 are collinear.

Suppose next that $\langle x_2, x_4 \rangle \cap H \neq \emptyset$. Then the line $\langle x_2, x_4 \rangle$ intersects H in a unique point h. We show first that $h \notin (x_2, x_4)$. Assume on the contrary that $h \in (x_2, x_4)$. Since x_2, x_3, x_4 are not collinear, it follows from Proposition 16 that $h \notin \langle h_2, h_3 \rangle$. Hence $\langle h_2, h_3, h \rangle = \langle x_2, x_3, x_4 \rangle$, which is a contradiction.

Thus either $x_4 \in (x_2, h)$ or $x_2 \in (x_4, h)$. If $x_4 \in (x_2, h)$ then, by **(P)**$'$, there exists a point $h' \in (h, h_1) \cap (x_1, x_4)$. If $x_2 \in (x_4, h)$ then, by Proposition $2'$, there exists a point $h' \in (x_1, x_4)$ such that $h_1 \in (h, h')$.

Thus we may assume that $\langle x_2, x_4 \rangle \cap H = \emptyset$, and similarly also that $\langle x_1, x_3 \rangle \cap H = \emptyset$. By **(P)**$'$, there exists a point $x \in (x_2, h_3) \cap (x_4, h_2)$ and a point $h \in (x, x_3) \cap (h_2, h_3)$. Since $\langle x_2, x \rangle \cap H \neq \emptyset$, it follows from the previous part of the proof (with x_4 replaced by x) that $(x_1, x) \cap H \neq \emptyset$. Since $\langle x, x_4 \rangle \cap H \neq \emptyset$, it further follows from the previous part of the proof (with x_2 replaced by x) that $(x_1, x_4) \cap H \neq \emptyset$. $\square$

PROPOSITION 18. *Let H be a hyperplane such that $X \setminus H$ is not convex. Then there exist unique nonempty convex sets H_+ and H_- such that $X \setminus H = H_+ \cup H_-$. Furthermore,*

(i) $H_+ \cap H_- = \emptyset$,
(ii) *if $y \in H_+$ and $z \in H_-$, then $(y, z) \cap H \neq \emptyset$,*
(iii) *$H \cup H_+$ and $H \cup H_-$ are also convex.*

PROOF: Since $X \setminus H$ is not convex, there exist points $a, b \in X \setminus H$ and $c \in H$ such that $c \in (a, b)$. Let H_+ denote the set of all points $x \in X$ such that $(b, x) \cap H \neq \emptyset$, and let H_- denote the set of all points $x \in X$ such that $(a, x) \cap H \neq \emptyset$. Then $a \in H_+$, $b \in H_-$ and $H_+ \cap H = \emptyset$, $H_- \cap H = \emptyset$.

We are going to show that also $H_+ \cap H_- = \emptyset$. Assume on the contrary that there exists a point $x \in H_+ \cap H_-$. If $x \in \langle a, b \rangle$ then $c \in (a, x) \cap (b, x)$, since $\langle a, b \rangle \cap H = \{c\}$. But this is impossible, since also $c \in (a, b)$. Thus a, b, x are not collinear. If (a, x), (b, x) intersect H in h, h' respectively, then $c \notin \langle h, h' \rangle$ by Proposition 16. Hence $\langle c, h, h' \rangle = \langle a, b, x \rangle$, which is a contradiction.

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We show next that H_- is convex. Assume on the contrary that there exist points $x', x'' \in H_-$ and a point $x \in (x', x'')$ such that $x \notin H_-$. Then there exist points $h', h'' \in H$ such that $h' \in (a, x')$, $h'' \in (a, x'')$. Hence a, x', x'' are not collinear and $h' \neq h''$. By Proposition 6′ the segment (h', h'') contains a point $h \in (a, x)$. Then $h \in H$ and $x \in H_-$, which is contradiction. Thus H_- is convex, and similarly also H_+.

It follows at once from Lemma 17 that if $y \in H_+$ and $z \in H_-$, then $(y, z) \cap H \neq \emptyset$. It will now be shown that $X = H \cup H_+ \cup H_-$. Since H is a hyperplane, it is sufficient to show that $X' := H \cup H_+ \cup H_-$ is affine. In fact it is enough to show that if $x' \in H_+$, $x'' \in X'$ and $x \in \langle x', x'' \rangle$, then $x \in X'$. Since $x' \in H_+$, the segment (b, x') contains a point $h' \in H$.

Suppose first that $x'' = h'' \in H$. If $h'' \in (x, x')$ then $x \in H_-$, since $x' \in H_+$. If $x \in (x', h'')$ then, by **(P)**, there exists a point $h \in [h', h''] \cap (b, x)$ and hence $x \in H_+$. If $x' \in (x, h'')$ and b, x, x' are collinear, then $h' = h'' \in (b, x)$ and hence $x \in H_+$. If $x' \in (x, h'')$ and b, x, x' are not collinear then, by Proposition 2′, there exists a point $h \in (b, x)$ such that $h' \in (h, h'')$ and again $x \in H_+$. This proves that a line containing a point of H and a point of H_+ is entirely contained in X', and similarly a line containing a point of H and a point of H_- is entirely contained in X'.

Suppose next that $x'' \in H_+ \cup H_-$ and the line $\langle x', x'' \rangle$ contains no point of H. Then $x'' \in H_+$, since $x'' \in H_-$ would imply $(x', x'') \cap H \neq \emptyset$. Hence the segment (b, x'') contains a point $h'' \in H$. If $x \in (x', x'')$ then $x \in H_+$, since H_+ is convex. Now consider the case $x' \in (x, x'')$. By (P)′, there exists a point $y \in (b, x') \cap (h'', x)$. Moreover $y \notin H$. If $h' \in (b, y)$, then the segment (b, x) contains a point h such that $h' \in (h, h'')$ and hence $x \in H_+$. If $h' \in (x', y)$, then the segment (x, x') contains a point h such that $h' \in (h, h'')$ and hence $x \in X'$ by what we have already proved. Since the same argument applies in the case $x'' \in (x, x')$, this completes the proof that $X = H \cup H_+ \cup H_-$.

It now follows that $H \cup H_+$ and $H \cup H_-$ are convex. For if (say) $x_1 \in H$ and $x_2 \in H_+$, then $(x_1, x_2) \cap H = \emptyset$ and hence $(x_1, x_2) \cap H_- = \emptyset$.

Finally, suppose $X \setminus H$ is the union of two nonempty convex sets G_+ and G_-. We may assume the notation chosen so that $G_+ \cap H_+ \neq \emptyset$. But then $G_+ \subseteq H_+$, by (ii). Hence $G_- \cap H_- \neq \emptyset$ and so, in the same way, $G_- \subseteq H_-$. Since $G_+ \cup G_- = H_+ \cup H_-$, we must actually have $G_+ = H_+$ and $G_- = H_-$. ∎

Proposition 18 says that a hyperplane has two 'sides', if its complement is not convex. The convex sets H_+ and H_- in the statement of Proposition 18 will be called the *open half-spaces* associated with the hyperplane H, and the convex sets $H_+ \cup H$ and $H_- \cup H$ will be called the *closed half-spaces* associated with H. When $X \setminus H$ is convex, we shall call $X \setminus H$ and X the open and closed half-spaces associated with the

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hyperplane H.

5. Faces

If $S \subseteq X$ and $e \in S$, then e is said to be an *extreme point* of the subset S if $e \notin [S \setminus e]$. Clearly if e is an extreme point of S, then it is an extreme point of every subset of S which contains it. Extreme points may also be characterised in the following way:

Proposition 19. *If $S \subseteq X$ and $e \in S$, then e is an extreme point of S if and only if $e \in [x, y]$, where $x, y \in S$, implies $x = e$ or $y = e$.*

Proof: If $e \in [x, y]$, where $x, y \in S \setminus e$, then $e \in [S \setminus e]$ and hence e is not an extreme point of S.

Suppose, on the other hand, that $e \in [x, y]$, where $x, y \in S$, implies $x = e$ or $y = e$. If e is not an extreme point of S, then $e \in [S \setminus e]$. Hence $e \in [F]$ for some finite set $F \subseteq S \setminus e$. We may assume that $e \notin [F']$ for every proper subset F' of F. If $u \in F$ then, by Proposition 5, $e \in [u, v]$ for some $v \in [F \setminus u]$. Since $v \neq e$, it follows that $u = e$. Since $u \in S \setminus e$, this is a contradiction. $\square$

Proposition 20. *A set and its convex hull have the same extreme points.*

Proof: We need only prove that if e is an extreme point of S, then it is also an extreme point of $[S]$. Assume on the contrary that $e \in [x, y]$, where $x, y \in [S] \setminus e$. By Proposition 5 we have $x \in [e, u]$ and $y \in [e, v]$ for some $u, v \in [S \setminus e]$. From $e \in [x, y]$, $y \in [e, v]$ and $y \neq e$ we obtain, by **(L2)**, $e \in [x, v]$. From $e \in [x, v]$, $x \in [e, u]$ and $x \neq e$ we obtain similarly $e \in [u, v]$. Since $e \notin [S \setminus e]$, this is a contradiction. $\square$

We shall denote by $\mathbf{E}(S)$ the set of all extreme points of the set S. If S is *finite*, then $[S] = [\mathbf{E}(S)]$ since, by Proposition 5, $x \in [S \setminus x]$ implies $[S] = [S \setminus x]$.

A convex set A is said to be a *face* of a convex set C if $A \subseteq C$ and if $a \in (c, c')$, where $a \in A$ and $c, c' \in C$, implies $c, c' \in A$.

Thus a singleton $\{e\}$ is a face of C if and only if e is an extreme point of C. It follows at once from the definition that if A is a face of C, then $C \setminus A$ is convex. Furthermore,

(i) the empty set $\emptyset$ and the whole set C are faces of C,
(ii) the intersection of any family of faces of C is again a face of C,
(iii) the union of any family of faces of C which is totally ordered by inclusion is again a face of C.

The following two properties are also immediate consequences of the definition:

Proposition 21. *If A, B, C are convex sets such that A is a face of B and B is a face of C, then A is a face of C.*

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Proposition 22. *If B, C are convex sets such that $B \subseteq C$ and if A is a face of C, then $A \cap B$ is a face of B. In particular, if $A \subseteq B$ then A is a face of B.*

Faces may also be characterised in the following way:

Proposition 23. *A set A is a face of a convex set C if and only if $C \cap \langle A \rangle = A$ and $C \setminus A$ is convex.*

Proof: Let A be a set such that $C \cap \langle A \rangle = A$ and $C \setminus A$ is convex. Then $A \subseteq C$ and A is convex, since it is the intersection of two convex sets. If $a \in (c, c')$, where $a \in A$ and $c, c' \in C$, then at least one of c, c' is in A, since $C \setminus A$ is convex, and in fact they both are, since $C \cap \langle A \rangle = A$. Thus A is a face of C.

On the other hand, if A is a face of C then $C \setminus A$ is certainly convex. Put $B = C \cap \langle A \rangle$. Then A is a face of B, since B is convex and $A \subseteq B \subseteq C$. By Proposition 13, if $b \in B$ then $b \in \langle a, a' \rangle$ for some $a, a' \in A$. If $b \in [a, a']$ then $b \in A$, since A is convex. If $a' \in (a, b)$ or $a \in (a', b)$ then $b \in A$, since A is a face of B. Thus $B = A$. ∎

Corollary 24. *If A and B are faces of a convex set C, then A properly contains B if and only if $\langle A \rangle$ properly contains $\langle B \rangle$.*

Proposition 25. *If a convex set C is contained in a closed half-space of $\langle C \rangle$ associated with a hyperplane H, then $C \cap H$ is a face of C.*

Proof: Put $D = C \cap H$ and let H_+ be the open half-space of $\langle C \rangle$ associated with H such that $C \subseteq H \cup H_+$. Then $C \cap \langle D \rangle = D$, since $D \subseteq C \cap \langle D \rangle \subseteq C \cap H$, and $C \setminus D$ is convex, since $C \setminus D = C \cap H_+$. ∎

Proposition 26. *Let C be a convex set and H_+ an open half-space of $\langle C \rangle$ associated with a hyperplane H. Also, let A be a subset of $B = C \cap (H \cup H_+)$ such that $A \cap H$ is a face of C (possibly $\emptyset$). Then A is a face of C if and only if A is a face of B.*

Proof: By Proposition 22 we need only show that if A is a face of B, then A is also a face of C. Put $H_- = \langle C \rangle \setminus (H \cup H_+)$.

We show first that $C \cap \langle A \rangle = A$. Since $B \cap \langle A \rangle = A$, it is enough to show that if $y \in H_-$, then $y \notin \langle A \rangle$. Assume on the contrary that $y \in \langle A \rangle$. Then $y \in \langle x, z \rangle$ for some $x, z \in A$ and we may choose the notation so that $z \in (x, y)$. Hence there exists a point $w \in (y, z] \cap H$. Moreover $w \notin A$, since $A \cap H$ is a face of C. Hence $w \neq z$ and $z \in (x, w)$. But, since A is a face of B, this contradicts $w \notin A$.

We show next that $C \setminus A$ is convex. Assume on the contrary that, for some points $x, y \in C \setminus A$, there exists a point $z \in (x, y) \cap A$. Since $B \setminus A$ is convex, we may assume that $y \in H_-$. Then $x \in H_+$, since $H \cup H_-$ is convex. Hence there exists a point $w \in (x, y) \cap H$, and $z \in [w, x)$, which leads to a contradiction in the same way

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as before. $\square$

Proposition 27. *Let C be a convex set and S an arbitrary subset of C. Then the subset A_S of C, consisting of the union of $C \cap \langle S \rangle$ with the set of all points $x, x' \in C$ such that $(x, x') \cap \langle S \rangle \neq \emptyset$, is a face of C, and is contained in every face of C which contains S.*

Proof: We show first that A_S is convex. Suppose $z \in (x, y)$, where x, y are distinct elements of A_S with $x \notin \langle S \rangle$ and $z \notin \langle S \rangle$. Then $z \in C$ and there exist points $x' \in A_S$, $a \in \langle S \rangle$ such that $a \in (x, x')$. If a, x, y are collinear, then $a \in (z, z')$ for some $z' \in \{x, y, x'\} \subseteq C$ and so $z \in A_S$. Hence we now suppose that a, x, y are not collinear. Then, by $(\mathbf{P})'$, there exists a point $c \in (a, y) \cap (x', z)$. If $y \in \langle S \rangle$, then $c \in \langle S \rangle$ and hence $z \in A_S$. If $y \notin \langle S \rangle$, there exist $y' \in A_S$ and $b \in \langle S \rangle$ such that $b \in (y, y')$.

If $b = a$ then, by Proposition $8'$, $a \in (z, z')$ for some $z' \in (x', y')$ and hence $z \in A_S$. Thus we now suppose $b \neq a$ and hence $y \notin \langle a, b \rangle$. By $(\mathbf{P})'$, there exists a point $d \in (a, b) \cap (c, y')$. Moreover $d \in C \cap \langle S \rangle$, since $a, b \in C \cap \langle S \rangle$. Thus we may suppose $d \neq z$. We may also suppose that $y' \notin \langle x', z \rangle$, since otherwise $d \in (z, x')$ or $d \in (z, y')$. Then by Proposition $2'$, there exists a point $e \in (x', y')$ such that $d \in (e, z)$. Thus $z \in A_S$ in every case.

We now show that A_S is a face of C. Suppose $x \in A_S$ and $x \in (c, c')$, where $c, c' \in C$. If $x \in \langle S \rangle$ then $c, c' \in A_S$, by the definition of A_S. If $x \notin \langle S \rangle$, then there exist $x' \in A_S$ and $a \in \langle S \rangle$ such that $a \in (x, x')$. If a, c, c' are collinear, then $c = a$ or $a \in (c, x)$ or $a \in (c, x')$. In any event $c \in A_S$, and likewise $c' \in A_S$. Thus we now suppose that a, c, c' are not collinear. Then, by Proposition $2'$, $a \in (c, y)$ for some $y \in (c', x')$. Hence $c \in A_S$, and likewise $c' \in A_S$.

It follows at once from Proposition 23 that any face of C which contains S must also contain A_S. $\square$

Corollary 28. *Let C be a convex set and $a \in C$. Then the set A_a consisting of a and all points $x, x' \in C$ such that $a \in (x, x')$ is a face of C, and is contained in every face of C which contains a.*

6. Polytopes

We define a *polytope* to be the convex hull of a finite set. We shall show that the notion of 'face' is especially significant for polytopes.

Proposition 29. *Let P be a polytope and S the (finite) set of extreme points of P. Then the faces of P are the sets $[T]$, where $T \subseteq S$ and $[S \setminus T] \cap \langle T \rangle = \emptyset$.*

Proof: We show first that if F is a face of P, then $F = [T]$ for some $T \subseteq S$ such that $[S \setminus T] \cap \langle T \rangle = \emptyset$. Evidently we may assume that $F \neq \emptyset, P$. Let T denote

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the set of all extreme points of P which are contained in F. Then $[S \setminus T] \subseteq P \setminus F$, since $P \setminus F$ is convex, and thus T is a nonempty proper subset of S. If $x \in F$, then $x \in [y, z]$ for some $y \in [S \setminus T]$ and $z \in [T]$. Thus $y \in P \setminus F$ and $z \in F$. Since $x \neq z$ would imply $y \in \langle F \rangle$, which contradicts $P \cap \langle F \rangle = F$, we must have $x = z \in [T]$. Hence $F = [T]$ and $[S \setminus T] \cap \langle T \rangle = \emptyset$.

We show next that $[T]$ is a face of P if T is a subset of S such that $[S \setminus T] \cap \langle T \rangle = \emptyset$. Let $x \in P \cap \langle T \rangle$. Then $x \in [y, z]$, where $y \in [S \setminus T]$ and $z \in [T]$. In fact $x = z$, since $x \neq z$ would imply $y \in \langle T \rangle$. Thus $P \cap \langle T \rangle = [T]$. It remains to show that $P \setminus [T]$ is convex.

Suppose $z \in (x, y)$, where $x \in [S]$, $y \in [S \setminus T]$ and $z \in [T]$. Then $x \in [y', z']$, where $y' \in [S \setminus T]$ and $z' \in [T]$. Since $[S \setminus T] \cap \langle T \rangle = \emptyset$, we must actually have $x \in (y', z')$ and x, y, z' are not collinear. Hence there exists a point $z'' \in (y, y')$ such that $z \in (z', z'')$. Since $z'' \in [S \setminus T] \cap \langle T \rangle$, this is a contradiction.

If $P \setminus [T]$ is not convex then, for some points $x', x'' \in P \setminus [T]$, there exists a point $z \in (x', x'') \cap [T]$. Then $x' \in [y', z']$, where $y' \in [S \setminus T]$ and $z' \in [T]$. Moreover, by what we have already proved, $x' \in (y', z')$ and x'', y', z' are not collinear. Hence there exists a point $z'' \in (x'', y')$ such that $z \in (z', z'')$. Since $z'' \in P \cap \langle T \rangle = [T]$ this yields a contradiction, as we have already seen. ∎

COROLLARY 30. *If P is a polytope and F a face of P, then F is a polytope and $\mathbf{E}(F) = \mathbf{E}(P) \cap F$.*

COROLLARY 31. *A polytope has only finitely many faces. More precisely, a polytope with n extreme points has at most 2^n faces.*

There is one important case in which the bound 2^n in Corollary 31 is actually attained:

PROPOSITION 32. *Let $P = [S]$, where S is a finite affine independent set. Then a set F is a face of P if and only if $F = [T]$ for some $T \subseteq S$.*

PROOF: To show that every set $[T]$, where $T \subseteq S$, is a face of P it is sufficient to show that $[S \setminus s]$ is a face of P, for every $s \in S$. But this follows at once from Proposition 29. ∎

We establish next some further porperties, showing that the polytopes considered here behave in the manner to which we are accustomed.

PROPOSITION 33. *If P is a polytope and ℓ a line, then $P \cap \ell$ is a segment.*

PROOF: Suppose $P = [S]$, where $S = \{s_1, \ldots, s_n\}$. Since the result is obvious if $n \leqslant 2$, we assume that $n > 2$ and the result holds for all smaller values of n. Obviously we may assume also that ℓ contains two distinct points x, y of P. It follows from Proposition 10 that $P \cap \ell = C \cap \ell$, where $C = \bigcup_{i=1}^{n} [S \setminus s_i]$. Furthermore, by

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the induction hypothesis, $[S \setminus s_i] \cap \ell$ is either empty or has the form $[a_i, b_i]$, for each $i \in \{1, \dots, n\}$. But the points of the line ℓ may be totally ordered, as in Proposition 12. If we put $a = \min_i\{a_i, b_i\}$ and $b = \max_i\{a_i, b_i\}$, then $P \cap \ell = [a, b]$. ∎

Proposition 34. *If P is a polytope and L an affine set, then $P \cap L$ is a polytope.*

Proof: Suppose $P = [S]$, where $S = \{x_1, \dots, x_n\}$. Since the result is obvious if $n = 1$, we assume that $n > 1$ and the result holds for all smaller values of n. Let $p \in P \cap L$. By Proposition 9, we have $P = \bigcup_{i=1}^{n} P_i$, where $P_i = [p \cup (S \setminus x_i)]$. It is enough to show that the sets $P_i \cap L$ $(i = 1, \dots, n)$ are all polytopes. For if $P_i \cap L$ is the convex hull of a finite set S_i $(i = 1, \dots, n)$, then

$$P \cap L = \bigcup_{i=1}^{n} P_i \cap L \subseteq [S_1 \cup \ldots \cup S_n].$$

Since $S_1 \cup \ldots \cup S_n \subseteq P \cap L$ and $P \cap L$ is convex, we must in fact have $[S_1 \cup \ldots \cup S_n] = P \cap L$.

Without loss of generality, we show only that the set $P_1 \cap L$ is a polytope. Put $P_1' = [x_2, \dots, x_n]$ and $P' = P_1' \cap L$. Since P' is the convex hull of a finite set, by the induction hypothesis, we shall complete the proof by showing that

$$P_1 \cap L = [p \cup P'].$$

The right side is certainly contained in the left, since $p \cup P' = (p \cup P_1') \cap L$ and $P_1 = [p \cup P_1']$. On the other hand, the left side is contained in the right. For if $x \in P_1 \setminus p$ then $x \in (p, y]$ for some $y \in P_1'$, and if $x \in (P_1 \cap L) \setminus p$ then $y \in P_1' \cap L$. ∎

Proposition 35. *The intersection of a polytope P with a closed half-space of $\langle P \rangle$ is again a polytope.*

Proof: Let H be a hyperplane of $\langle P \rangle$ and H_+, H_- the open half-spaces associated with H. Let S' be the set of extreme points of the polytope $P' = P \cap H$ and let S_+, S_- be the sets of extreme points of P in H_+, H_- respectively. Obviously we may assume that both S_+ and S_- are nonempty. Then the intersections of P with the closed half-spaces $H \cup H_+$, $H \cup H_-$ are the polytopes $P_+ = [S' \cup S_+]$, $P_- = [S' \cup S_-]$, since $P_+ \cup P_- = P$, $P_+ \cap P_- = P'$. ∎

We say that F is a *facet* of a polytope P if F is a face of P and $\dim F = \dim P - 1$. Equivalently, F is a facet of P if $\langle F \rangle$ is a hyperplane of $\langle P \rangle$, $P \cap \langle F \rangle = F$, and P is contained in a closed half-space of $\langle P \rangle$ associated with the hyperplane $\langle F \rangle$. We say that F is an *edge* of a polytope P if F is a face of P and $\dim F = 1$. The *proper* faces of a polytope P are the faces other than P itself.

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Proposition 36. *Any proper face F of a polytope P is contained in a facet of P. Moreover, if $\dim F = \dim P - 2$, then F is the intersection of two facets of P.*

Proof: We begin by proving the following assertion:

Let S be the set of extreme points of P, let S_1 be a subset of S such that $L_1 = \langle S_1 \rangle \subset \langle S \rangle$ and let F_1 be a facet of the polytope $P_1 = P \cap L_1$. If $L_2 = \langle b, S_1 \rangle$, where $b \in S \setminus L_1$, then the polytope $P_2 = P \cap L_2$ has a facet F_2 such that $F_1 \subset F_2$.

By hypothesis $H_1 = \langle F_1 \rangle$ is a hyperplane of L_1 and, if $a \in S_1 \setminus H_1$, P_1 is contained in the closed half-space of L_1 associated with H_1 which contains a. Let T be the set of extreme points of P_2 and let M be the open half-space of L_2 associated with the hyperplane L_1 which contains b. We can choose $b_1, \ldots, b_m \in T \cap M$ so that every element of $T \cap M$ belongs to $\langle b_i, H_1 \rangle$ for some i and $b_j \notin \langle b_i, H_1 \rangle$ if $j \neq i$. We shall show that the set $\{b_1, \ldots, b_m\}$ is partially ordered by writing $b_i \leqslant b_j$ if $[a, b_j] \cap \langle b_i, H_1 \rangle \neq \emptyset$.

It is evident that $b_i \leqslant b_i$ for every i. Suppose $b_i \leqslant b_j$ and $b_j \leqslant b_i$. Then there exist points $c_i \in [a, b_j] \cap \langle b_i, H_1 \rangle$ and $c_j \in [a, b_i] \cap \langle b_j, H_1 \rangle$. Hence there exists a point $x \in [b_i, c_i] \cap [b_j, c_j]$ and a point $y \in [b_i, b_j]$ such that $x \in [a, y]$. Then $x \in \langle b_i, H_1 \rangle \cap \langle b_j, H_1 \rangle$. Moreover $x \notin H_1$, since $y \notin L_1$. Hence $\langle b_i, H_1 \rangle = \langle x, H_1 \rangle = \langle b_j, H_1 \rangle$, which implies $b_i = b_j$.

Suppose next that $b_i \leqslant b_j$ and $b_j \leqslant b_k$. We wish to show that $b_i \leqslant b_k$. There exist points $c_i \in [a, b_j] \cap \langle b_i, H_1 \rangle$ and $c_j \in [a, b_k] \cap \langle b_j, H_1 \rangle$. If there exists a point $x \in [b_j, c_j] \cap \langle b_i, H_1 \rangle$, then $x \in [a, y]$ for some $y \in [b_j, b_k]$. Moreover $x \notin H_1$, since $y \notin L_1$, and hence, as before, $b_i = b_j \leqslant b_k$. Thus we may now assume $[b_j, c_j] \cap \langle b_i, H_1 \rangle = \emptyset$. Then b_j and c_j lie in the same open half-space of L_2 associated with the hyperplane $\langle b_i, H_1 \rangle$. Since a and b_j lie in different open half-spaces, it follows that a and c_j lie in different open half-spaces, and hence so also do a and b_k. Thus $[a, b_k] \cap \langle b_i, H_1 \rangle \neq \emptyset$ and $b_i \leqslant b_k$.

This completes the proof that the set $\{b_1, \ldots, b_m\}$ is partially ordered. We now choose the notation so that b_m is a maximal element of this partial ordered set. Thus, putting $H_2 = \langle b_m, H_1 \rangle$, we have $[a, b_i] \cap H_2 = \emptyset$ for all $i < m$. We are going to show that $F_2 = P \cap H_2$ is a facet of the polytope $P_2 = P \cap L_2$. Since $\langle F_2 \rangle = H_2$ is a hyperplane of L_2, we need only show that P_2 is contained in the closed half-space of L_2 associated with the hyperplane H_2 which contains a.

Assume on the contrary that, for some $a' \in T$, there exists a point $x \in (a, a') \cap H_2$. Then, by construction, $a' \notin M$. If $a' \in L_1$ then $x \in H_1$, since $b_m \notin L_1$. But then a and a' lie in different open half-spaces of L_1 associated with the hyperplane H_1, which is a contradiction because F_1 is a facet of P_1. Thus we now suppose $a' \notin L_1$. Then $a' \in N$, where N is an open half-space of L_2 associated with the hyperplane L_1 and $N \neq M$. Thus there exists a point $y \in (a', b_m) \cap L_1$. Hence a, a', b_m are not collinear

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and there exists a point $z \in (a, y) \cap (b_m, x)$. Since $z \in L_1 \cap H_2$, we must actually have $z \in H_1$. Since $y \in P_1$, this again contradicts the fact that F_1 is a facet of P_1. This completes the proof that F_2 is a facet of P_2, and it is obvious that $F_1 \subset F_2$.

If $F_2' = P \cap L_1$ is a facet of P_2, then $F_2 \cap F_2' = F_1$ since $H_2 \cap L_1 = H_1$. If $P \cap L_1$ is not a facet of P_2, then $T \cap N \neq \emptyset$ and we can in the same way choose points $d_1, \ldots, d_n \in T \cap N$ and show that $F_2' = P \cap H_2'$, where $H_2' = \langle d_n, H_1 \rangle$, is also a facet of P_2 containing F_1. Since the segment (b_m, d_n) contains a point $x \in L_1$, and $H_2' = H_2$ implies $x \in H_1$, the facets F_2 and F_2' will be distinct if $(b_m, d_n) \cap H_1 = \emptyset$. Thus they are certainly distinct if F_1 is a face of P_2. Again, $F_2 \cap F_2' = F_1$ since $H_2 \cap H_2' = H_1$. If we take $F_1 = F$ to be a face of P such that $\dim F = \dim P - 2$, then $P_2 = P$ and thus we obtain the second statement of the proposition.

To prove the first statement, take $F_1 = F$ to be a face of P such that $\dim F \leqslant \dim P - 2$ and put $H_1 = \langle F_1 \rangle$. Then $H_1 = \langle R \rangle$, where R is the set of extreme points of P which are contained in F. If $L_1 = \langle a, R \rangle$, where $a \in S \setminus H_1$, then F_1 is a facet of $P_1 = P \cap L_1$. By repeatedly applying the initial assertion, we see that F is contained in a facet of P. □

From Proposition 36 we can deduce many other properties of polytopes:

Corollary 37. *If F and G are faces of a polytope P such that $F \subset G$, then there exists a finite sequence $F_0, F_1, \ldots, F_r$ of faces of P, with $F_0 = F$ and $F_r = G$, such that F_{i-1} is a facet of F_i $(i = 1, \ldots, r)$.*

Proposition 38. *Any proper face F of a polytope P is the intersection of all facets of P which contain it.*

Proof: Put $d = \dim P - \dim F$. If $d = 1$ the result is obvious and if $d = 2$ it holds by Proposition 36. We assume that $d > 2$ and the result holds for all smaller values of d. There exist faces G, G' of P such that $G \cap G' = F$ and F is a facet of both G and G'. Since, by the induction hypothesis, G and G' are the intersections of all facets of P which contain them, so also is F. □

Proposition 39. *If F, G, H are faces of P such that $F \subseteq G \subseteq H$, then there exists a face G' of P such that $F \subseteq G' \subseteq H$, $F = G \cap G'$, and every face of P which contains both G and G' also contains H.*

Proof: We may suppose $F \subset G \subset H$, since if $G = F$ we can take $G' = H$ and if $G = H$ we can take $G' = F$. Put $n = \dim H - \dim F$. If $n = 2$ the result holds by Proposition 36. We assume that $n > 2$ and the result holds for all smaller values of n. If $m = \dim G - \dim F$ then, since $n > 2$, either $m > 1$ or $m < n - 1$ (or both). Without loss of generality suppose $m > 1$ and let F_1 be a face of P such that $F \subset F_1 \subset G$ and F is a facet of F_1. Then by the induction hypothesis there is a face G'' of P such that $F_1 \subseteq G'' \subseteq H$, $F_1 = G \cap G''$ and every face which contains both

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G and G'' also contains H. Since $G'' \neq H$, there also exists a face G' of P such that $F \subseteq G' \subseteq G''$, $F = F_1 \cap G'$ and every face which contains both F_1 and G' also contains G''. Since $G' \subseteq G''$,

$$G \cap G' = (G \cap G'') \cap G' = F_1 \cap G' = F$$

and, since $F_1 \subseteq G$, any face which contains both G and G' also contains G'' and hence also contains H. ∎

The next result shows, in particular, that polytopes are *polyhedra*:

PROPOSITION 40. *Let P be a polytope and $F_1, \ldots, F_m$ its facets. If Q_i is the closed half-space of $\langle P \rangle$ associated with the hyperplane $H_i = \langle F_i \rangle$ that contains P $(i = 1, \ldots, m)$, then*

$$P = \bigcap_{i=1}^{m} Q_i.$$

PROOF: Put $Q = \bigcap_{i=1}^{m} Q_i$. Then obviously $P \subseteq Q$. We shall assume that there exists a point $a \in Q \setminus P$ and derive a contradiction. Since $a \in \langle P \rangle$, we have $a \in \langle x, y \rangle$ for some distinct points $x, y \in P$, by Proposition 13. Moreover, by Proposition 33, we may choose x, y so that $P \cap \langle x, y \rangle = [x, y]$ and we may choose the notation so that $x \in (a, y)$. Then x and y belong to proper faces of P, by Corollary 28. Hence, by Proposition 36, x and y belong to facets of P. Thus $x \in H_k$ for some k. Since a and y cannot lie in different open half-spaces of $\langle P \rangle$ associated with the hyperplane H_k, it follows that $a, y \in H_k$ and hence y belongs to the same facet of P as x. Put $n = \dim P$. If $n = 1$, then $P = [x, y]$ and we already have a contradiction. If $n > 1$, then the polytope $P' = P \cap H_k$ has dimension $n-1$ and x, y belong to facets of P'. By the same argument as before, they must belong to the same facet of P'. This process can be continued until we arrive at a one-dimensional polytope, and a contradiction. ∎

From Propositions 35 and 40 we immediately obtain

PROPOSITION 41. *The intersection of two polytopes is again a polytope.*

We can now show also that two-dimensional polytopes are *polygons*:

PROPOSITION 42. *If P is a two-dimensional polytope, then the $n \geqslant 3$ extreme points of P can be numbered $e_1, \ldots, e_n$ so that the facets of P are precisely the segments $[e_i, e_{i+1}]$ $(i = 1, \ldots, n-1)$ and $[e_n, e_1]$.*

PROOF: Since the result is obvious if $n = 3$, we assume that $n > 3$ and the result holds for two-dimensional polytopes with fewer than n extreme points. Let S denote the set of extreme points of P. If $e \in S$ then, by Proposition 36, there exist distinct $e', e'' \in S$ such that $[e, e']$ and $[e, e'']$ are facets of P. Suppose $f \in S$ and

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$f \neq e, e', e''$. Then $\langle e, e'\rangle \cap [e'', f] = \emptyset$ and also $\langle e, e''\rangle \cap [e', f] = \emptyset$. Hence, by Lemma 15, $\langle e, f\rangle \cap [e', e''] \neq \emptyset$. Since all points involved are extreme points of P, we must actually have $(e, f) \cap (e', e'') \neq \emptyset$. It follows that $[e', e'']$ is a facet of the polytope $[S \setminus e]$. The result now follows from the induction hypothesis. ∎

Finally we shall show that the theorem of Balinski [1], that the graph of a d-dimensional polytope in Euclidean space is *d-connected*, remains valid for the polytopes considered here.

LEMMA 43. *Let P be a polytope, G a facet of P and F a face of G. Then there exists an extreme point e' of P such that $e' \notin \langle G\rangle$ and $P \cap \langle e', F\rangle$ is a face of P.*

PROOF: Put $d = \dim P$. Since the result is trivial when $d = 0$ or 1, we assume that $d \geqslant 2$ and the result holds for all smaller values of d.

Let F' be a facet of G and let G' be a facet of P such that $F' = G \cap G'$. Also, let e' be an extreme point of G' which is not in F'. Then e' is an extreme point of P, $e' \notin \langle G\rangle$ and $\langle e', G\rangle = \langle P\rangle$.

Thus we may now assume that F is a proper face of G and that F' is a facet of G which contains F. By the induction hypothesis there exists an extreme point e' of G' such that $e' \notin \langle F'\rangle$ and $G' \cap \langle e', F\rangle$ is a face of G'. Then e' is an extreme point of P, $e' \notin \langle G\rangle$ and $P \cap \langle e', F\rangle = G \cap \langle e', F\rangle$ is a face of P. ∎

PROPOSITION 44. *Let P be a d-dimensional polytope, S the set of extreme points of P and E a subset of S such that $0 \leqslant |S \setminus E| < d$. Then any two distinct points $e, e' \in E$ can be connected in E by edges of P, that is there exists a finite sequence $e_0, e_1, \ldots, e_m$ of elements of E, with $e_0 = e$ and $e_m = e'$, such that $[e_{i-1}, e_i]$ is an edge of P for $i = 1, \ldots, m$.*

PROOF: Fix $e \in E$. It follows from Lemma 43, with $F = \{e\}$, that there exists an affine independent set V of $d+1$ extreme points of P, including e, such that $[e, f]$ is an edge of P for every $f \in V \setminus e$. Hence we may assume that $e' \notin V$ and that the result holds for all d-dimensional polytopes with fewer extreme points than P. Since $\langle V\rangle = \langle P\rangle$, it follows that $e' \in \langle S \setminus e'\rangle$.

By applying Proposition 40 to the polytope $Q = [S \setminus e']$, we see that there exists a facet F of Q such that e' does not lie in the closed half-space of $\langle P\rangle = \langle Q\rangle$ associated with the hyperplane $\langle F\rangle$ which contains Q. We have $F = [T]$, where $T \subset S \setminus e'$. If $f \in T$ then $[e', f]$ is an edge of the polytope $[e' \cup T]$, by Proposition 29, and hence also of P, by Proposition 26. Thus we may assume that $e \in U$, where $U = S \setminus (e' \cup T)$. By the induction hypothesis any two distinct extreme points of Q which are in E can be connected in E by edges of Q. Hence, by Proposition 26, e can be connected in E to some extreme point $f \in T \cap E$ by edges of P. Consequently e can also be connected

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to e'. $\square$

Since this proof of Balinski's theorem appears simpler than previous proofs, and was necessitated by our weaker hypotheses, it shows to advantage the axiomatic method.

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