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RESEARCH ARTICLE

Cohomological Descent for Faltings Ringed Topos

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Abstract
Faltings ringed topos, the keystone of Faltings' approach to p-adic Hodge theory for a smooth variety over a local field, relies on the choice of an integral model, and its good properties depend on the (logarithmic) smoothness of this model. Inspired by Deligne's approach to classical Hodge theory for singular varieties, we establish a cohomological descent result for the structural sheaf of Faltings topos, which makes it possible to extend Faltings' approach to any integral model, that is, without any smoothness assumption. An essential ingredient of our proof is a variation of Bhatt–Scholze's arc-descent of perfectoid rings.

Contents

1. Introduction

1.1. Faltings ringed topos, the keystone of Faltings' approach to p-adic Hodge theory, was originally introduced by Faltings in his proof of the Hodge–Tate decomposition [Fal88, Fal02] and since then became a fundamental tool in p-adic Hodge theory, in particular for p-adic comparison theorems and the p-adic Simpson correspondence (see [Fal05, AGT16]). Faltings topos builds on an integral model of the p-adic variety, which has both benefits and limitations. On the one hand, Faltings' approach uses only standard techniques from scheme theory and seems appropriate for cohomology with integral coefficients. But on the other hand, the (log-)smoothness of the integral model seems necessary for good properties of Faltings topos.

1.2. The goal of this work is to get rid of the (log-)smoothness assumption on integral models for Faltings' approach to p-adic Hodge theory. For this, we establish a *cohomological descent for Faltings ringed topos* along a proper hypercovering of integral models, which allows us to descend important results for Faltings topos associated to nice models to Faltings topos associated to general models using alterations of de Jong et al. In order to avoid the complexity raised by proper hypercoverings, we introduce a variant of Faltings site in v-topology (where both faithfully flat morphisms and proper surjective morphisms are coverings), that we call the *v-site of integrally closed schemes*. We show that both p-adic étale cohomology and the cohomology of Faltings ringed topos can be computed by this new site (see 3.27, 8.12), which automatically implies the cohomological descent for Faltings ringed topos along proper hypercoverings. For this purpose, we prove a variation of Bhatt–Scholze's arc-descent of perfectoid rings [BS22]. More precisely, we prove an almost arc-descent of almost perfectoid algebras (see 5.35), which couldn't be obtained directly from their results but by adjusting their proof.

1.3. Our cohomological descent result has interesting applications in p-adic Hodge theory. Firstly, I extend in [He21] Faltings' main p-adic comparison theorem, both in the absolute and the relative cases, to general integral models without any smoothness condition. Notably, the relative comparison takes place in our v-site of integrally closed schemes and remains valid for torsion abelian étale coefficients (not necessarily finite locally constant). Secondly, I deduce in [He21] an explicit local version of the relative Hodge–Tate filtration from the global version constructed by Abbes–Gros. Thirdly, Xu [Xu22] recently deduced from our cohomological descent a descent result for the p-adic Simpson correspondence. Finally, we would like to mention that our v-site of integrally closed schemes is a scheme theoretic analogue of the v-site of an adic space introduced by Scholze and that our cohomological descent is an analogue of the cohomological descent from the v-topos to the pro-étale topos of an adic space established by Scholze [Sch17]. The advantage of our v-site is that it remains in the framework of algebraic geometry and uses only scheme theoretic arguments. Moreover, it may lead to an explicit comparison between Faltings and Scholze's approaches to p-adic Hodge theory.

1.4. In order to state our cohomological descent result, we recall now the definition of the Faltings site associated to a morphism of coherent schemes $Y \to X$ (see 7.7), where 'coherent' stands for 'quasi-compact and quasi-separated'. Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the category of morphisms of coherent schemes $V \to U$ over $Y \to X$, that is, commutative diagrams

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{1.4.1}$$

such that U is étale over X and that V is finite étale over $Y \times_X U$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N} U_n \to U$ is surjective.

Consider the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ defined by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{1.4.2}$$

where U^V is the integral closure of U in V. It is indeed a sheaf of rings, called the structural sheaf of $\mathbf{E}^{\text{ét}}_{Y\to X}$ (see 7.6).

1.5. Recall that the cohomological descent of étale cohomology along proper hypercoverings can be generalized as follows: For a coherent S-scheme, we endow the category of coherent S-schemes $\mathbf{Sch}^{\text{coh}}_{/S}$ with *h-topology* which is generated by étale coverings and proper surjective morphisms of finite presentation. Then, for any torsion abelian sheaf $\mathcal{F}$ on $S_{\text{ét}}$, denoting by $a : (\mathbf{Sch}^{\text{coh}}_{/S})_{\text{h}} \to S_{\text{ét}}$ the natural morphism of sites, the adjunction morphism $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ is an isomorphism.

This result remains true for a finer topology, the *v-topology*. A morphism of coherent schemes $T \to S$ is called a v-covering if for any morphism $\mathrm{Spec}(A) \to S$ with A a valuation ring, there exists an extension

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of valuation rings $A \to B$ and a lifting $\mathrm{Spec}(B) \to T$. In fact, a v-covering is a limit of h-coverings (see 3.6). We will describe the cohomological descent for $\overline{\mathscr{B}}$ using a new site built from the v-topology defined as follows:

Definition 1.6 (see 3.23). Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is integrally closed in S°. We define a site $\mathbf{I}_{S^\circ\to S}$ as follows:

(1) The underlying category is formed by coherent S-schemes T which are integrally closed in $S^\circ \times_S T$.
(2) The topology is generated by covering families $\{T_i \to T\}_{i\in I}$ in the v-topology.

We call $\mathbf{I}_{S^\circ\to S}$ the *v-site of S°-integrally closed coherent S-schemes*, and we call the sheaf $\mathscr{O}$ on $\mathbf{I}_{S^\circ\to S}$ associated to the presheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ the structural sheaf of $\mathbf{I}_{S^\circ\to S}$.

1.7. Let p be a prime number, $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We take $S^\circ = \mathrm{Spec}(\overline{\mathbb{Q}_p})$ and $S = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Consider a diagram of coherent schemes

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X^Y \arrow[r] \arrow[d] & X \\ \mathrm{Spec}(\overline{\mathbb{Q}_p}) \arrow[r] & \mathrm{Spec}(\overline{\mathbb{Z}_p}) & \end{tikzcd} , \tag{1.7.1}$$

where X^Y is the integral closure of X in Y and the square is Cartesian (we don't impose any condition on the regularity or finiteness of Y or X). The functor $\varepsilon^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{I}_{Y\to X^Y}$ sending $V \to U$ to U^V defines a natural morphism of ringed sites

$$\varepsilon : (\mathbf{I}_{Y\to X^Y}, \mathscr{O}) \longrightarrow (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}). \tag{1.7.2}$$

Our cohomological descent for Faltings ringed topos is stated as follows:

Theorem 1.8 (see 8.14). *For any finite locally constant abelian sheaf $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{1.8.1}$$

is an almost isomorphism, that is, the cohomology groups of its cone are killed by p^r for any rational number $r > 0$ (see 5.7).

Corollary 1.9 (see 8.18). *For any proper hypercovering $X_\bullet \to X$, if $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ denotes the augmentation of simplicial site where $Y_\bullet = Y \times_X X_\bullet$, then the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{1.9.1}$$

is an almost isomorphism.

The key ingredient of our proof of 1.8 is the almost descent of almost perfectoid algebras in arc-topology (a topology finer than the v-topology) (see 5.35). The analogue in characteristic p of 1.8 is Gabber's computation of the cohomology of the structural sheaf in h-topology (see Section 4). Theorem 1.8 allows us to descend important results for Faltings sites associated to nice models to Faltings sites associated to general models. On the other hand, one important step of its proof, which has its own interests, is a characterization of 'acyclic objects' for Faltings ringed site in terms of almost perfectoid algebras. This result holds in the open case, that is, the complement of a normal crossings divisor in the generic fibre, using Abhyankar's lemma (see 8.24).

1.10. The paper is structured as follows. In Section 3, we establish the foundation of the v-site $\mathbf{I}_{S^\circ\to S}$ of integrally closed schemes, where Proposition 3.27 proves that the étale cohomology of S° can be computed by this v-site. Sections 4 and 5 are devoted to a detailed proof of the almost arc-descent for almost perfectoid algebras. Since we use the language of schemes, the terminology 'pre-perfectoid' is introduced for those algebras whose p-adic completions are perfectoid. In Sections 6 and 7, we include some preliminaries about Faltings sites and we introduce a pro-version of Faltings site to evaluate the structural sheaf on the spectrums of pre-perfectoid algebras. Finally, we prove our cohomological descent results in Section 8.

2. Notation and conventions

2.1. We fix a prime number p throughout this paper. For any monoid M, we denote by $\mathrm{Frob}: M \to M$ the map sending an element x to x^p, and we call it the *Frobenius* of M. For a ring R, we denote by $R^\times$ the group of units of R. A ring R is called *absolutely integrally closed* if any monic polynomial $f \in R[T]$ has a root in R ([Sta23, 0DCK]). We remark that quotients, localizations and products of absolutely integrally closed rings are still absolutely integrally closed.

Recall that a valuation ring is a domain V such that for any element x in its fraction field, if $x \notin V$ then $x^{-1} \in V$. The family of ideals of V is totally ordered by the inclusion relation ([Bou06, VI.§1.2, Thm.1]). In particular, a radical ideal of V is a prime ideal. Moreover, any quotient of V by a prime ideal and any localization of V are still valuations rings ([Sta23, 088Y]). We remark that V is normal, and that V is absolutely integrally closed if and only if its fraction field is algebraically closed. An *extension of valuation rings* is an injective and local homomorphism of valuation rings.

2.2. Following [SGA 4_{II}, VI.1.22], a *coherent* scheme (resp. morphism of schemes) stands for a quasi-compact and quasi-separated scheme (resp. morphism of schemes). For a coherent morphism $Y \to X$ of schemes, we denote by X^Y the integral closure of X in Y ([Sta23, 0BAK]). For an X-scheme Z, we say that Z is *Y-integrally closed* if $Z = Z^{Y\times_X Z}$.

2.3. Throughout this paper, we fix two universes $\mathbb{U}$ and $\mathbb{V}$ such that the set of natural numbers $\mathbb{N}$ is an element of $\mathbb{U}$ and that $\mathbb{U}$ is an element of $\mathbb{V}$ ([SGA 4_{I}, I.0]). In most cases, we won't emphasize this set theoretical issue. Unless stated otherwise, we only consider $\mathbb{U}$-small schemes and we denote by **Sch** the category of $\mathbb{U}$-small schemes, which is a $\mathbb{V}$-small category.

2.4. Let C be a category. We denote by $\widehat{C}$ the category of presheaves of $\mathbb{V}$-small sets on C. If C is a $\mathbb{V}$-site ([SGA 4_{I}, II.3.0.2]), we denote by $\widetilde{C}$ the topos of sheaves of $\mathbb{V}$-small sets on C. We denote by $h^C : C \to \widehat{C}$, $x \mapsto h^C_x$ the Yoneda embedding ([SGA 4_{I}, I.1.3]) and by $\widehat{C} \to \widetilde{C}$, $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{a}}$ the sheafification functor ([SGA 4_{I}, II.3.4]).

2.5. Let $u^+ : C \to D$ be a functor of categories. We denote by $u^{\mathrm{p}} : \widehat{D} \to \widehat{C}$ the functor that associates to a presheaf $\mathcal{G}$ of $\mathbb{V}$-small sets on D the presheaf $u^{\mathrm{p}}\mathcal{G} = \mathcal{G} \circ u^+$. If C is $\mathbb{V}$-small and D is a $\mathbb{V}$-category, then u^{p} admits a left adjoint u_{p} [Sta23, 00VC] and a right adjoint ${}_{\mathrm{p}}u$ [Sta23, 00XF] (cf. [SGA 4_{I}, I.5]). So we have a sequence of adjoint functors

$$u_{\mathrm{p}},\ u^{\mathrm{p}},\ {}_{\mathrm{p}}u. \tag{2.5.1}$$

If moreover C and D are $\mathbb{V}$-sites, then we denote by $u_{\mathrm{s}}, u^{\mathrm{s}}, {}_{\mathrm{s}}u$ the functors of the topoi $\widetilde{C}$ and $\widetilde{D}$ of sheaves of $\mathbb{V}$-small sets induced by composing the sheafification functor with the functors $u_{\mathrm{p}}, u^{\mathrm{p}}, {}_{\mathrm{p}}u$, respectively. If finite limits are representable in C and D and if u^+ is left exact and continuous, then u^+ gives a morphism of sites $u : D \to C$ ([SGA 4_{I}, IV.4.9.2]) and we also denote by

$$u = (u^{-1}, u_*) : \widetilde{D} \to \widetilde{C} \tag{2.5.2}$$

the associated morphism of topoi, where $u^{-1} = u_{\mathrm{s}}$ and $u_* = u^{\mathrm{s}} = u^{\mathrm{p}}|_{\widetilde{D}}$. If moreover u is a morphism of ringed sites $u : (D, \mathcal{O}_D) \to (C, \mathcal{O}_C)$, then we denote by $u^* = \mathcal{O}_D \otimes_{u^{-1}\mathcal{O}_C} u^{-1}$ the pullback functor of modules. We remark that the notation here, adopted by [Sta23], is slightly different with that in [SGA 4_{I}] (see [Sta23, 0CMZ]).

3. The v-site of integrally closed schemes

Definition 3.1. Let $X \to Y$ be a quasi-compact morphism of schemes.

(1) We say that $X \to Y$ is a *v-covering* if for any valuation ring V and any morphism $\mathrm{Spec}(V) \to Y$, there exists an extension of valuation rings $V \to W$ (2.1) and a commutative diagram (cf. [Sta23, 0ETN])

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Y. \end{array} \tag{3.1.1}$$

(2) Let π be an element of $\Gamma(Y, \mathcal{O}_Y)$. We say that $X \to Y$ is an *arc-covering* (resp. *π-complete arc-covering*) if for any valuation ring (resp. π-adically complete valuation ring) V of height ≤ 1 and any morphism $\mathrm{Spec}(V) \to Y$, there exists an extension of valuation rings (resp. π-adically complete valuation rings) $V \to W$ of height ≤ 1 and a commutative diagram (3.1.1) (cf. [BM21, 1.2], [CS19, 2.2.1]).

(3) We say that $X \to Y$ is an *h-covering* if it is a v-covering and locally of finite presentation (cf. [Sta23, 0ETS]).

We note that an arc-covering is simply a 0-complete arc-covering.

Lemma 3.2. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be quasi-compact morphisms of schemes, $\pi \in \Gamma(X, \mathcal{O}_X)$, $\tau \in \{h, v, \pi\text{-complete arc}\}$.*

(1) *If f is a τ-covering, then any base change of f is also a τ-covering.*
(2) *If f and g are τ-coverings, then $f \circ g$ is also a τ-covering.*
(3) *If $f \circ g$ is a τ-covering (and if f is locally of finite presentation when $\tau = h$), then f is also a τ-covering.*

Proof. It follows directly from the definitions. □

3.3. Let $\mathbf{Sch}^{\mathrm{coh}}$ be the category of coherent $\mathbb{U}$-small schemes, $\tau \in \{\mathrm{h, v, arc}\}$. We endow $\mathbf{Sch}^{\mathrm{coh}}$ with the τ-topology generated by the pretopology formed by families of morphisms $\{X_i \to X\}_{i \in I}$ with I finite such that $\coprod_{i \in I} X_i \to X$ is a τ-covering, and we denote the corresponding site by $\mathbf{Sch}^{\mathrm{coh}}_\tau$. It is clear that a morphism $Y \to X$ (which is locally of finite presentation if $\tau = \mathrm{h}$) is a τ-covering if and only if $\{Y \to X\}$ is a covering family in $\mathbf{Sch}^{\mathrm{coh}}_\tau$ by 3.2 and [SGA 4_{I}, II.1.4].

For any coherent $\mathbb{U}$-small scheme X, we endow the category $\mathbf{Sch}^{\mathrm{coh}}_{/X}$ of objects of $\mathbf{Sch}^{\mathrm{coh}}$ over X with the topology induced by the τ-topology of $\mathbf{Sch}^{\mathrm{coh}}$, that is, the topology generated by the pretopology formed by families of X-morphisms $\{Y_i \to Y\}_{i \in I}$ with I finite such that $\coprod_{i \in I} Y_i \to Y$ is a τ-covering ([SGA 4_{I}, III.5.2]). For any sheaf $\mathcal{F}$ of $\mathbb{V}$-small abelian groups on the site $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_\tau$, we denote its q-th cohomology by $H^q_\tau(X, \mathcal{F})$.

Lemma 3.4. *Let $f : X \to Y$ be a quasi-compact morphism of schemes, $\pi \in \Gamma(Y, \mathcal{O}_Y)$.*

(1) *If f is proper surjective or faithfully flat, then f is a v-covering.*
(2) *If f is an h-covering and Y is affine, then there exists a proper surjective morphism $Y' \to Y$ of finite presentation and a finite affine open covering $Y' = \bigcup_{r=1}^n Y'_r$ such that $Y'_r \to Y$ factors through f for each r.*
(3) *If f is an h-covering and if there exists a directed inverse system $(f_\lambda : X_\lambda \to Y_\lambda)_{\lambda \in \Lambda}$ of finitely presented morphisms of coherent schemes with affine transition morphisms $\psi_{\lambda'\lambda} : X_{\lambda'} \to X_\lambda$ and $\phi_{\lambda'\lambda} : Y_{\lambda'} \to Y_\lambda$ such that $X = \lim X_\lambda$, $Y = \lim Y_\lambda$ and that f_λ is the base change of f_{λ_0} by $\phi_{\lambda\lambda_0}$ for some index $\lambda_0 \in \Lambda$ and any $\lambda \geq \lambda_0$, then there exists an index $\lambda_1 \geq \lambda_0$ such that f_λ is an h-covering for any $\lambda \geq \lambda_1$.*
(4) *If f is a v-covering, then it is a π-complete arc-covering and is particularly an arc-covering.*

(5) *Let π' be another element of $\Gamma(Y, \mathcal{O}_Y)$ which divides π. If f is a π-complete arc-covering, then it is a π'-complete arc-covering.*
(6) *If* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is a π-complete arc-covering, then the morphism* $\mathrm{Spec}(\widehat{B}) \to \mathrm{Spec}(\widehat{A})$ *between the spectrums of their π-adic completions is also a π-complete arc-covering.*

Proof. (1), (2) are proved in [Sta23, 0ETK, 0ETU], respectively.

(3) To show that one can take $\lambda_1 \geq \lambda_0$ such that f_{λ_1} is an h-covering, we may assume that Y_{λ_0} is affine by replacing it by a finite affine open covering by 3.2 and (1). Thus, applying (2) to the h-covering f and using [EGA IV$_3$, 8.8.2, 8.10.5], there exists an index $\lambda_1 \geq \lambda_0$, a proper surjective morphism $Y'_{\lambda_1} \to Y_{\lambda_1}$ and a finite affine open covering $Y'_{\lambda_1} = \bigcup_{r=1}^n Y'_{r\lambda_1}$ such that the morphisms $Y'_r \to Y' \to Y$ are the base changes of the morphisms $Y'_{r\lambda_1} \to Y'_{\lambda_1} \to Y_{\lambda_1}$ by the transition morphism $Y \to Y_{\lambda_1}$, and that $Y'_{r\lambda_1} \to Y_{\lambda_1}$ factors through X_{λ_1}. This shows that f_{λ_1} is an h-covering by 3.2 and (1).

(4) With the notation in Equation (3.1.1) if V is a π-adically complete valuation ring of height ≤ 1 with maximal ideal $\mathfrak{m}$, then since the family of prime ideals of W is totally ordered by the inclusion relation (2.1), we take the maximal prime ideal $\mathfrak{p} \subseteq W$ over $0 \subseteq V$ and the minimal prime ideal $\mathfrak{q} \subseteq W$ over $\mathfrak{m} \subseteq V$. Then, $\mathfrak{p} \subseteq \mathfrak{q}$ and $W' = (W/\mathfrak{p})_{\mathfrak{q}}$ over V is an extension of valuation rings of height ≤ 1. Since $\pi \in \mathfrak{m}$ and W' is of height ≤ 1, the π-adic completion $\widehat{W'}$ is still a valuation ring extension of V of height ≤ 1 (see [Bou06, VI.§5.3, Prop.5]), which proves (4) (as arc-coverings are just 0-complete arc-coverings).

(5) Since a π'-adically complete valuation ring V is also π-adically complete ([Sta23, 090T]), there exists a lifting $\mathrm{Spec}(W) \to X$ for any morphism $\mathrm{Spec}(V) \to Y$. After replacing W by its π'-adic completion, the conclusion follows.

(6) Let V be a π-adically complete valuation ring of height ≤ 1. Given a morphism $\widehat{A} \to V$, there exists a lifting $B \to W$ where $V \to W$ is an extension of π-adically complete valuation rings of height ≤ 1. It is clear that $B \to W$ factors through $\widehat{B}$, which proves (6). □

3.5. Let X be a coherent scheme, $\mathbf{Sch}^{\mathrm{fp}}_{/X}$ the full subcategory of $\mathbf{Sch}^{\mathrm{coh}}_{/X}$ formed by finitely presented X-schemes. We endow it with the topology generated by the pretopology formed by families of morphisms $\{Y_i \to Y\}_{i\in I}$ with I finite such that $\coprod_{i\in I} Y_i \to Y$ is an h-covering, and we denote the corresponding site by $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. It is clear that this topology coincides with the topologies induced from $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ and from $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}}$. The inclusion functors $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}} \xrightarrow{\xi^+} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\zeta^+} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ define morphisms of sites (2.5)

$$(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \xrightarrow{\zeta} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\xi} (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}. \tag{3.5.1}$$

Lemma 3.6. *Let X be a coherent scheme. Then, for any covering family $\mathfrak{U} = \{Y_i \to Y\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ with I finite,*

(i) *there exists a directed inverse system $(Y_\lambda)_{\lambda\in\Lambda}$ of finitely presented X-schemes with affine transition morphisms such that $Y = \lim Y_\lambda$, and*
(ii) *for each $i \in I$, there exists a directed inverse system $(Y_{i\lambda})_{\lambda\in\Lambda}$ of finitely presented X-schemes with affine transition morphisms over the inverse system $(Y_\lambda)_{\lambda\in\Lambda}$ such that $Y_i = \lim Y_{i\lambda}$ and*
(iii) *for each $\lambda \in \Lambda$, the family $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ is a covering in $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$.*

Proof. We take a directed set A such that for each $i \in I$, we can write Y_i as a cofiltered limit of finitely presented Y-schemes $Y_i = \lim_{\alpha\in A} Y_{i\alpha}$ with affine transition morphisms ([Sta23, 09MV]). We see that $\coprod_{i\in I} Y_{i\alpha} \to Y$ is an h-covering for each $\alpha \in A$ by 3.2.

We write Y as a cofiltered limit of finitely presented X-schemes $Y = \lim_{\beta\in B} Y_\beta$ with affine transition morphisms ([Sta23, 09MV]). By [EGA IV$_3$, 8.8.2, 8.10.5] and 3.4.(3), for each $\alpha \in A$, there exists an index $\beta_\alpha \in B$ such that the morphism $Y_{i\alpha} \to Y$ is the base change of a finitely presented morphism $Y_{i\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ by the transition morphism $Y \to Y_{\beta_\alpha}$ for each $i \in I$ and that $\coprod_{i\in I} Y_{i\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ is an h-covering. For each $\beta \geq \beta_\alpha$, let $Y_{i\alpha\beta}$ be the base change of $Y_{i\alpha\beta_\alpha}$ by $Y_\beta \to Y_{\beta_\alpha}$.

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We define a category Λ^{op}, whose set of objects is $\{(\alpha,\beta)\in A\times B \mid \beta\geq\beta_\alpha\}$, and for any two objects $\lambda'=(\alpha',\beta')$, $\lambda=(\alpha,\beta)$, the set $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\lambda',\lambda)$ is

(i) the subset of $\prod_{i\in I}\mathrm{Hom}_{Y_{\beta'}}(Y_{i\alpha'\beta'},Y_{i\alpha\beta'})$ formed by elements $f=(f_i)_{i\in I}$ such that for each $i\in I$, $f_i: Y_{i\alpha'\beta'}\to Y_{i\alpha\beta'}$ is affine and the base change of f_i by $Y\to Y_{\beta'}$ is the transition morphism $Y_{i\alpha'}\to Y_{i\alpha}$ if $\alpha'\geq\alpha$ and $\beta'\geq\beta$;

(ii) empty, if else.

The composition of morphisms $(g_i: Y_{i\alpha''\beta''}\to Y_{i\alpha'\beta''})_{i\in I}$ with $(f_i: Y_{i\alpha'\beta'}\to Y_{i\alpha\beta'})_{i\in I}$ in Λ^{op} is $(g_i\circ f_i': Y_{i\alpha''\beta''}\to Y_{i\alpha\beta''})$, where f_i' is the base change of f_i by the transition morphism $Y_{\beta''}\to Y_{\beta'}$. We see that Λ^{op} is cofiltered by [EGA IV_3, 8.8.2]. Let Λ be the opposite category of Λ^{op}. For each $i\in I$ and $\lambda=(\alpha,\beta)\in\Lambda$, we set $Y_\lambda=Y_\beta$ and $Y_{i\lambda}=Y_{i\alpha\beta}$. It is clear that the natural functors $\Lambda\to A$ and $\Lambda\to B$ are cofinal ([SGA 4_{I}, I.8.1.3]). After replacing Λ by a directed set ([Sta23, 0032]), the families $\mathfrak{U}_\lambda=\{Y_{i\lambda}\to Y_\lambda\}_{i\in I}$ satisfy the required conditions. □

Lemma 3.7. *With the notation in 3.5, let $\mathcal{F}$ be a presheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, (Y_λ) a directed inverse system of finitely presented X-schemes with affine transition morphisms, $Y=\lim Y_\lambda$. Then, we have $\nu_{\mathrm{p}}\mathcal{F}(Y)=\operatorname{colim}\mathcal{F}(Y_\lambda)$, where $\nu^+=\xi^+$ (resp. $\nu^+=\zeta^+\circ\xi^+$).*

Proof. Notice that the presheaf $\mathcal{F}$ is a filtered colimit of representable presheaves by [SGA 4_{I}, I.3.4]

$$\mathcal{F}=\operatorname*{colim}_{Y'\in(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{/\mathcal{F}}} h_{Y'}. \tag{3.7.1}$$

Thus, we may assume that $\mathcal{F}$ is representable by a finitely presented X-scheme Y' since the section functor $\Gamma(Y,-)$ commutes with colimits of presheaves ([Sta23, 00VB]). Then, we have

$$\nu_{\mathrm{p}}h_{Y'}(Y)=h_{\nu^+(Y')}(Y)=\mathrm{Hom}_X(Y,Y')=\operatorname{colim}\mathrm{Hom}_X(Y_\lambda,Y')=\operatorname{colim}h_{Y'}(Y_\lambda), \tag{3.7.2}$$

where the first equality follows from [Sta23, 04D2], and the third equality follows from [EGA IV_3, 8.14.2]. □

Proposition 3.8. *With the notation in 3.5, let $\mathcal{F}$ be an abelian sheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, (Y_λ) a directed inverse system of finitely presented X-schemes with affine transition morphisms, $Y=\lim Y_\lambda$. Let $\tau=\mathrm{h}$ and $\nu^+=\xi^+$ (resp. $\tau=\mathrm{v}$ and $\nu^+=\zeta^+\circ\xi^+$). Then, for any integer q, we have*

$$H^q_\tau(Y,\nu^{-1}\mathcal{F})=\operatorname{colim}H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y_\lambda})_{\mathrm{h}},\mathcal{F}). \tag{3.8.1}$$

In particular, the canonical morphism $\mathcal{F}\longrightarrow\mathrm{R}\nu_*\nu^{-1}\mathcal{F}$ is an isomorphism.

Proof. For the second assertion, the sheaf $\mathrm{R}^q\nu_*\nu^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y\mapsto H^q_\tau(Y,\nu^{-1}\mathcal{F})=H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y})_{\mathrm{h}},\mathcal{F})$ by the first assertion, which is $\mathcal{F}$ if $q=0$ and vanishes otherwise.

We claim that it suffices to show that Equation (3.8.1) holds for any injective abelian sheaf $\mathcal{F}=\mathcal{I}$ on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. Indeed, if so, then we prove by induction on q that Equation (3.8.1) holds in general. The case where $q\leq-1$ is trivial. We set $H^q_1(\mathcal{F})=H^q_\tau(Y,\nu^{-1}\mathcal{F})$ and $H^q_2(\mathcal{F})=\operatorname{colim}H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y_\lambda})_{\mathrm{h}},\mathcal{F})$. We embed an abelian sheaf $\mathcal{F}$ to an injective abelian sheaf $\mathcal{I}$. Consider the exact sequence $0\to\mathcal{F}\to\mathcal{I}\to\mathcal{G}\to0$ and the morphism of long exact sequences

$$\begin{tikzcd}
H^{q-1}_1(\mathcal{I}) \arrow[r] \arrow[d, "\gamma_1"] & H^{q-1}_1(\mathcal{G}) \arrow[r] \arrow[d, "\gamma_2"] & H^q_1(\mathcal{F}) \arrow[r] \arrow[d, "\gamma_3"] & H^q_1(\mathcal{I}) \arrow[r] \arrow[d, "\gamma_4"] & H^q_1(\mathcal{G}) \arrow[d, "\gamma_5"] \\
H^{q-1}_2(\mathcal{I}) \arrow[r] & H^{q-1}_2(\mathcal{G}) \arrow[r] & H^q_2(\mathcal{F}) \arrow[r] & H^q_2(\mathcal{I}) \arrow[r] & H^q_2(\mathcal{G}).
\end{tikzcd} \tag{3.8.2}$$

If Equation (3.8.1) holds for any abelian sheaf $\mathcal{F}$ for degree $q-1$, then $\gamma_1, \gamma_2, \gamma_4$ are isomorphisms and thus γ_3 is injective by the 5-lemma ([Sta23, 05QA]). Thus, γ_5 is also injective since $\mathcal{F}$ is an arbitrary abelian sheaf. Then, we see that γ_3 is an isomorphism, which completes the induction procedure.

For an injective abelian sheaf $\mathcal{I}$ on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, we claim that for any covering family $\mathfrak{U} = \{(Y_i \to Y)\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_\tau$ with I finite, the augmented Čech complex associated to the presheaf $\nu_{\mathrm{p}}\mathcal{I}$

$$\nu_{\mathrm{p}}\mathcal{I}(Y) \to \prod_{i\in I} \nu_{\mathrm{p}}\mathcal{I}(Y_i) \to \prod_{i,j\in I} \nu_{\mathrm{p}}\mathcal{I}(Y_i \times_Y Y_j) \to \cdots \tag{3.8.3}$$

is exact. Admitting this claim, we see that $\nu_{\mathrm{p}}\mathcal{I}$ is indeed a sheaf, that is, $\nu^{-1}\mathcal{I} = \nu_{\mathrm{p}}\mathcal{I}$, and the vanishing of higher Čech cohomologies implies that $H^q_\tau(Y, \nu^{-1}\mathcal{I}) = 0$ for $q > 0$ by 3.6 ([Sta23, 03F9]), which completes the proof together with 3.7. For the claim, we take the covering families $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$ constructed by 3.6. By 3.7, the sequence (3.8.3) is the filtered colimit of the augmented Čech complexes

$$\mathcal{I}(Y_\lambda) \to \prod_{i\in I} \mathcal{I}(Y_{i\lambda}) \to \prod_{i,j\in I} \mathcal{I}(Y_{i\lambda} \times_{Y_\lambda} Y_{j\lambda}) \to \cdots, \tag{3.8.4}$$

which are exact since $\mathcal{I}$ is an injective abelian sheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. □

Corollary 3.9. *Let X be a coherent scheme, $\mathcal{F}$ a torsion abelian sheaf on the site $X_{\text{ét}}$ formed by coherent étale X-schemes endowed with the étale topology, $a : (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \to X_{\text{ét}}$ the morphism of sites defined by the inclusion functor. Then, the canonical morphism $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ is an isomorphism.*

Proof. Consider the morphisms of sites defined by inclusion functors

$$(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \xrightarrow{\zeta} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\xi} (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}} \xrightarrow{\mu} X_{\text{ét}}. \tag{3.9.1}$$

Notice that the morphism $\mathcal{F} \to \mathrm{R}(\mu\circ\xi)_*(\mu\circ\xi)^{-1}\mathcal{F}$ is an isomorphism by [Sta23, 0EWG]. Hence, $\mathcal{F} \to \mathrm{R}\mu_*\mu^{-1}\mathcal{F}$ is an isomorphism by 3.8 and thus so is $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ by 3.8. □

Corollary 3.10. *Let $f : X \to Y$ be a proper morphism of coherent schemes, $\mathcal{F}$ a torsion abelian sheaf on $X_{\text{ét}}$. Consider the commutative diagram*

$$\begin{tikzcd} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \arrow[r, "a_X"] \arrow[d, "f_{\mathrm{v}}"'] & X_{\text{ét}} \arrow[d, "f_{\text{ét}}"] \\ (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \arrow[r, "a_Y"] & Y_{\text{ét}}, \end{tikzcd} \tag{3.10.1}$$

where f_{v} and $f_{\text{ét}}$ are defined by the base change by f. Then, the canonical morphism

$$a_Y^{-1}\mathrm{R}f_{\text{ét}*}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{v}*}a_X^{-1}\mathcal{F} \tag{3.10.2}$$

is an isomorphism.

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Proof. Consider the commutative diagram

$$\begin{tikzcd} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \arrow[r, "\zeta_X"] \arrow[d, "f_{\mathrm{v}}"] & (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \arrow[r, "b_X"] \arrow[d, "f_{\mathrm{h}}"] & X_{\mathrm{\acute{e}t}} \arrow[d, "f_{\mathrm{\acute{e}t}}"] \\ (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \arrow[r, "\zeta_Y"] & (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{h}} \arrow[r, "b_Y"] & Y_{\mathrm{\acute{e}t}}. \end{tikzcd} \tag{3.10.3}$$

The canonical morphism $b_Y^{-1}\mathrm{R}f_{\mathrm{\acute{e}t}*}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{h}*}b_X^{-1}\mathcal{F}$ is an isomorphism by [Sta23, 0EWF]. It remains to show that the canonical morphism $\zeta_Y^{-1}\mathrm{R}f_{\mathrm{h}*}b_X^{-1}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{v}*}a_X^{-1}\mathcal{F}$ is an isomorphism. Let Y' be a coherent Y-scheme, and we set $g : X' = Y' \times_Y X \to X$. For each integer q, $\zeta_Y^{-1}\mathrm{R}^q f_{\mathrm{h}*}b_X^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y' \mapsto H^q_{\mathrm{h}}(X', b_{X'}^{-1}g_{\mathrm{\acute{e}t}}^{-1}\mathcal{F}) = H^q_{\mathrm{\acute{e}t}}(X', g_{\mathrm{\acute{e}t}}^{-1}\mathcal{F})$ by [Sta23, 0EWH], and $\mathrm{R}^q f_{\mathrm{v}*}a_X^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y' \mapsto H^q_{\mathrm{v}}(X', a_{X'}^{-1}g_{\mathrm{\acute{e}t}}^{-1}\mathcal{F}) = H^q_{\mathrm{\acute{e}t}}(X', g_{\mathrm{\acute{e}t}}^{-1}\mathcal{F})$ by 3.9. □

Lemma 3.11. *Let A be a product of (resp. absolutely integrally closed) valuation rings (2.1).*

(1) *Any finitely generated ideal of A is principal.*
(2) *Any connected component of* $\mathrm{Spec}(A)$ *with the reduced closed subscheme structure is isomorphic to the spectrum of a (resp. absolutely integrally closed) valuation ring.*

Proof. (1) is proved in [Sta23, 092T], and (2) follows from the proof of [BS17, 6.2]. □

Lemma 3.12. *Let X be a $\mathbb{U}$-small scheme, $y \rightsquigarrow x$ a specialization of points of X. Then, there exists a $\mathbb{U}$-small family $\{f_\lambda : \mathrm{Spec}(V_\lambda) \to X\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ of morphisms of schemes such that*

(i) *the ring V_λ is a $\mathbb{U}$-small (resp. absolutely integrally closed) valuation ring and that*
(ii) *the morphism f_λ maps the generic point and closed point of* $\mathrm{Spec}(V_\lambda)$ *to y and x respectively and that*
(iii) *for any morphism of schemes $f : \mathrm{Spec}(V) \to X$, where V is a (resp. absolutely integrally closed) valuation ring such that f maps the generic point and closed point of V to y and x, respectively, there exists an element $\lambda \in \Lambda_{y\rightsquigarrow x}$ such that f factors through f_λ and that $V_\lambda \to V$ is an extension of valuation rings.*

Proof. Let K_y be the residue field $\kappa(y)$ of y (resp. an algebraic closure of $\kappa(y)$). Let $\mathfrak{p}_y$ be the prime ideal of the local ring $\mathcal{O}_{X,x}$ corresponding to the point y, and let $\{V_\lambda\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ be the set of all valuation rings with fraction field K_y which contain $\mathcal{O}_{X,x}/\mathfrak{p}_y$ such that the injective homomorphism $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V_\lambda$ is local. The smallness of $\Lambda_{y\rightsquigarrow x}$ and V_λ is clear, and the inclusion $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V_\lambda$ induces a morphism $f_\lambda : \mathrm{Spec}(V_\lambda) \to X$ satisfying (ii). It remains to check (iii). The morphism f induces an injective and local homomorphism $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V$. Notice that $\mathcal{O}_{X,x}/\mathfrak{p}_y \to \mathrm{Frac}(V)$ factors through K_y and that $K_y \cap V$ is a valuation ring with fraction field K_y. It is clear that $K_y \cap V \to V$ is local and injective, which shows that $K_y \cap V$ belongs to the set $\{V_\lambda\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ constructed before. □

Lemma 3.13. *Let $f : \mathrm{Spec}(V) \to X$ be a morphism of schemes where V is a valuation ring. We denote by a and b the closed point and generic point of* $\mathrm{Spec}(V)$*, respectively. If $c \in X$ is a generalization of $f(b)$, then there exists an absolutely integrally closed valuation ring W, a prime ideal $\mathfrak{p}$ of W and a morphism $g : \mathrm{Spec}(W) \to X$ satisfying the following conditions:*

(i) *If z, y, x denote, respectively, the generic point, the point $\mathfrak{p}$ and the closed point of* $\mathrm{Spec}(W)$*, then $g(z) = c$, $g(y) = f(b)$ and $g(x) = f(a)$.*
(ii) *The induced morphism* $\mathrm{Spec}(W/\mathfrak{p}) \to X$ *factors through f and the induced morphism $V \to W/\mathfrak{p}$ is an extension of valuation rings.*

Proof. According to [EGA II, 7.1.4], there exists an absolutely integrally closed valuation ring U and a morphism $\mathrm{Spec}(U) \to X$ which maps the generic point z and the closed point y of $\mathrm{Spec}(U)$ to c and $f(b)$, respectively. After extending U, we may assume that the morphism $y \to f(b)$ factors through b

([EGA II, 7.1.2]). We denote by $\kappa(y)$ the residue field of the point y. Let V' be a valuation ring extension of V with fraction field $\kappa(y)$, and let W be the preimage of V' by the surjection $U \to \kappa(y)$. Then, the maximal ideal $\mathfrak{p} = \mathrm{Ker}(U \to \kappa(y))$ of U is a prime ideal of W, and $W/\mathfrak{p} = V'$. We claim that W is an absolutely integrally closed valuation ring such that $W_{\mathfrak{p}} = U$. Indeed, firstly note that the fraction fields of U and W are equal as $\mathfrak{p} \subseteq W$. Let γ be an element of $\mathrm{Frac}(W) \setminus W$. If $\gamma \in U$, then $\gamma^{-1} \in W \setminus \mathfrak{p}$ by definition since $\gamma^{-1} \in U \setminus \mathfrak{p}$ and V is a valuation ring, and then $\gamma \in W_{\mathfrak{p}}$. If $\gamma \notin U$, then $\gamma^{-1} \in \mathfrak{p}$ since U is a valuation ring, and then $\gamma \notin W_{\mathfrak{p}}$. Thus, we have proved the claim, which shows that W satisfies the required conditions. □

Proposition 3.14. *Let X be a coherent $\mathbb{U}$-small scheme, X° a quasi-compact dense open subset of X. Then, there exists a $\mathbb{U}$-small product A of absolutely integrally closed $\mathbb{U}$-small valuation rings and a v-covering* $\mathrm{Spec}(A) \to X$ *such that* $\mathrm{Spec}(A)$ *is X°-integrally closed (2.2).*

Proof. After replacing X by a finite affine open covering, we may assume that $X = \mathrm{Spec}(R)$. For a specialization $y \rightsquigarrow x$ of points of X, let $\{R \to V_\lambda\}_{\lambda \in \Lambda_{y \rightsquigarrow x}}$ be the $\mathbb{U}$-small set constructed in 3.12. Let $\Lambda = \coprod_{y \in X^\circ} \Lambda_{y \rightsquigarrow x}$, where $y \rightsquigarrow x$ runs through all specializations in X such that $y \in X^\circ$. We take $A = \prod_{\lambda \in \Lambda} V_\lambda$ and $R \to A$ the natural homomorphism. As a quasi-compact open subscheme of $\mathrm{Spec}(A)$, $X^\circ \times_X \mathrm{Spec}(A)$ is the spectrum of $A[1/\pi]$ for an element $\pi = (\pi_\lambda)_{\lambda \in \Lambda} \in A$ by 3.11.(1) ([Sta23, 01PH]). Notice that $\pi_\lambda \neq 0$ for any $\lambda \in \Lambda$. We see that A is integrally closed in $A[1/\pi]$. It remains to check that $\mathrm{Spec}(A) \to X$ is a v-covering. For any morphism $f : \mathrm{Spec}(V) \to X$, where V is a valuation ring, by 3.13, there exists an absolutely integrally closed valuation ring W, a prime ideal $\mathfrak{p}$ of W and a morphism $g : \mathrm{Spec}(W) \to X$ such that g maps the generic point of W into X° and that $W/\mathfrak{p}$ is a valuation ring extension of V. By construction, there exists $\lambda \in \Lambda$ such that g factors through $\mathrm{Spec}(V_\lambda) \to X$. We see that f lifts to the composition of $\mathrm{Spec}(W/\mathfrak{p}) \to \mathrm{Spec}(V_\lambda) \to \mathrm{Spec}(A)$. □

Proposition 3.15. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Z \arrow[r] & X. \end{tikzcd} \tag{3.15.1}$$

Assume the following conditions hold:

(1) *$Y \to Z$ is dominant and $Y' \to Y \times_X X'$ is surjective.*
(2) *$Z \to X$ is separated, $Z' \to Z$ is quasi-compact and $Z' \to X'$ is integral.*
(3) *For any valuation ring W and any morphism* $\mathrm{Spec}(W) \to X$ *such that the generic point of* $\mathrm{Spec}(W)$ *lies over Y, there exists an extension of valuation rings $W \to W'$ and a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(W') \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(W) \arrow[r] & X. \end{tikzcd} \tag{3.15.2}$$

Then, $Z' \to Z$ is a v-covering.

Proof. Notice that $Z' \to Z \times_X X'$ is still integral as $Z \to X$ is separated. After replacing $X' \to X$ by $Z \times_X X' \to Z$, we may assume that $Z = X$. Let $\mathrm{Spec}(V) \to Z$ be a morphism of schemes where V is a valuation ring. Since $Y \to Z$ is dominant, by 3.13, there exists a morphism $\mathrm{Spec}(W) \to Z$, where W is an absolutely integrally closed valuation ring, a prime ideal $\mathfrak{p}$ of W such that $W/\mathfrak{p}$ is a valuation ring extension of V and that the generic point ξ of $\mathrm{Spec}(W)$ is over the image of $Y \to Z$. After extending W ([Sta23, 00IA]), we may assume that there exists a lifting $\xi \to Y$ of $\xi \to Z$. Thus, by assumption (3.15), the morphism $\mathrm{Spec}(W) \to Z = X$ admits a lifting $\mathrm{Spec}(W') \to X'$, where $W \to W'$ is an extension of valuation rings. We claim that after extending W', $\mathrm{Spec}(W') \to X'$ factors through Z'. Indeed, if

ξ' denotes the generic point of $\mathrm{Spec}(W')$, as $Y' \to Y \times_X X'$ is surjective, after extending W', we may assume that there exists an X'-morphism $\xi' \to Y'$ which is over $\xi \to Y$. Since $\mathrm{Spec}(W')$ is integrally closed in ξ' and Z' is integral over X', the morphism $\mathrm{Spec}(W') \to X'$ factors through Z' ([Sta23, 035I]). Finally, let $\mathfrak{q} \in \mathrm{Spec}(W')$ which lies over $\mathfrak{p} \in \mathrm{Spec}(W)$, then we get a lifting $\mathrm{Spec}(W'/\mathfrak{q}) \to Z'$ of $\mathrm{Spec}(V) \to Z$, which shows that $Z' \to Z$ is a v-covering (as we assume that $Z' \to Z$ is quasi-compact, cf. 3.1). □

Definition 3.16. Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is S°-integrally closed (2.2). For any S-scheme X, we set $X^\circ = S^\circ \times_S X$. We denote by $\mathbf{I}_{S^\circ\to S}$ the category formed by coherent S-schemes which are S°-integrally closed.

Note that any S°-integrally closed coherent S-scheme X is also X°-integrally closed by definition. It is clear that the category $(\mathbf{I}_{S^\circ\to S})_{/X}$ of objects of $\mathbf{I}_{S^\circ\to S}$ over X is canonically equivalent to the category $\mathbf{I}_{X^\circ\to X}$.

Lemma 3.17 [Sta23, 03GV]. *Let $Y \to X$ be a coherent morphism of schemes, $X' \to X$ a smooth morphism of schemes, $Y' = Y \times_X X'$. Then, we have $X'^{Y'} = X^Y \times_X X'$.*

Lemma 3.18. *Let $(Y_\lambda \to X_\lambda)_{\lambda\in\Lambda}$ be a directed inverse system of morphisms of coherent schemes with affine transition morphisms $Y_{\lambda'} \to Y_\lambda$ and $X_{\lambda'} \to X_\lambda$ ($\lambda' \geq \lambda$). We set $Y = \lim Y_\lambda$ and $X = \lim X_\lambda$. Then, $(X_\lambda^{Y_\lambda})_{\lambda\in\Lambda}$ is a directed inverse system of coherent schemes with affine transition morphisms and we have $X^Y = \lim X_\lambda^{Y_\lambda}$.*

Proof. We fix an index $\lambda_0 \in \Lambda$. After replacing X_{λ_0} by an affine open covering, we may assume that X_{λ_0} is affine (3.17). We write $X_\lambda = \mathrm{Spec}(A_\lambda)$ and $B_\lambda = \Gamma(Y_\lambda, \mathcal{O}_{Y_\lambda})$ for each $\lambda \geq \lambda_0$, and we set $A = \mathrm{colim}\, A_\lambda$ and $B = \mathrm{colim}\, B_\lambda$. Then, we have $X = \mathrm{Spec}(A)$ and $B = \Gamma(Y, \mathcal{O}_Y)$ ([Sta23, 009F]). Let R_λ (resp. R) be the integral closure of A_λ in B_λ (resp. A in B). By definition, we have $X_\lambda^{Y_\lambda} = \mathrm{Spec}(R_\lambda)$ and $X^Y = \mathrm{Spec}(R)$. The conclusion follows from the fact that $R = \mathrm{colim}\, R_\lambda$. □

Lemma 3.19. *Let $S^\circ \to S$ be an open immersion of coherent schemes.*

(1) *If X is an S°-integrally closed coherent S-scheme, then the open subscheme X° is scheme theoretically dense in X.*
(2) *If X is an S°-integrally closed coherent S-scheme and X' is a coherent smooth X-scheme, then X' is also S°-integrally closed.*
(3) *If $(X_\lambda)_{\lambda\in\Lambda}$ is a directed inverse system of S°-integrally closed coherent S-scheme with affine transition morphisms, then $X = \lim_{\lambda\in\Lambda} X_\lambda$ is also S°-integrally closed.*
(4) *If $Y \to X$ is a morphism of coherent schemes over $S^\circ \to S$ such that Y is integral over X°, then the integral closure X^Y is S°-integrally closed with $(X^Y)^\circ = Y$.*

Proof. (1), (2), (3) follow from [Sta23, 035I], 3.17 and 3.18, respectively. For (4), $(X^Y)^\circ = X^\circ \times_X X^Y$ is the integral closure of X° in $X^\circ \times_X Y = Y$ by 3.17, which is Y itself. □

3.20. We take the notation in 3.16. The inclusion functor

$$\Phi^+ : \mathbf{I}_{S^\circ\to S} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/S},\ X \longmapsto X, \tag{3.20.1}$$

admits a right adjoint

$$\sigma^+ : \mathbf{Sch}^{\mathrm{coh}}_{/S} \longrightarrow \mathbf{I}_{S^\circ\to S},\ X \longmapsto \overline{X} = X^{X^\circ}. \tag{3.20.2}$$

Indeed, σ^+ is well defined by 3.19.(4), and the adjointness follows from the functoriality of taking integral closures. We remark that $\overline{X}^\circ = X^\circ$. On the other hand, the functor

$$\Psi^+ : \mathbf{I}_{S^\circ\to S} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/S^\circ},\ X \longmapsto X^\circ, \tag{3.20.3}$$

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admits a left adjoint

$$\alpha^+ : \mathbf{Sch}^{\mathrm{coh}}_{/S^\circ} \longrightarrow \mathbf{I}_{S^\circ\to S},\ Y \longmapsto Y. \tag{3.20.4}$$

Lemma 3.21. *With the notation in 3.16, let $\varphi : I \to \mathbf{I}_{S^\circ\to S}$ be a functor sending i to X_i. If $X = \lim X_i$ represents the limit of φ in the category of coherent S-schemes, then the integral closure $\overline{X} = X^{X^\circ}$ represents the limit of φ in $\mathbf{I}_{S^\circ\to S}$ with $\overline{X}^\circ = X^\circ$.*

Proof. It follows directly from the adjoint pair (Φ^+, σ^+) (3.20). □

It follows from 3.21 that for a diagram $X_1 \to X_0 \leftarrow X_2$ in $\mathbf{I}_{S^\circ\to S}$, the fibred product is representable by

$$X_1 \overline{\times}_{X_0} X_2 = (X_1 \times_{X_0} X_2)^{X_1^\circ \times_{X_0^\circ} X_2^\circ}. \tag{3.21.1}$$

Proposition 3.22. *With the notation in 3.16, let $\mathscr{C}$ be the set of families of morphisms $\{X_i \to X\}_{i\in I}$ of $\mathbf{I}_{S^\circ\to S}$ with I finite such that $\coprod_{i\in I} X_i \to X$ is a v-covering. Then, $\mathscr{C}$ forms a pretopology of $\mathbf{I}_{S^\circ\to S}$.*

Proof. Let $\{X_i \to X\}_{i\in I}$ be an element of $\mathscr{C}$. Firstly, we check that for a morphism $X' \to X$ of $\mathbf{I}_{S^\circ\to S}$, the base change $\{X_i' \to X'\}_{i\in I}$ also lies in $\mathscr{C}$, where $Z_i = X_i \times_X X'$ and $X_i' = Z_i^{Z_i^\circ}$ by 3.21. Applying 3.15 to the following diagram

$$\begin{tikzcd}
\coprod_{i\in I} Z_i^\circ \arrow[r] \arrow[d] & \coprod_{i\in I} X_i' \arrow[r] \arrow[d] & \coprod_{i\in I} Z_i \arrow[d] \\
X'^\circ \arrow[r] & X' \arrow[r] & X',
\end{tikzcd} \tag{3.22.1}$$

we deduce that $\coprod_{i\in I} X_i' \to X'$ is also a v-covering, which shows the stability of $\mathscr{C}$ under base change.

For each $i \in I$, let $\{X_{ij} \to X_i\}_{j\in J_i}$ be an element of $\mathscr{C}$. We need to show that the composition $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ also lies in $\mathscr{C}$. This follows immediately from the stability of v-coverings under composition. We conclude that $\mathscr{C}$ forms a pretopology. □

Definition 3.23. With the notation in 3.16, we endow the category $\mathbf{I}_{S^\circ\to S}$ with the topology generated by the pretopology defined in 3.22, and we call $\mathbf{I}_{S^\circ\to S}$ the *v-site of S°-integrally closed coherent S-schemes*.

By definition, any object in $\mathbf{I}_{S^\circ\to S}$ is quasi-compact. Let $\mathscr{O}$ be the sheaf on $\mathbf{I}_{S^\circ\to S}$ associated to the presheaf $X \mapsto \Gamma(X, \mathcal{O}_X)$. We call $\mathscr{O}$ the structural sheaf of $\mathbf{I}_{S^\circ\to S}$.

Proposition 3.24. *With the notation in 3.16, let $f : X' \to X$ be a covering in $\mathbf{I}_{S^\circ\to S}$ such that f is separated and that the diagonal morphism $X'^\circ \to X'^\circ \times_{X^\circ} X'^\circ$ is surjective. Then, the morphism of representable sheaves $h^{\mathrm{a}}_{X'} \to h^{\mathrm{a}}_X$ is an isomorphism.*

Proof. We need to show that for any sheaf $\mathcal{F}$ on $\mathbf{I}_{S^\circ\to S}$, $\mathcal{F}(X) \to \mathcal{F}(X')$ is an isomorphism. Since the composition of $X'^\circ \to X'^\circ \times_{X^\circ} X'^\circ \to X' \overline{\times}_X X'$ factors through the closed immersion $X' \to X' \overline{\times}_X X'$ (as f is separated), the closed immersion $X' \to X' \overline{\times}_X X'$ is surjective (3.19.(1)). Consider the following sequence

$$\mathcal{F}(X) \to \mathcal{F}(X') \rightrightarrows \mathcal{F}(X' \overline{\times}_X X') \to \mathcal{F}(X'). \tag{3.24.1}$$

The right arrow is injective as $X' \to X' \overline{\times}_X X'$ is a v-covering. Thus, the middle two arrows are actually the same. Thus, the first arrow is an isomorphism by the sheaf property of $\mathcal{F}$. □

Proposition 3.25. *With the notation in 3.16, let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of presheaves on $\mathbf{I}_{S^\circ\to S}$. Assume that*

(i) *the morphism $\mathcal{F}_1(\mathrm{Spec}(V)) \to \mathcal{F}_2(\mathrm{Spec}(V))$ is an isomorphism for any S°-integrally closed S-scheme* $\mathrm{Spec}(V)$, *where V is an absolutely integrally closed valuation ring and that*

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(ii) *for any directed inverse system of S°-integrally closed affine schemes* $(\mathrm{Spec}(A_\lambda))_{\lambda\in\Lambda}$ *over S the natural morphism* $\operatorname{colim}\mathcal{F}_i(\mathrm{Spec}(A_\lambda)) \to \mathcal{F}_i(\mathrm{Spec}(\operatorname{colim}A_\lambda))$ *is an isomorphism for $i = 1, 2$ (cf. 3.19.(3)).*

Then, the morphism of the associated sheaves $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is an isomorphism.

Assume moreover that

(iii) *$\mathcal{F}_i$ sends finite coproducts to finite products for $i = 1, 2$.*

Then, for any product A of absolutely integrally closed valuation rings such that $\mathrm{Spec}(A)$ *is an S°-integrally closed S-scheme, the map* $\mathcal{F}_1(\mathrm{Spec}(A)) \to \mathcal{F}_2(\mathrm{Spec}(A))$ *is bijective.*

Proof. Let A be a product of absolutely integrally closed valuation rings such that $X = \mathrm{Spec}(A)$ is an S°-integrally closed S-scheme. Let $\mathrm{Spec}(V)$ be a connected component of $\mathrm{Spec}(A)$ with the reduced closed subscheme structure. Then, V is an absolutely integrally closed valuation ring by 3.11.(2), and $\mathrm{Spec}(V)$ is also S°-integrally closed since it has nonempty intersection with the dense open subset X° of X. Notice that each connected component of an affine scheme is the intersection of some open and closed subsets ([Sta23, 04PP]). Moreover, since A is reduced, we have $V = \operatorname{colim} A'$, where the colimit is taken over all the open and closed subschemes $X' = \mathrm{Spec}(A')$ of X which contain $\mathrm{Spec}(V)$. By assumptions (i) and (ii), we have an isomorphism

$$\operatorname{colim}\mathcal{F}_1(X') \xrightarrow{\sim} \operatorname{colim}\mathcal{F}_2(X'). \tag{3.25.1}$$

For two elements $\xi_1, \xi_1' \in \mathcal{F}_1(X)$ with $\alpha_X(\xi_1) = \alpha_X(\xi_1')$ in $\mathcal{F}_2(X)$, by Equation (3.25.1) and a limit argument, there exists a finite disjoin union $X = \coprod_{i=1}^r X_i'$ such that the images of ξ_1 and ξ_1' in each $\mathcal{F}_1(X_i')$ are the same. Therefore, $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is injective by 3.14. Moreover, under the assumption (iii), we have $\xi_1 = \xi_1'$ in $\mathcal{F}_1(X) = \prod_{i=1}^r \mathcal{F}_1(X_i')$ (so that $\mathcal{F}_1(X) \to \mathcal{F}_2(X)$ is injective).

On the other hand, for an element $\xi_2 \in \mathcal{F}_2(X)$, by Equation (3.25.1) and a limit argument, there exists a finite disjoin union $X = \coprod_{i=1}^r X_i'$ such that there exists an element $\xi_{1,i} \in \mathcal{F}_1(X_i')$ for each i such that the image of ξ_2 in $\mathcal{F}_2(X_i')$ is equal to $\alpha_{X_i'}(\xi_{1,i})$. Therefore, $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is surjective by 3.14. Moreover, under the assumption (iii), let ξ_1 be the section $(\xi_{1,i})_{1\le i\le r} \in \mathcal{F}_1(X) = \prod_{i=1}^r \mathcal{F}_1(X_i')$. Then, $\alpha_X(\xi_1) = \xi_2 \in \mathcal{F}_2(X) = \prod_{i=1}^r \mathcal{F}_2(X_i')$ (so that $\mathcal{F}_1(X) \to \mathcal{F}_2(X)$ is surjective). □

3.26. We take the notation in 3.16. Endowing $\mathbf{Sch}^{\mathrm{coh}}$ with the v-topology (3.3), we see that the functors σ^+ and Ψ^+ defined in 3.20 are left exact (as they have left adjoints) and continuous by 3.15 and 3.22. Therefore, they define morphisms of sites (2.5)

$$(\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \xrightarrow{\Psi} \mathbf{I}_{S^\circ\to S} \xrightarrow{\sigma} (\mathbf{Sch}^{\mathrm{coh}}_{/S})_{\mathrm{v}}. \tag{3.26.1}$$

Proposition 3.27. *With the notation in 3.26, let $a : (\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \to S^\circ_{\text{ét}}$ be the morphism of site defined by the inclusion functor (3.9).*

(1) *For any torsion abelian sheaf $\mathcal{F}$ on $S^\circ_{\text{ét}}$, the canonical morphism $\Psi_*(a^{-1}\mathcal{F}) \to \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$ is an isomorphism.*

(2) *For any locally constant torsion abelian sheaf $\mathbb{L}$ on $\mathbf{I}_{S^\circ\to S}$, the canonical morphism $\mathbb{L} \to \mathrm{R}\Psi_*\Psi^{-1}\mathbb{L}$ is an isomorphism.*

Proof. (1) For each integer q, the sheaf $\mathrm{R}^q\Psi_*(a^{-1}\mathcal{F})$ is the sheaf associated to the presheaf $X \mapsto H^q_{\mathrm{v}}(X^\circ, a^{-1}\mathcal{F}) = H^q_{\text{ét}}(X^\circ, f_{\text{ét}}^{-1}\mathcal{F})$ by 3.9, where $f_{\text{ét}} : X^\circ_{\text{ét}} \to S^\circ_{\text{ét}}$ is the natural morphism. If X is the spectrum of a nonzero absolutely integrally closed valuation ring V, then $X^\circ = \mathrm{Spec}(V[1/\pi])$ for a nonzero element $\pi \in V$ by 3.11.(1) and 3.19.(1), which is also the spectrum of an absolutely integrally closed valuation ring (2.1). In this case, $H^q_{\text{ét}}(X^\circ, f_{\text{ét}}^{-1}\mathcal{F}) = 0$ for $q > 0$, which proves (1) by 3.25 and [SGA 4_{II}, VII.5.8].

(2) The problem is local on $\mathbf{I}_{S^\circ\to S}$. We may assume that $\mathbb{L}$ is the constant sheaf with value L. Then, $\mathrm{R}^q\Psi_*\Psi^{-1}\mathbb{L} = 0$ for $q > 0$ by applying (1) on the constant sheaf with value L on $S^\circ_{\text{ét}}$. For $q = 0$,

notice that $\mathbb{L}$ is also the sheaf associated to the presheaf $X \mapsto H^0_{\text{ét}}(X, L)$, while $\Psi_*\Psi^{-1}\mathbb{L}$ is the sheaf $X \mapsto H^0_{\text{ét}}(X^\circ, L)$ by the discussion in (1). If X is the spectrum of a nonzero absolutely integrally closed valuation ring, then so is X° and so that $H^0_{\text{ét}}(X, L) = H^0_{\text{ét}}(X^\circ, L) = L$. The conclusion follows from 3.25 and [SGA 4_{II}, VII.5.8]. $\square$

4. The arc-descent of perfect algebras

Definition 4.1. For any $\mathbb{F}_p$-algebra R, we denote by R_{perf} the filtered colimit

$$R_{\text{perf}} = \operatorname*{colim}_{\text{Frob}} R \tag{4.1.1}$$

indexed by $(\mathbb{N}, \leq)$, where the transition map associated to $i \leq (i+1)$ is the Frobenius of R.

It is clear that the endo-functor of the category of $\mathbb{F}_p$-algebras, $R \mapsto R_{\text{perf}}$, commutes with colimits.

4.2. We define a presheaf $\mathcal{O}_{\text{perf}}$ on the category $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ of coherent $\mathbb{U}$-small $\mathbb{F}_p$-schemes X by

$$\mathcal{O}_{\text{perf}}(X) = \Gamma(X, \mathcal{O}_X)_{\text{perf}}. \tag{4.2.1}$$

For any morphism $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ of affine $\mathbb{F}_p$-schemes, we consider the augmented Čech complex of the presheaf $\mathcal{O}_{\text{perf}}$,

$$0 \longrightarrow A_{\text{perf}} \longrightarrow B_{\text{perf}} \longrightarrow B_{\text{perf}} \otimes_{A_{\text{perf}}} B_{\text{perf}} \longrightarrow \cdots. \tag{4.2.2}$$

Lemma 4.3 [Sta23, 0EWT]. *The presheaf $\mathcal{O}_{\text{perf}}$ is a sheaf on $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ with respect to the fppf topology ([Sta23, 021L]). Moreover, for any coherent $\mathbb{F}_p$-scheme X and any integer q,*

$$H^q_{\text{fppf}}(X, \mathcal{O}_{\text{perf}}) = \operatorname*{colim}_{\text{Frob}} H^q(X, \mathcal{O}_X). \tag{4.3.1}$$

Proof. Firstly, we remark that for any integer q, the functor $H^q_{\text{fppf}}(X, -)$ commutes with filtered colimit of abelian sheaves on $(\mathbf{Sch}^{\text{coh}}_{/X})_{\text{fppf}}$ for any coherent scheme X ([Sta23, 0739]). Since the presheaf $\mathcal{O}$ sending X to $\Gamma(X, \mathcal{O}_X)$ on $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ is an fppf-sheaf, we have $H^0_{\text{fppf}}(X, \mathcal{O}^{\text{a}}_{\text{perf}}) = \operatorname{colim}_{\text{Frob}} H^0_{\text{fppf}}(X, \mathcal{O}) = \mathcal{O}_{\text{perf}}(X)$. Thus, $\mathcal{O}_{\text{perf}}$ is an fppf-sheaf. Moreover, $H^q_{\text{fppf}}(X, \mathcal{O}_{\text{perf}}) = \operatorname{colim}_{\text{Frob}} H^q_{\text{fppf}}(X, \mathcal{O}) = \operatorname{colim}_{\text{Frob}} H^q(X, \mathcal{O}_X)$ by faithfully flat descent ([Sta23, 03DW]). $\square$

Lemma 4.4. *Let $\tau \in \{$fppf, h, v, arc$\}$. The following propositions are equivalent:*

(1) *The presheaf $\mathcal{O}_{\text{perf}}$ on $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ is a τ-sheaf and $H^q_\tau(X, \mathcal{O}_{\text{perf}}) = \operatorname{colim}_{\text{Frob}} H^q(X, \mathcal{O}_X)$ for any coherent $\mathbb{F}_p$-scheme X and any integer q.*
(2) *For any τ-covering $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ of affine $\mathbb{F}_p$-schemes, the augmented Čech complex (4.2.2) is exact.*

Proof. For an affine scheme $X = \operatorname{Spec}(A)$, $H^q(X, \mathcal{O}_X)$ vanishes for $q > 0$ and $H^0(X, \mathcal{O}_X) = A$. For (1) $\Rightarrow$ (2), the exactness of Equation (4.2.2) follows from the Čech-cohomology-to-cohomology spectral sequence associated to the τ-covering $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ [Sta23, 03AZ]. Therefore, (1) and (2) hold for τ = fppf by 4.3. Conversely, the exactness of Equation (4.2.2) shows the sheaf property for any τ-covering of an affine scheme by affine schemes, which implies the fppf-sheaf $\mathcal{O}_{\text{perf}}$ is a τ-sheaf (cf. [Sta23, 0ETM]). The vanishing of higher Čech cohomologies implies that $H^q_\tau(X, \mathcal{O}_{\text{perf}}) = 0$ for any affine $\mathbb{F}_p$-scheme X and any $q > 0$ ([Sta23, 03F9]). Therefore, for a coherent $\mathbb{F}_p$-scheme X, $H^q_\tau(X, \mathcal{O}_{\text{perf}})$ can be computed by the hyper-Čech cohomology of a hypercovering of X formed by affine open subschemes ([Sta23, 01GY]). In particular, we have $H^q_\tau(X, \mathcal{O}_{\text{perf}}) = H^q_{\text{fppf}}(X, \mathcal{O}_{\text{perf}})$ for any integer q, which completes the proof by 4.3. $\square$

Lemma 4.5 (Gabber). *The augmented Čech complex (4.2.2) is exact for any h-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes.*

Proof. This is a result of Gabber; see [BST17, 3.3] or [Sta23, 0EWU], and 4.4. □

Lemma 4.6 [BS17, 4.1]. *The augmented Čech complex (4.2.2) is exact for any v-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes.*

Proof. We write B as a filtered colimit of finitely presented A-algebras $B = \mathrm{colim}\, B_\lambda$. Then, $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is an h-covering for each λ by 3.2. Notice that $B_{\mathrm{perf}} = \mathrm{colim}\, B_{\lambda,\mathrm{perf}}$, then the conclusion follows from applying 4.5 on $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ and taking colimit. □

Lemma 4.7 [BS17, 6.3]. *For any valuation ring V and any prime ideal $\mathfrak{p}$ of V, the sequence*

$$0 \longrightarrow V \xrightarrow{\alpha} V/\mathfrak{p} \oplus V_{\mathfrak{p}} \xrightarrow{\beta} V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \longrightarrow 0 \tag{4.7.1}$$

is exact, where $\alpha(a) = (a, a)$ and $\beta(a, b) = a - b$. If moreover V is a perfect $\mathbb{F}_p$-algebra, then for any perfect V-algebra R, the base change of Equation (4.7.1) by $V \to R$,

$$0 \longrightarrow R \longrightarrow R/\mathfrak{p}R \oplus R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \longrightarrow 0 \tag{4.7.2}$$

is exact.

Proof. The sequence (4.7.1) is exact if and only if $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$. Let $a \in \mathfrak{p}$ and $s \in V \setminus \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, $s/a \notin V$, thus $a/s \in V$ as V is a valuation ring. Moreover, we must have $a/s \in \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal. This shows the equality $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$.

The second assertion follows directly from the fact that $\mathrm{Tor}_q^A(B, C) = 0$ for any $q > 0$ and any diagram $B \leftarrow A \to C$ of perfect $\mathbb{F}_p$-algebras ([BS17, 3.16]). □

Lemma 4.8 [BM21, 4.8]. *The augmented Čech complex (4.2.2) is exact for any arc-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes with A a valuation ring.*

Proof. We follow the proof of Bhatt–Mathew [BM21, 4.8]. Let $B = \mathrm{colim}\, B_\lambda$ be a filtered colimit of finitely presented A-algebras. Then, $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is also an arc-covering by 3.2. Thus, we may assume that B is a finitely presented A-algebra.

An interval $I = [\mathfrak{p}, \mathfrak{q}]$ of a valuation ring A is a pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of A. We denote by $A_I = (A/\mathfrak{p})_{\mathfrak{q}}$. The set $\mathcal{I}$ of intervals of A is partially ordered under inclusion. Let $\mathcal{P}$ be the subset consisting of intervals I such that the lemma holds for $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$. It suffices to show that $\mathcal{P} = \mathcal{I}$.

(1) If the valuation ring A_I is of height ≤ 1, we claim that $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$ is automatically a v-covering. Indeed, there is an extension of valuation rings $A_I \to V$ of height ≤ 1 which factors through $B \otimes_A A_I$. As $A_I \to V$ is faithfully flat, $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$ is a v-covering by 3.2 and 3.4.(1). Therefore, $I \in \mathcal{P}$ by 4.6.
(2) For any interval $J \subseteq I$ if $I \in \mathcal{P}$, then $J \in \mathcal{P}$. Indeed, applying $\otimes_{\mathbb{F}_p} (A_J)_{\mathrm{perf}}$ to the exact sequence (4.2.2) for $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$, we still get an exact sequence by the Tor-independence of perfect $\mathbb{F}_p$-algebras ([BS17, 3.16]).
(3) If $\mathfrak{p} \subseteq A$ is not maximal, then there exists $\mathfrak{q} \supsetneq \mathfrak{p}$ with $I = [\mathfrak{p}, \mathfrak{q}] \in \mathcal{P}$. Indeed, if there is no such I with the height of A_I no more than 1, then $\mathfrak{p} = \bigcap_{\mathfrak{q} \supsetneq \mathfrak{p}} \mathfrak{q}$, and thus,

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \operatorname*{colim}_{I = [\mathfrak{p}, \mathfrak{q}], \mathfrak{q} \supsetneq \mathfrak{p}} A_I. \tag{4.8.1}$$

Since $\mathrm{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \to \mathrm{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ is an h-covering as $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a field (and we have assumed that B is of finite presentation over A), there exists an interval I in the above colimit such that $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$ is also an h-covering by 3.4.(3). Therefore, this I lies in $\mathcal{P}$ by 4.6.

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(4) If $\mathfrak{p} \subseteq A$ is nonzero, then there exists $\mathfrak{q} \subsetneq \mathfrak{p}$ with $I = [\mathfrak{q}, \mathfrak{p}] \in \mathcal{P}$. This is similar to (3).
(5) If $I, J \in \mathcal{P}$ are overlapping, then $I \cup J \in \mathcal{P}$. Indeed, by (2) and replacing A by $A_{I \cup J}$, we may assume that $I = [0, \mathfrak{p}]$, $J = [\mathfrak{p}, \mathfrak{m}]$ with $\mathfrak{m}$ the maximal ideal. In particular, $A_I = A_{\mathfrak{p}}$, $A_J = A/\mathfrak{p}$ and $A_{I \cap J} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Since for each $R = \otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$ we have the short exact sequence (4.7.2), we get $I \cup J \in \mathcal{P}$.

In general, by Zorn's lemma, the above five properties of $\mathcal{P}$ guarantee that $\mathcal{P} = \mathcal{I}$ (see [BM21, 4.7]). □

Lemma 4.9 (cf. [BM21, 3.30]). *The augmented Čech complex (4.2.2) is exact for any arc-covering* $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes with* A *a product of valuation rings.*

Proof. We follow closely the proof of 3.25. Let $\mathrm{Spec}(V)$ be a connected component of $\mathrm{Spec}(A)$ with the reduced closed subscheme structure. Then, V is a valuation ring by 3.11.(2). By 4.8, the augmented Čech complex

$$0 \longrightarrow V_{\mathrm{perf}} \longrightarrow (B \otimes_A V)_{\mathrm{perf}} \longrightarrow (B \otimes_A V)_{\mathrm{perf}} \otimes_{V_{\mathrm{perf}}} (B \otimes_A V)_{\mathrm{perf}} \longrightarrow \cdots \tag{4.9.1}$$

is exact. Notice that each connected component of an affine scheme is the intersection of some open and closed subsets ([Sta23, 04PP]). Moreover, since A is reduced, we have $V = \mathrm{colim}\, A'$, where the colimit is taken over all the open and closed subschemes $\mathrm{Spec}(A')$ which contain $\mathrm{Spec}(V)$.

Therefore, by a limit argument, for an element $f \in \otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$ which maps to zero in $\otimes^{n+1}_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$, as $\mathrm{Spec}(A)$ is quasi-compact, we can decompose $\mathrm{Spec}(A)$ into a finite disjoint union $\coprod_{i=1}^N \mathrm{Spec}(A_i)$ such that there exists $g_i \in \otimes^{n-1}_{A_{i,\mathrm{perf}}} (B \otimes_A A_i)_{\mathrm{perf}}$ which maps to the image f_i of f in $\otimes^n_{A_{i,\mathrm{perf}}} (B \otimes_A A_i)_{\mathrm{perf}}$. Since we have

$$\otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}} = \prod_{i=1}^N \otimes^n_{A_{i,\mathrm{perf}}} (B \otimes_A A_i)_{\mathrm{perf}}, \tag{4.9.2}$$

the element $g = (g_i)_{i=1}^N$ maps to f, which shows the exactness of Equation (4.2.2). □

Proposition 4.10 [BS22, 8.10]. *Let* $\tau \in \{$*fppf, h, v, arc*$\}$.

(1) *The presheaf* $\mathcal{O}_{\mathrm{perf}}$ *is a* τ*-sheaf over* $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$*, and for any coherent* $\mathbb{F}_p$*-scheme* X *and any integer* q*,*

$$H^q_\tau(X, \mathcal{O}_{\mathrm{perf}}) = \operatorname*{colim}_{\mathrm{Frob}} H^q(X, \mathcal{O}_X). \tag{4.10.1}$$

(2) *For any* τ*-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes, the augmented Čech complex*

$$0 \longrightarrow A_{\mathrm{perf}} \longrightarrow B_{\mathrm{perf}} \longrightarrow B_{\mathrm{perf}} \otimes_{A_{\mathrm{perf}}} B_{\mathrm{perf}} \longrightarrow \cdots \tag{4.10.2}$$

is exact.

Proof. We follow closely the proof of Bhatt–Scholze [BS22, 8.10]. (1) and (2) are equivalent by 4.4, and they hold for $\tau \in \{\mathrm{fppf, h, v}\}$ by 4.3, 4.5 and 4.6. In particular,

$$H^0_{\mathrm{v}}(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{perf}}) = A_{\mathrm{perf}} \text{ and } H^q_{\mathrm{v}}(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{perf}}) = 0,\ \forall q > 0. \tag{4.10.3}$$

We take a hypercovering in the v-topology $\mathrm{Spec}(A_\bullet) \to \mathrm{Spec}(A)$ such that A_n is a product of valuation rings for each degree n by 3.14 and [Sta23, 094K and 0DB1]. The associated sequence

$$0 \longrightarrow A_{\mathrm{perf}} \longrightarrow A_{0,\mathrm{perf}} \longrightarrow A_{1,\mathrm{perf}} \longrightarrow \cdots \tag{4.10.4}$$

is exact by the hyper-Čech-cohomology-to-cohomology spectral sequence [Sta23, 01GY].

Consider the double complex (A_i^j) where the i-th row $A_i^\bullet$ is the base change of Equation (4.10.2) by $A_{\mathrm{perf}} \to A_{i,\mathrm{perf}}$, that is, the augmented Čech complex (4.2.2) associated to $\mathrm{Spec}(B \otimes_A A_i) \to \mathrm{Spec}(A_i)$

(we set $A_{-1} = A$). On the other hand, the j-th column $A^j_\bullet$ is the associated sequence (4.10.4) to the hypercovering $\mathrm{Spec}(A_\bullet \otimes_A (\otimes^j_A B)) \to \mathrm{Spec}(\otimes^j_A B)$, which is exact by the previous discussion. Since $A^\bullet_{-1} \to \mathrm{Tot}(A^j_i)^{j\geq 0}_{i\geq 0}$ is a quasi-isomorphism ([Sta23, 0133]), for the exactness of the (-1)-row $A^\bullet_{-1}$, we only need to show the exactness of the i-th row $A^\bullet_i$ for any $i \geq 0$ but this has been proved in 4.9 thanks to our choice of the hypercovering, which completes the proof. $\square$

5. Almost pre-perfectoid algebras

Definition 5.1.

(1) A *pre-perfectoid field* K is a valuation field whose valuation ring $\mathcal{O}_K$ is nondiscrete, of height 1 and of residue characteristic p, and such that the Frobenius map on $\mathcal{O}_K/p\mathcal{O}_K$ is surjective.
(2) A *perfectoid field* K is a pre-perfectoid field which is complete for the topology defined by its valuation (cf. [Sch12, 3.1]).
(3) A *pseudo-uniformizer* π of a pre-perfectoid field K is a nonzero element of the maximal ideal $\mathfrak{m}_K$ of $\mathcal{O}_K$.

A *morphism of pre-perfectoid fields* $K \to L$ is a homomorphism of fields which induces an extension of valuation rings $\mathcal{O}_K \to \mathcal{O}_L$.

Lemma 5.2. *Let K be a pre-perfectoid field with a pseudo-uniformizer π. Then, the fraction field $\widehat{K}$ of the π-adic completion of $\mathcal{O}_K$ is a perfectoid field.*

Proof. The π-adic completion $\widehat{\mathcal{O}_K}$ of $\mathcal{O}_K$ is still a nondiscrete valuation ring of height 1 with residue characteristic p (see [Bou06, VI.§5.3, Prop.5]). If $p \neq 0$ in $\mathcal{O}_K$, then it is canonically isomorphic to the p-adic completion of $\mathcal{O}_K$ so that there is a canonical isomorphism $\mathcal{O}_K/p\mathcal{O}_K \xrightarrow{\sim} \widehat{\mathcal{O}_K}/p\widehat{\mathcal{O}_K}$, from which we see that $\widehat{K}$ is a perfectoid field. If $p = 0$ in $\mathcal{O}_K$, then the Frobenius induces a surjection $\mathcal{O}_K \to \mathcal{O}_K$ if and only if $\mathcal{O}_K$ is perfect. Thus, $\widehat{\mathcal{O}_K}$ is also perfect, and we see that $\widehat{K}$ is a perfectoid field. $\square$

5.3. Let K be a pre-perfectoid field. There is a unique (up to scalar) ordered group homomorphism $v_K : K^\times \to \mathbb{R}$ such that $v_K^{-1}(0) = \mathcal{O}_K^\times$, where the group structure on $\mathbb{R}$ is given by the addition. In particular, $\mathcal{O}_K \setminus 0 = v_K^{-1}(\mathbb{R}_{\geq 0})$ and $\mathfrak{m}_K \setminus 0 = v_K^{-1}(\mathbb{R}_{>0})$ (see [Bou06, VI.§4.5 Prop.7] and [Bou07, V.§2 Prop.1, Rem.2]). The nondiscrete assumption on $\mathcal{O}_K$ implies that the image $v_K(K^\times) \subseteq \mathbb{R}$ is dense. We set $v_K(0) = +\infty$.

Lemma 5.4 [Sch12, 3.2]. *Let K be a pre-perfectoid field. Then, for any pseudo-uniformizer π of K, there exists $\pi_n \in \mathfrak{m}_K$ for each integer $n \geq 0$ such that $\pi_0 = \pi$ and $\pi_n = u_n \cdot \pi_{n+1}^p$ for some unit $u_n \in \mathcal{O}_K^\times$, and $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$. We call π_{n+1} a p-th root of π_n up to a unit for simplicity.*

Proof. If $v_K(\pi) < v_K(p)$, since the Frobenius is surjective on $\mathcal{O}_K/p$, there exists $\pi_1 \in \mathcal{O}_K$ such that $v_K(\pi - \pi_1^p) \geq v_K(p)$. Then, $v_K(\pi) = v_K(\pi_1^p)$ and thus $\pi = u \cdot \pi_1^p$ with $u \in \mathcal{O}_K^\times$. In general, since $v_K(K^\times) \subseteq \mathbb{R}$ is dense, any pseudo-uniformizer π is a finite product of pseudo-uniformizers whose valuation values are strictly less than $v_K(p)$, from which we get a p-th root π_1 of π up to a unit. Since π_1 is also a pseudo-uniformizer, we get π_n inductively. As $v_K(\pi_n)$ tends to zero when n tends to infinity, $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$. $\square$

5.5. Let K be a pre-perfectoid field. We briefly review almost algebra over $(\mathcal{O}_K, \mathfrak{m}_K)$ for which we mainly refer to [AG20, 2.6], [AGT16, V] and [GR03]. Remark that $\mathfrak{m}_K \otimes_{\mathcal{O}_K} \mathfrak{m}_K \cong \mathfrak{m}_K^2 = \mathfrak{m}_K$ is flat over $\mathcal{O}_K$.

An $\mathcal{O}_K$-module M is called *almost zero* if $\mathfrak{m}_K M = 0$. A morphism of $\mathcal{O}_K$-modules $M \to N$ is called an *almost isomorphism* if its kernel and cokernel are almost zero. Let $\mathscr{N}$ be the full subcategory of the category $\mathcal{O}_K$-**Mod** of $\mathcal{O}_K$-modules formed by almost zero objects. It is clear that $\mathscr{N}$ is a Serre subcategory of $\mathcal{O}_K$-**Mod** ([Sta23, 02MO]). Let $\mathcal{S}$ be the set of almost isomorphisms in $\mathcal{O}_K$-**Mod**. Since $\mathscr{N}$ is a Serre subcategory, $\mathcal{S}$ is a multiplicative system, and moreover the quotient abelian category $\mathcal{O}_K$-**Mod**$/\mathscr{N}$ is representable by the localized category $\mathcal{S}^{-1}\mathcal{O}_K$-**Mod** (cf. [Sta23, 02MS]). We denote

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$\mathcal{S}^{-1}\mathcal{O}_K$-**Mod** by $\mathcal{O}_K^{\mathrm{al}}$-**Mod**, whose objects are called *almost $\mathcal{O}_K$-modules* or simply *$\mathcal{O}_K^{\mathrm{al}}$-modules* (cf. [AG20, 2.6.2]). We denote by

$$\alpha^* : \mathcal{O}_K\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod},\ M \longmapsto M^{\mathrm{al}} \tag{5.5.1}$$

the localization functor. It induces an $\mathcal{O}_K$-linear structure on $\mathcal{O}_K^{\mathrm{al}}$-**Mod**. For any two $\mathcal{O}_K$-modules M and N, we have a natural $\mathcal{O}_K$-linear isomorphism ([AG20, 2.6.7.1])

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(M^{\mathrm{al}}, N^{\mathrm{al}}) = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K \otimes_{\mathcal{O}_K} M, N). \tag{5.5.2}$$

The localization functor α^* admits a right adjoint

$$\alpha_* : \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K\text{-}\mathbf{Mod},\ M \longmapsto M_* = \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(\mathcal{O}_K^{\mathrm{al}}, M), \tag{5.5.3}$$

and a left adjoint

$$\alpha_! : \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K\text{-}\mathbf{Mod},\ M \longmapsto M_! = \mathfrak{m}_K \otimes_{\mathcal{O}_K} M_*. \tag{5.5.4}$$

Moreover, the natural morphisms

$$(M_*)^{\mathrm{al}} \xrightarrow{\sim} M,\ M \xrightarrow{\sim} (M_!)^{\mathrm{al}} \tag{5.5.5}$$

are isomorphisms for any $\mathcal{O}_K^{\mathrm{al}}$-module M (cf. [AG20, 2.6.8]). In particular, for any functor $\varphi : I \to \mathcal{O}_K^{\mathrm{al}}$-**Mod** sending i to M_i, the colimit and limit of φ are representable by

$$\mathrm{colim}\, M_i = (\mathrm{colim}\, M_{i*})^{\mathrm{al}},\ \lim M_i = (\lim M_{i*})^{\mathrm{al}}. \tag{5.5.6}$$

The tensor product in $\mathcal{O}_K$-**Mod** induces a tensor product in $\mathcal{O}_K^{\mathrm{al}}$-**Mod** by

$$M^{\mathrm{al}} \otimes_{\mathcal{O}_K^{\mathrm{al}}} N^{\mathrm{al}} = (M \otimes_{\mathcal{O}_K} N)^{\mathrm{al}} \tag{5.5.7}$$

making $\mathcal{O}_K^{\mathrm{al}}$-**Mod** an abelian tensor category ([AG20, 2.6.4]). We denote by $\mathcal{O}_K^{\mathrm{al}}$-**Alg** the category of commutative unitary monoids in $\mathcal{O}_K^{\mathrm{al}}$-**Mod** induced by the tensor structure, whose objects are called *almost $\mathcal{O}_K$-algebras* or simply *$\mathcal{O}_K^{\mathrm{al}}$-algebras* (cf. [AG20, 2.6.11]). Notice that R^{al} (resp. R_*) admits a canonical algebra structure for any $\mathcal{O}_K$-algebra (resp. $\mathcal{O}_K^{\mathrm{al}}$-algebra) R. Moreover, α^* and α_* induce adjoint functors between $\mathcal{O}_K$-**Alg** and $\mathcal{O}_K^{\mathrm{al}}$-**Alg** ([AG20, 2.6.12]). Combining with Equations (5.5.5) and (5.5.6), we see that for any functor $\varphi : I \to \mathcal{O}_K^{\mathrm{al}}$-**Alg** sending i to R_i, the colimit and limit of φ are representable by (cf. [GR03, 2.2.16])

$$\mathrm{colim}\, R_i = (\mathrm{colim}\, R_{i*})^{\mathrm{al}},\ \lim R_i = (\lim R_{i*})^{\mathrm{al}}. \tag{5.5.8}$$

In particular, for any diagram $B \leftarrow A \to C$ of $\mathcal{O}_K^{\mathrm{al}}$-algebras, we denote its colimit by

$$B \otimes_A C = (B_* \otimes_{A_*} C_*)^{\mathrm{al}}, \tag{5.5.9}$$

which is clearly compatible with the tensor products of modules. We remark that α^* commutes with arbitrary colimits (resp. limits), since it has a right adjoint α_* (resp. since the forgetful functor $\mathcal{O}_K^{\mathrm{al}}$-**Alg** $\to$ $\mathcal{O}_K^{\mathrm{al}}$-**Mod** and the localization functor $\alpha^* : \mathcal{O}_K$-**Mod** $\to \mathcal{O}_K^{\mathrm{al}}$-**Mod** commute with arbitrary limits).

5.6. For an element a of $\mathcal{O}_K$, we denote by $(\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-**Mod** the full subcategory of $\mathcal{O}_K^{\mathrm{al}}$-**Mod** formed by the objects on which the morphism induced by multiplication by a is zero. Notice that for an $(\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-module M, M_* is an $\mathcal{O}_K/a\mathcal{O}_K$-module. Thus, the localization functor α^* induces an

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essentially surjective exact functor $(\mathcal{O}_K/a\mathcal{O}_K)$-**Mod** $\to (\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-**Mod**, which identifies the latter with the quotient abelian category $(\mathcal{O}_K/a\mathcal{O}_K)$-**Mod**$/\mathscr{N} \cap (\mathcal{O}_K/a\mathcal{O}_K)$-**Mod**.

Let π be a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4). The Frobenius on $\mathcal{O}_K/\pi\mathcal{O}_K$ induces an isomorphism $\mathcal{O}_K/\pi_1\mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K$. The Frobenius on $(\mathcal{O}_K/\pi)$-algebras and the localization functor α^* induce a natural transformation from the base change functor $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** $\to (\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg**, $R \mapsto (\mathcal{O}_K/\pi) \otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R$ to the identity functor.

$$\begin{tikzcd}(\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Alg} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"', ""{name=I}] & (\mathcal{O}_K/\pi_1)^{\mathrm{al}}\text{-}\mathbf{Alg} \arrow[r, "\sim"] \arrow[d, phantom, "\Downarrow\mathrm{Frob}"] & (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Alg}\end{tikzcd} \tag{5.6.1}$$

For an $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra R, we usually identify the $(\mathcal{O}_K/\pi_1)^{\mathrm{al}}$-algebra R/π_1R with the $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra $(\mathcal{O}_K/\pi) \otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R$, and we denote by $R/\pi_1R \to R$ the natural morphism $(\mathcal{O}_K/\pi) \otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R \to R$ induced by the Frobenius (cf. [GR03, 3.5.6]). Moreover, the natural transformations induced by Frobenius for $(\mathcal{O}_K/\pi)$-**Alg** and $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** are also compatible with the functor α_*. Indeed, it follows from the fact that for any $(\mathcal{O}_K/\pi)$-algebra R, the composition of

$$(\mathcal{O}_K/\pi) \otimes_{(\mathcal{O}_K/\pi)} \mathrm{Hom}(\mathfrak{m}_K, R) \longrightarrow \mathrm{Hom}(\mathfrak{m}_K, (\mathcal{O}_K/\pi) \otimes_{(\mathcal{O}_K/\pi)} R) \xrightarrow{\mathrm{Hom}(\mathfrak{m}_K,\mathrm{Frob})} \mathrm{Hom}(\mathfrak{m}_K, R) \tag{5.6.2}$$

is the relative Frobenius on $(R^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, R)$.

5.7. Let C be a site. A presheaf $\mathcal{F}$ of $\mathcal{O}_K$-modules on C is called *almost zero* if $\mathcal{F}(U)$ is almost zero for any object U of C. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_K$-modules on C is called an *almost isomorphism* if $\mathcal{F}(U) \to \mathcal{G}(U)$ is an almost isomorphism for any object U of C (cf. [AG20, 2.6.23]). Let $\mathscr{N}$ be the full subcategory of the category $\mathcal{O}_K$-$\mathbf{Mod}_C$ of sheaves of $\mathcal{O}_K$-modules on C formed by almost zero objects. Similarly, $\mathscr{N}$ is a Serre subcategory of $\mathcal{O}_K$-$\mathbf{Mod}_C$. Let $\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ be the full subcategory of the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ formed by complexes with almost zero cohomologies. It is a strictly full saturated triangulated subcategory ([Sta23, 06UQ]). We also say that the objects of $\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ are *almost zero*. Let $\mathcal{S}$ be the set of arrows in $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ which induce almost isomorphisms on cohomologies. We also call the elements of $\mathcal{S}$ *almost isomorphisms*. Then, $\mathcal{S}$ is a saturated multiplicative system ([Sta23, 05RG]), and moreover the quotient triangulated category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)/\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ is representable by the localized triangulated category $\mathcal{S}^{-1}\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ ([Sta23, 05RI]). The natural functor

$$\mathcal{S}^{-1}\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C) \longrightarrow \mathbf{D}(\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}_C) \tag{5.7.1}$$

is an equivalence by [Sta23, 06XM] and Equation (5.5.5) (cf. [GR03, 2.4.9]).

Lemma 5.8. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, M a flat $\mathcal{O}_K$-module. We fix a system of p^n-th roots $(\pi_n)_{n\geq 0}$ of π up to units (5.4), then the map*

$$\bigcap_{n\geq 0} \pi_n^{-1}M \to (M^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, M),\ a \mapsto (x \mapsto xa), \tag{5.8.1}$$

where $\pi_n^{-1}M \subseteq M[1/\pi]$, is an isomorphism of $\mathcal{O}_K$-modules. Moreover, for an extension of valuation rings $\mathcal{O}_K \to R$ of height 1, we have $R = \bigcap_{n\geq 0} \pi_n^{-1}R$ and the above isomorphism coincides with the unit map $R \to (R^{\mathrm{al}})_*$.

Proof. Since $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$, any $\mathcal{O}_K$-linear morphism $f : \mathfrak{m}_K \to M$ is determined by its values $f(\pi_n) \in M$. Notice that $(\pi/\pi_n) \cdot f(\pi_n) = f(\pi)$ and M is π-torsion free so that f must be given by the multiplication by an element $a = f(\pi)/\pi \in M[1/\pi]$. It is clear that such a multiplication sends

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$\mathfrak{m}_K$ to M if and only if $a \in \bigcap_{n\geq 0} \pi_n^{-1} M$, which shows the first assertion. If $\mathcal{O}_K \to R$ is an extension of valuation rings of height 1, then we directly deduce from the valuation map $v : R[1/\pi] \setminus 0 \to \mathbb{R}$ (5.3) the equality $R = \bigcap_{n\geq 0} \pi_n^{-1} R$. □

Lemma 5.9. *Let K be a pre-perfectoid field, R an $\mathcal{O}_K$-algebra, $\mathcal{O}_K \to V$ an extension of valuation rings of height* 1*. Then, the canonical map*

$$\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V) \longrightarrow \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}}(R^{\mathrm{al}}, V^{\mathrm{al}}) \tag{5.9.1}$$

is bijective.

Proof. There are natural maps

$$\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V) \to \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}}(R^{\mathrm{al}}, V^{\mathrm{al}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, (V^{\mathrm{al}})_*) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V), \tag{5.9.2}$$

where the middle isomorphism is given by adjunction and the last isomorphism is induced by the inverse of the unit map $V \to (V^{\mathrm{al}})_*$ by 5.8. The composition is the identity map, which completes the proof. □

Definition 5.10. Let K be a pre-perfectoid field. We say that an $\mathcal{O}_K^{\mathrm{al}}$-module M (resp. an $\mathcal{O}_K$-module M) is *flat* (resp. *almost flat*) if the functor $\mathcal{O}_K^{\mathrm{al}}$-**Mod** $\to \mathcal{O}_K^{\mathrm{al}}$-**Mod** given by tensoring with M is exact (resp. M^{al} is flat).

Remark 5.11. In general, one can define the flatness of a morphism of $\mathcal{O}_K^{\mathrm{al}}$-algebras (see [GR03, 3.1.1.(i)]). We say that a morphism of $\mathcal{O}_K$-algebras $A \to B$ is *almost flat* if $A^{\mathrm{al}} \to B^{\mathrm{al}}$ is flat.

Lemma 5.12. *Let K be a pre-perfectoid field with a pseudo-uniformizer π. Then, an $\mathcal{O}_K^{\mathrm{al}}$-module M is flat if and only if M_* is π-torsion free. In particular, an $\mathcal{O}_K$-module N is almost flat if and only if the submodule of π-torsion elements of N is almost zero.*

Proof. First of all, for any $\mathcal{O}_K^{\mathrm{al}}$-modules L_1 and L_2, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(M \otimes_{\mathcal{O}_K^{\mathrm{al}}} L_1, L_2) = \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(L_1, \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(M_*, L_{2*})^{\mathrm{al}}) \tag{5.12.1}$$

by Equations (5.5.2), (5.5.5) and (5.5.7). Therefore, the functor defined by tensoring with M admits a right adjoint, and thus it is right exact. Consider the sequence

$$0 \longrightarrow \mathcal{O}_K^{\mathrm{al}} \xrightarrow{\cdot\pi} \mathcal{O}_K^{\mathrm{al}} \longrightarrow (\mathcal{O}_K/\pi\mathcal{O}_K)^{\mathrm{al}} \longrightarrow 0, \tag{5.12.2}$$

which is exact since the localization functor α^* is exact. If M is flat, tensoring the above sequence with M and applying α_*, we deduce that M_* is π-torsion free since α_* is left exact (as a right adjoint to α^*). Conversely, if M_* is π-torsion free, then it is flat over $\mathcal{O}_K$. For any injective morphism $L_1 \to L_2$ of $\mathcal{O}_K^{\mathrm{al}}$-modules, $L_{1*} \to L_{2*}$ is also injective, and it remains injective after tensoring with M_*. Therefore, $L_1 \to L_2$ also remains injective after tensoring with M since α^* is exact. This shows that M is flat.

The second assertion follows from the almost isomorphism $N \to (N^{\mathrm{al}})_*$ and the fact that $(N^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, N)$ has no nonzero almost zero element. □

Lemma 5.13. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, M a flat $\mathcal{O}_K^{\mathrm{al}}$-module, x an element of $\mathcal{O}_K$. Then, the canonical morphism $M_*/xM_* \to (M/xM)_*$ is injective, and for any $\epsilon \in \mathfrak{m}_K$, the image of $\varphi_\epsilon : (M/\epsilon xM)_* \to (M/xM)_*$ is M_*/xM_*. In particular, the canonical morphism*

$$\varprojlim_n M_*/\pi^n M_* \longrightarrow (\varprojlim_n M/\pi^n M)_* \tag{5.13.1}$$

is an isomorphism of $\mathcal{O}_K$-modules.

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Proof. We follow the proof of [Sch12, 5.3]. Applying the left exact functor α_* to the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0, \tag{5.13.2}$$

we see that $M_*/xM_* \to (M/xM)_*$ is injective.

To show that the image of φ_ϵ is M_*/xM_*, it suffices to show that φ_ϵ factors through M_*/xM_*. We identify $(M/xM)_*$ with $\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, M_*/xM_*)$ by Equations (5.5.5) and (5.5.2) so that M_*/xM_* identifies with the subset consisting of the $\mathcal{O}_K$-morphisms $\mathfrak{m}_K \to M_*/xM_*$ sending y to ya for some element $a \in M_*/xM_*$. For an $\mathcal{O}_K$-morphism $f : \mathfrak{m}_K \to M_*/\epsilon xM_*$, let b be an element of M_* which lifts $f(\epsilon)$. Notice that M_* is π-torsion free by 5.12. With notation in 5.8, we have $b \equiv (\epsilon/\pi_n) \cdot f(\pi_n)$ mod ϵxM_* for n big enough so that the element $b/\epsilon \in M_*[1/\pi]$ lies in $\bigcap_{n\geq 0} \pi_n^{-1}M_* = M_*$. Moreover, $\pi_n \cdot (b/\epsilon) \equiv f(\pi_n)$ mod xM_* for n big enough. As $\varphi_\epsilon(f)$ is determined by its values on π_n for n big enough, it follows that $\varphi_\epsilon(f) = a$, where a is the image of b/ϵ in M_*/xM_*.

Finally, the previous result implies that the inverse system $((M/\pi^n M)_*)_{n\geq 1}$ is Mittag–Leffler so that the 'in particular' part follows immediately from the fact that α_* commutes with arbitrary limits (as a right adjoint to α^*) ([Sta23, 0596]). □

Definition 5.14. Let K be a pre-perfectoid field. For any $\mathcal{O}_K$-algebra R, we define a perfect ring $R^\flat$ as the projective limit

$$R^\flat = \varprojlim_{\mathrm{Frob}} R/pR \tag{5.14.1}$$

indexed by $(\mathbb{N}, \leq)$, where transition map associated to $i \leq (i+1)$ is the Frobenius on R/pR. We call $R^\flat$ the *tilt* of R.

Lemma 5.15 [Sch12, 3.4]. *Let K be a perfectoid field with a pseudo-uniformizer π dividing p.*

(1) *The projection induces an isomorphism of multiplicative monoids*

$$\varprojlim_{\mathrm{Frob}} \mathcal{O}_K \longrightarrow \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/\pi\mathcal{O}_K. \tag{5.15.1}$$

In particular, the right-hand side is canonically isomorphic to $(\mathcal{O}_K)^\flat$ as a ring.

(2) *We denote by*

$$\sharp : (\mathcal{O}_K)^\flat \longrightarrow \mathcal{O}_K,\ x \mapsto x^\sharp, \tag{5.15.2}$$

the composition of the inverse of Equation (5.15.1) and the projection onto the first component. Then $v_K \circ \sharp : (\mathcal{O}_K)^\flat \setminus 0 \to \mathbb{R}_{\geq 0}$ defines a valuation of height 1 on $(\mathcal{O}_K)^\flat$.

(3) The fraction field $K^\flat$ of $(\mathcal{O}_K)^\flat$ is a perfectoid field of characteristic p and the element

$$\pi^\flat = (\cdots, \pi_1^{1/p^2}, \pi_1^{1/p}, \pi_1, 0) \in (\mathcal{O}_K)^\flat \tag{5.15.3}$$

is a pseudo-uniformizer of $K^\flat$, where $\pi = u \cdot \pi_1^p$ with $\pi_1 \in \mathfrak{m}_K$ and $u \in \mathcal{O}_K^\times$.

(4) We have $\mathcal{O}_{K^\flat} = (\mathcal{O}_K)^\flat$, and there is a canonical isomorphism

$$\mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat} \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K \tag{5.15.4}$$

induced by (1) and the projection onto the first component.

5.16. We see that the tilt defines a functor $\mathcal{O}_K$-**Alg** $\to \mathcal{O}_{K^\flat}$-**Alg**, $R \mapsto R^\flat$, which preserves almost zero objects and almost isomorphisms. For an $\mathcal{O}_K^{\mathrm{al}}$-algebra R, we set $R^\flat = ((R_*)^\flat)^{\mathrm{al}}$ and call it the *tilt* of R, which induces a functor $\mathcal{O}_K^{\mathrm{al}}$-**Alg** $\to \mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Alg**, $R \mapsto R^\flat$. Note that the tilt functor commutes with the localization functor α^* up to a canonical isomorphism and commutes with the functor α_* up to a canonical almost isomorphism.

Definition 5.17 [Sch12, 5.1]. Let K be a perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4).

(1) A *perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra* is an $\mathcal{O}_K^{\mathrm{al}}$-algebra R such that
  (i) R is flat over $\mathcal{O}_K^{\mathrm{al}}$;
  (ii) the Frobenius of $R/\pi R$ induces an isomorphism $R/\pi_1 R \to R/\pi R$ of $\mathcal{O}_K^{\mathrm{al}}$-algebras (5.6);
  (iii) the canonical morphism $R \to \varprojlim_n R/\pi^n R$ is an isomorphism in $\mathcal{O}_K^{\mathrm{al}}$-**Alg**.

  We denote by $\mathcal{O}_K^{\mathrm{al}}$-**Perf** the full subcategory of $\mathcal{O}_K^{\mathrm{al}}$-**Alg** formed by perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras.

(2) A *perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra* is a flat $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra R such that the Frobenius map induces an isomorphism $R/\pi_1 R \xrightarrow{\sim} R$. We denote by $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Perf** the full subcategory of $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** formed by perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebras.

Lemma 5.18. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4). Then, for an $\mathcal{O}_K$-algebra R, the following conditions are equivalent:*

(1) *The almost algebra $\widehat{R}^{\mathrm{al}}$ associated to the π-adic completion $\widehat{R}$ of R is a perfectoid $\mathcal{O}_{\widehat{K}}^{\mathrm{al}}$-algebra.*

(2) *The $\mathcal{O}_{\widehat{K}}$-module $\widehat{R}$ is almost flat, and the Frobenius of $R/\pi R$ induces an almost isomorphism $R/\pi_1 R \to R/\pi R$.*

Proof. We have seen that $\widehat{K}$ is a perfectoid field in 5.2 and π is obviously a pseudo-uniformizer of $\widehat{K}$. Since the localization functor $\alpha^* : \mathcal{O}_K$-**Alg** $\to \mathcal{O}_K^{\mathrm{al}}$-**Alg** commutes with arbitrary limits and colimits (5.5), we have a canonical isomorphism $\widehat{R}^{\mathrm{al}} \xrightarrow{\sim} \varprojlim_n \widehat{R}^{\mathrm{al}}/\pi^n \widehat{R}^{\mathrm{al}}$. Thus, the third condition in 5.17.(1) holds for $\widehat{R}^{\mathrm{al}}$. Since there are canonical isomorphisms

$$R/\pi_1 R \xrightarrow{\sim} \widehat{R}/\pi_1 \widehat{R},\ R/\pi R \xrightarrow{\sim} \widehat{R}/\pi \widehat{R}, \tag{5.18.1}$$

the conditions (1) and (2) are clearly equivalent. □

Definition 5.19. Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4). We say that an $\mathcal{O}_K$-algebra is *almost pre-perfectoid* if it satisfies the equivalent conditions in 5.18.

We remark that in 5.19, if a morphism of $\mathcal{O}_K$-algebras $R \to R'$ induces an almost isomorphism $R/\pi^n R \to R'/\pi^n R'$ for each $n \geq 1$, then the morphism of the π-adic completions $\widehat{R} \to \widehat{R'}$ is an almost isomorphism since α^* commutes with limits. In particular, R is almost pre-perfectoid if and only if R' is almost pre-perfectoid.

Lemma 5.20. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra. If R is almost flat (resp. flat) over $\mathcal{O}_K$, then the π-adic completion $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$.*

Proof. For any integer $n > 0$, there is a canonical isomorphism

$$R/\pi^n R \xrightarrow{\sim} \widehat{R}/\pi^n \widehat{R}. \tag{5.20.1}$$

Let $x \in \widehat{R}$ be a π-torsion element. Since any π-torsion element of R is almost zero (resp. zero) by 5.12, for any $\epsilon \in \mathfrak{m}_K$ (resp. $\epsilon = 1$), the image of ϵx in $\widehat{R}/\pi^n \widehat{R}$ lies in $\pi^{n-1}\widehat{R}/\pi^n \widehat{R}$. Therefore, $\epsilon x \in \bigcap_{n>0} \pi^{n-1}\widehat{R} = 0$, which amounts to say that $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$. □

Lemma 5.21. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4), R a flat $\mathcal{O}_K$-algebra. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an injection $R/\pi_1 R \to R/\pi R$.*
(2) *For any $x \in R[1/\pi]$, if $x^p \in R$, then $x \in R$.*

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Proof. We follow the proof of [Sch12, 5.7]. Assume first that $R/\pi_1 R \to R/\pi R$ is injective. Let $x \in R[1/\pi]$ with $x^p \in R$, k the minimal natural number such that $y = \pi_1^k x \in R$. If $k \geq 1$, then $y^p = \pi_1^{pk} x^p \in \pi R$. Therefore, $y \in \pi_1 R$ by the injectivity of the Frobenius. However, as R is π-torsion free, we have $y' = y/\pi_1 = \pi_1^{k-1} x \in R$ which contradicts the minimality of k.

Conversely, for any $x \in R$ with $x^p \in \pi R$, we have $(x/\pi_1)^p \in R$. Thus, $x/\pi_1 \in R$ by assumption, that is, $x \in \pi_1 R$, which implies the injectivity of the Frobenius. □

Lemma 5.22. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4), R an $\mathcal{O}_K$-algebra which is almost flat. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an almost injection (resp. almost isomorphism) $R/\pi_1 R \to R/\pi R$.*
(2) *The Frobenius induces an injection (resp. isomorphism) $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$.*

Proof. We follow the proof of [Sch12, 5.6]. Notice that the Frobenius is compatible with the functors α^* and α_* (5.6). (2) $\Rightarrow$ (1) follows from the almost isomorphism $R \to (R^{\mathrm{al}})_*$. The 'injection' part of (1) $\Rightarrow$ (2) follows from the inclusions (5.13)

$$(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \subseteq ((R/\pi_1 R)^{\mathrm{al}})_*,\ (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_* \subseteq ((R/\pi R)^{\mathrm{al}})_*. \tag{5.22.1}$$

For the 'isomorphism' part of (1) $\Rightarrow$ (2), notice that $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$ is almost surjective. Let π_2 be a p-th root of π_1 up to a unit (5.4). Then, for an element x of $(R^{\mathrm{al}})_*$, there exist elements y and x' of $(R^{\mathrm{al}})_*$ such that $\pi_2^p x = y^p + \pi_2^{p^2} x'$. Thus, $x = y'^p + \pi_2^{p^2-p} x'$, where $y' = y/\pi_2 \in (R^{\mathrm{al}})_*[1/\pi]$ (as $(R^{\mathrm{al}})_*$ is flat over $\mathcal{O}_K$ by 5.12). In fact, y' lies in $(R^{\mathrm{al}})_*$ by 5.21 and the 'injection' part of (1) $\Rightarrow$ (2). By applying this process to x', there exist elements y'' and x'' of $(R^{\mathrm{al}})_*$ such that $x' = y''^p + \pi_2^{p^2-p} x''$. In conclusion, we have $x = y'^p + \pi_2^{p^2-p}(y''^p + \pi_2^{p^2-p} x'') \equiv (y' + \pi_2^{p-1} y'')^p$ mod $\pi(R^{\mathrm{al}})_*$, which shows the surjectivity of $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$. □

Lemma 5.23. *Let K be a pre-perfectoid field, R an almost flat $\mathcal{O}_K$-algebra, π, π' pseudo-uniformizers dividing p with p-th roots π_1, π_1', respectively, up to units. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an almost injection (resp. almost surjection) $R/\pi_1 R \to R/\pi R$.*
(2) *The Frobenius induces an almost injection (resp. almost surjection) $R/\pi_1' R \to R/\pi' R$.*

In particular, the definitions 5.17.(1) and 5.19 do not depend on the choice of the pseudo-uniformizer.

Proof. Notice that $(R^{\mathrm{al}})_*$ is flat over $\mathcal{O}_K$ by 5.12. The 'injection' part follows from 5.21 and 5.22. For the 'surjection' part, we assume that $R/\pi_1 R \to R/\pi R$ is almost surjective. Let $\epsilon \in \mathfrak{m}_K$. We can take a pseudo-uniformizer $\widetilde{\pi}$ of K dividing p with $\widetilde{\pi}_1^p = \widetilde{\pi}$ and $v_K(\pi)/3 < v_K(\widetilde{\pi}) < v_K(\pi)/2$. For any $x \in R$, by the almost surjectivity, we have $\epsilon x = y^p + \widetilde{\pi}^2 z$ for some $y, z \in R$. We also have $\widetilde{\pi} z = v^p + \pi w$ for some $v, w \in R$, then $\epsilon x = y^p + \widetilde{\pi} v^p + \widetilde{\pi}\pi w$. Since $y^p + \widetilde{\pi} v^p \equiv (y + \widetilde{\pi}_1 v)^p \mathrm{mod}\ pR$, $R'/\pi_1' R \to R/\pi' R$ is almost surjective for any pseudo-uniformizer π' dividing p with $v_K(\pi') < 4v_K(\pi)/3$. By induction, we see that $R'/\pi_1' R \to R/\pi' R$ is almost surjective in general. □

Proposition 5.24. *Let K be a pre-perfectoid field of characteristic p with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra, $\widehat{R}$ the π-adic completion of R. Then, R is almost pre-perfectoid if and only if $(\widehat{R}^{\mathrm{al}})_*$ is perfect.*

Proof. Note that $\mathcal{O}_K$ is perfect by definition. If R is almost pre-perfectoid, then $\widehat{R}$ is almost flat so that $(\widehat{R}^{\mathrm{al}})_*$ is π-adically complete by taking $M = \widehat{R}^{\mathrm{al}}$ in 5.13. Moreover, the Frobenius induces an isomorphism $(\widehat{R}^{\mathrm{al}})_*/\pi^n(\widehat{R}^{\mathrm{al}})_* \to (\widehat{R}^{\mathrm{al}})_*/\pi^{pn}(\widehat{R}^{\mathrm{al}})_*$ for any integer $n \geq 1$ by 5.22 and 5.23, which implies that $(\widehat{R}^{\mathrm{al}})_*$ is perfect. Conversely, assume that $(\widehat{R}^{\mathrm{al}})_*$ is perfect. For any π-torsion element $f \in (\widehat{R}^{\mathrm{al}})_*$, we have $\pi^{1/p^n} f = 0$ for any integer $n \geq 0$, which shows that $\widehat{R}$ is almost flat by 5.12. Moreover, it is clear that the Frobenius induces an isomorphism $(\widehat{R}^{\mathrm{al}})_*/\pi(\widehat{R}^{\mathrm{al}})_* \to (\widehat{R}^{\mathrm{al}})_*/\pi^p(\widehat{R}^{\mathrm{al}})_*$, which shows that R is almost pre-perfectoid by 5.22 and 5.23. □

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Proposition 5.25. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost flat, R' the integral closure of R in $R[1/\pi]$. If the Frobenius induces an almost injection $R/\pi_1 R \to R/\pi R$, then $R \to R'$ is an almost isomorphism.*

Proof. Since $R \to (R^{\mathrm{al}})_*$ is an almost isomorphism, we may replace R by $(R^{\mathrm{al}})_*$ so that we may assume that $R = (R^{\mathrm{al}})_*$, $R \subseteq R[1/\pi]$ by 5.12 and for any $x \in R[1/\pi]$ such that $x^p \in R$, then $x \in R$ by 5.21 and 5.22. It suffices to show that R is integrally closed in $R[1/\pi]$. Suppose that $x \in R[1/\pi]$ is integral over R. There is an integer $N > 0$ such that x^r is an R-linear combination of $1, x, \dots, x^N$ for any $r > 0$. Therefore, there exists an integer $k > 0$ such that $\pi^k x^r \in R$ for any $r > 0$. Taking $r = p^n$, we get $x \in \bigcap_{n \geq 0} \pi_n^{-1} R = (R^{\mathrm{al}})_* = R$ by 5.8, which completes our proof. □

Lemma 5.26. *Let R be a ring, π a nonzero divisor of R, $\widehat{R}$ the π-adic completion of R, $\varphi : R[1/\pi] \to \widehat{R}[1/\pi]$ the canonical morphism. Then, $\varphi^{-1}(\pi^n \widehat{R}) = \pi^n R$ for any integer n.*

Proof. Remark that $\widehat{R}$ is also π-torsion free by 5.20. Take an element x/π^k of $R[1/\pi]$ (where $x \in R$, $k \geq 0$) such that $\varphi(x/\pi^k) = \pi^n y$ for some $y \in \widehat{R}$. After enlarging k, we may assume that $k+n > 0$. Thus, we deduce from the canonical isomorphism $R/\pi^{k+n} R \to \widehat{R}/\pi^{k+n}\widehat{R}$ that $x \in \pi^{k+n} R$, which completes the proof. □

Lemma 5.27. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra such that its π-adic completion $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$, $R[\pi^\infty]$ the R-submodule of elements of R killed by some power of π. Then, $(R[\pi^\infty])^\wedge$ is almost zero (resp. zero) and the canonical morphism $\widehat{R} \to (R/R[\pi^\infty])^\wedge$ is surjective and is an almost isomorphism (resp. an isomorphism).*

Proof. The exact sequence $0 \to R[\pi^\infty] \to R \to R/R[\pi^\infty] \to 0$ induces an exact sequence of the π-adic completions

$$0 \longrightarrow (R[\pi^\infty])^\wedge \longrightarrow \widehat{R} \longrightarrow (R/R[\pi^\infty])^\wedge \longrightarrow 0, \tag{5.27.1}$$

since $R/R[\pi^\infty]$ is flat over $\mathcal{O}_K$ ([Sta23, 0315]). As $\widehat{R}[\pi^\infty]$ is almost zero (resp. zero) by assumption (5.12), the canonical morphism $R[\pi^\infty]^{\mathrm{al}} \to \widehat{R}^{\mathrm{al}}$ (resp. $R[\pi^\infty] \to \widehat{R}$) factors through 0, and thus so is the morphism $(R[\pi^\infty])^{\wedge\mathrm{al}} \to \widehat{R}^{\mathrm{al}}$ (resp. $(R[\pi^\infty])^\wedge \to \widehat{R}$). The conclusion follows from the exactness of Equation (5.27.1). □

Lemma 5.28. *Let K be a pre-perfectoid field. Given a commutative diagram of $\mathcal{O}_K$-algebras*

$$\begin{tikzcd} B \arrow[r, "g"] & B' \\ A \arrow[u] \arrow[r, "f"] & A' \arrow[u] \end{tikzcd} \tag{5.28.1}$$

we denote by C (resp. C') the integral closure of A in B (resp. of A' in B'). Assume that f and g are almost isomorphisms. Then, the morphism $C \to C'$ is an almost isomorphism.

Proof. Since $C \to C'$ is almost injective as g is, it remains to show the almost surjectivity. For any $\epsilon \in \mathfrak{m}_K$ and $x' \in C'$ with identity $x'^n + a'_{n-1}x'^{n-1} + \cdots + a'_1 x' + a'_0 = 0$ in B', where $a'_{n-1}, \dots, a'_0 \in A'$, there exist $a_{n-1}, \dots, a_0 \in A$ and $x \in B$ such that $f(a_i) = \epsilon^{n-i} a'_i$ $(0 \leq i < n)$ and $g(x) = \epsilon x'$. Thus, $g(x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0) = 0$. Since g is almost injective, we see that $\epsilon x \in C$. It follows that $C \to C'$ is almost surjective. □

Proposition 5.29. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, A an $\mathcal{O}_K$-algebra such that its π-adic completion $\widehat{A}$ is almost flat over $\mathcal{O}_{\widehat{K}}$. We denote by B (resp. B') the integral closure of A in $A[1/\pi]$ (resp. of $\widehat{A}$ in $\widehat{A}[1/\pi]$). Then, the canonical morphism of π-adic completions $\widehat{B} \to \widehat{B'}$ is an almost isomorphism of $\mathcal{O}_K$-algebras.*

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Proof. We take a system of p^k-th roots $(\pi_k)_{k\geq 0}$ of π up to units (5.4). By 5.27 and 5.28, we can replace A by its image $A/A[\pi^\infty]$ in $A[1/\pi]$ so that we may assume that A is π-torsion free (and thus so is $\widehat{A}$). Let $\varphi : A[1/\pi] \to \widehat{A}[1/\pi]$ be the canonical morphism. It suffices to show that φ induces an almost isomorphism $B/\pi^n B \to B'/\pi^n B'$ for any $n > 0$.

For any element $x' \in B'$, there exists $r > 0$ such that $\pi^r x'^{p^k} \in \widehat{A}$ for any $k > 0$. We take an element $x_{ki} \in A$ such that $\varphi(x_{ki}) - \pi^r x'^{p^i} \in \pi^{rp^k}\widehat{A}$ for $i = 0, k$. Thus, $\varphi(x_{k0}^{p^k}) - \varphi(\pi^{r(p^k-1)}x_{kk}) \in \pi^{rp^k}\widehat{A}$. By 5.26, we see that $x_{k0}^{p^k}/\pi^{r(p^k-1)} - x_{kk} \in \pi^r A$. In particular, $(x_{k0}/\pi_k^{r(p^k-1)})^{p^k} \in A$, which implies that $x_{k0}/\pi_k^{r(p^k-1)} \in B$. Notice that $\varphi(x_{k0}/\pi_k^{r(p^k-1)}) - (\pi/\pi_k^{p^k-1})^r x' \in \pi^{r(p^k-1)}\widehat{A}$. Since k is an arbitrary positive integer, we see that $B/\pi^n B \to B'/\pi^n B'$ is almost surjective.

For any element $x \in B$ such that $\varphi(x/\pi^n) \in B'$, there exists $r > 0$ such that $\pi^r\varphi(x/\pi^n)^{p^k} \in \widehat{A}$ for any $k > 0$. We take $y \in A$ such that $\pi^r\varphi(x/\pi^n)^{p^k} - \varphi(y) \in \pi\widehat{A}$, and then we see that $\pi^r(x/\pi^n)^{p^k} - y \in \pi A$ by 5.26. In particular, $(x/\pi_k^{np^k-r})^{p^k} \in A$, which implies that $x/\pi_k^{np^k-r} \in B$. Since k is an arbitrary positive integer, we see that $B/\pi^n B \to B'/\pi^n B'$ is almost injective. □

Corollary 5.30. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, R' the integral closure of R in $R[1/\pi]$. Then, the morphism of π-adic completions $\widehat{R} \to \widehat{R'}$ is an almost isomorphism. In particular, R' is also almost pre-perfectoid.*

Proof. We consider the following commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R' & \longrightarrow & R[\frac{1}{\pi}] \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & R'' & \longrightarrow & \widehat{R}[\frac{1}{\pi}], \end{array} \tag{5.30.1}$$

where R'' is the integral closure of $\widehat{R}$ in $\widehat{R}[1/\pi]$. Since $\widehat{R} \to R''$ is an almost isomorphism by 5.25, R'' is also perfectoid. The conlusion follows from the fact that $\widehat{R'} \to \widehat{R''}$ is an almost isomorphism by 5.29. □

Theorem 5.31 (Tilting correspondence [Sch12, 5.2, 5.21]). *Let K be a perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4).*

(1) *The functor $\mathcal{O}_K^{\mathrm{al}}$-**Perf** $\to (\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Perf**, $R \mapsto R/\pi R$, is an equivalence of categories.*
(2) *The functor $\mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Perf** $\to (\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-**Perf**, $R \mapsto R/\pi^\flat R$ is an equivalence of categories, and the functor $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-**Perf** $\to \mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Perf**, $R \mapsto R^\flat$ is a quasi-inverse.*
(3) *Let R be a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra with tilt $R^\flat$. Then, R is isomorphic to $\mathcal{O}_L^{\mathrm{al}}$ for some perfectoid field L over K if and only if $R^\flat$ is isomorphic to $\mathcal{O}_{L'}^{\mathrm{al}}$ for some perfectoid field L' over $K^\flat$.*

In conclusion, we have natural equivalences

$$\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Perf} \xrightarrow{\sim} (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Perf} \xrightarrow{\sim} (\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}\text{-}\mathbf{Perf} \xleftarrow{\sim} \mathcal{O}_{K^\flat}^{\mathrm{al}}\text{-}\mathbf{Perf}, \tag{5.31.1}$$

where the middle equivalence is given by the isomorphism (5.15.4) $\mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat} \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K$. We remark that the natural isomorphisms of the equivalence in (2) are defined as follows: For a perfectoid $\mathcal{O}_{K^\flat}^{\mathrm{al}}$-algebra R, the natural isomorphism $R \xrightarrow{\sim} (R/\pi^\flat R)^\flat$ is induced by the homomorphism of $\mathcal{O}_{K^\flat}$-algebras $R_* \to (R_*/\pi^\flat R_*)^\flat$ sending x to $(\cdots, x^{1/p^2}, x^{1/p}, x)$ (notice that R_* is perfect by 5.24); for a perfectoid $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-algebra R, the natural isomorphism $R^\flat/\pi^\flat R^\flat \xrightarrow{\sim} R$ is induced by the projection on the first component $(R_*)^\flat \to R_*$ of $\mathcal{O}_{K^\flat}$-algebras (cf. [Sch12, 5.17]). Consequently, for a perfectoid

$\mathcal{O}_K^{\mathrm{al}}$-algebra R, the morphism of $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}} = (\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebras

$$R^\flat/\pi^\flat R^\flat \longrightarrow R/\pi R \tag{5.31.2}$$

induced by the projection on the first component is an isomorphism.

Proposition 5.32. *Let K be a perfectoid field with a pseudo-uniformizer π of K dividing p, $B \leftarrow A \rightarrow C$ a diagram of perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras. Then, the π-adically completed tensor product $B\widehat{\otimes}_A C$ is also perfectoid.*

Proof. We follow closely the proof of [Sch12, 6.18]. Firstly, we claim that $(B \otimes_A C)/\pi$ is flat over $(\mathcal{O}_K/\pi)^{\mathrm{al}}$. Since $(B \otimes_A C)/\pi = (B^\flat \otimes_{A^\flat} C^\flat)/\pi^\flat$, it suffices to show the flatness of $B^\flat \otimes_{A^\flat} C^\flat$ over $\mathcal{O}_{K^\flat}^{\mathrm{al}}$, which amounts to say that the submodule of $\pi^\flat$-torsion elements of $(B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$ is almost zero as $B^\flat \otimes_{A^\flat} C^\flat = ((B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat)^{\mathrm{al}}$. If $f \in (B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$ is a $\pi^\flat$-torsion element, then by perfectness of $(B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$, we have $(\pi^\flat)^{1/p^n} f = 0$ for any $n > 0$, which proves the claim.

Thus, $(B \otimes_A C)/\pi$ is a perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra. It corresponds to a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra D by 5.31 and it induces a morphism $B\widehat{\otimes}_A C \to D$ by universal property of π-adically completed tensor product. We use dévissage to show that $(B \otimes_A C)/\pi^n \to D/\pi^n$ is an isomorphism for any integer $n > 0$. By induction,

$$\begin{tikzcd}
& (B \otimes_A C)/\pi^n \arrow[r, "\cdot\pi"] \arrow[d] & (B \otimes_A C)/\pi^{n+1} \arrow[r] \arrow[d] & (B \otimes_A C)/\pi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D/\pi^n \arrow[r, "\cdot\pi"] & D/\pi^{n+1} \arrow[r] & D/\pi \arrow[r] & 0
\end{tikzcd} \tag{5.32.1}$$

the vertical arrows on the left and right are isomorphisms. By snake's lemma in the abelian category $\mathcal{O}_K^{\mathrm{al}}$-**Mod** ([Sta23, 010H]), we know that the vertical arrow in the middle is also an isomorphism. In conclusion, $B\widehat{\otimes}_A C \to D$ is an isomorphism, which completes the proof. □

Corollary 5.33. *Let K be a pre-perfectoid field, $B \leftarrow A \rightarrow C$ a diagram of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. Then, the tensor product $B \otimes_A C$ is also almost pre-perfectoid.*

Proof. Since α^* commutes with arbitrary limits and colimits (5.5), we have $(B\widehat{\otimes}_A C)^{\mathrm{al}} = \widehat{B}^{\mathrm{al}}\widehat{\otimes}_{\widehat{A}^{\mathrm{al}}}\widehat{C}^{\mathrm{al}}$, which is perfectoid by 5.32. □

Lemma 5.34. *Let K be a perfectoid field, $\mathcal{O}_K \to V$ an extension of valuation rings of height* 1. *Then, there exists an extension of perfectoid fields $K \to L$ and an extension of valuation rings $V \to \mathcal{O}_L$ over $\mathcal{O}_K$.*

Proof. Let π be a pseudo-uniformizer of K, E the fraction field of V, $\overline{E}$ an algebraic closure of E, $\overline{V}$ the integral closure of V in $\overline{E}$. Let $\mathfrak{m}$ be a maximal ideal of $\overline{V}$. It lies over the unique maximal ideal of V as $V \to \overline{V}$ is integral. Setting $W = \overline{V}_{\mathfrak{m}}$, according to [Bou06, VI.§8.6, Prop.6], $V \to W$ is an extension of valuation rings of height 1. Since W is integrally closed in the algebraically closed fraction field $\overline{E}$, the Frobenius is surjective on W/pW. Thus, the fraction field of W is a pre-perfectoid field over K. Passing to completion, we get an extension of perfectoid fields $K \to L$ by 5.2. □

Theorem 5.35 (cf. [BS22, 8.10]). *Let K be a pre-perfectoid field with a pseudo-uniformizer π dividing p, $R \to R'$ a homomorphism of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. If* $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ *is a π-complete arc-covering, then for any integer $n \geq 1$, the augmented Čech complex*

$$0 \to R/\pi^n \to R'/\pi^n \to (R' \otimes_R R')/\pi^n \to \cdots \tag{5.35.1}$$

is almost exact.

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Proof. We follow closely to the proof of [BS22, 8.10]. After replacing $\mathcal{O}_K$, R, R' by their π-adic completions, we may assume that K is a perfectoid field and that R^{al} and R'^{al} are perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras such that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a π-complete arc-covering by 3.4.(6). Moreover, since $R \to (R^{\mathrm{al}})_*$ is an almost isomorphism, after replacing R by $(R^{\mathrm{al}})_*$ and R' by $(R^{\mathrm{al}})_* \otimes_R R'$, we may assume further that $R = (R^{\mathrm{al}})_*$. Then, the Frobenius induces an isomorphism (resp. almost isomorphism) $R/\pi_1 R \xrightarrow{\sim} R/\pi R$ (resp. $R'/\pi_1 R' \to R'/\pi R'$) by 5.22, where π_1 is a p-th root of π up to a unit (5.4). Thus, we see that the projection on the first component induces an isomorphism (note that $R = (R^{\mathrm{al}})_*$ is π-torsion free by 5.12)

$$R^\flat/\pi^\flat R^\flat \xrightarrow{\sim} R/\pi R \tag{5.35.2}$$

and an almost isomorphism (by the preceding isomorphism for $(R'^{\mathrm{al}})_*$ or by Equation (5.31.2))

$$R'^\flat/\pi^\flat R'^\flat \longrightarrow R'/\pi R'. \tag{5.35.3}$$

In particular, $\mathrm{Spec}(R'^\flat/\pi^\flat) \to \mathrm{Spec}(R^\flat/\pi^\flat)$ is an arc-covering as $\mathrm{Spec}(R'/\pi) \to \mathrm{Spec}(R/\pi)$ is so.

On the other hand, since the localization functor α^* commutes with arbitrary limits and colimits (5.5), $(\widehat{\otimes}_R^k R')^{\mathrm{al}} = \widehat{\otimes}_{R^{\mathrm{al}}}^k R'^{\mathrm{al}}$ is still a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra by 5.32 for any $k \geq 0$. In particular, $\widehat{\otimes}_R^k R'$ is almost flat over $\mathcal{O}_K$. Then, by dévissage, it suffices to show the almost exactness of the augmented Čech complex when $n = 1$, that is, the almost exactness of

$$0 \to R^\flat/\pi^\flat \to R'^\flat/\pi^\flat \to (R'^\flat \otimes_{R^\flat} R'^\flat)/\pi^\flat \to \cdots . \tag{5.35.4}$$

We claim that the natural morphism $X = \mathrm{Spec}(R'^\flat) \coprod \mathrm{Spec}(R^\flat[1/\pi^\flat]) \to Y = \mathrm{Spec}(R^\flat)$ is an arc-covering. Firstly, we see that $X \to Y = \mathrm{Spec}(R^\flat/\pi^\flat) \bigcup \mathrm{Spec}(R^\flat[1/\pi^\flat])$ is surjective as we have shown that $\mathrm{Spec}(R'^\flat/\pi^\flat) \to \mathrm{Spec}(R^\flat/\pi^\flat)$ is an arc-covering. Therefore, we only need to consider the test map $\mathrm{Spec}(V) \to Y$, where V is a valuation ring of height 1. There are three cases:

(1) If $\pi^\flat$ is invertible in V, then we get a natural lifting $R^\flat[1/\pi^\flat] \to V$.
(2) If $\pi^\flat$ is zero in V, then $R^\flat \to V$ factors through $R^\flat/\pi^\flat \xrightarrow{\sim} R/\pi$, and thus there is a lifting $R'^\flat/\pi^\flat \to R'/\pi \to W$.
(3) Otherwise, $\mathcal{O}_{K^\flat} \to V$ is an extension of valuation rings. After replacing V by an extension (5.34), we may assume that $V[1/\pi^\flat]$ is a perfectoid field over $K^\flat$ with valuation ring V. By tilting correspondence 5.31, it corresponds to a perfectoid field over K with valuation ring $V^\sharp$, together with an $\mathcal{O}_K$-morphism $R \to V^\sharp$ by 5.9. Since $R \to R'$ gives a π-complete arc-covering, there is an extension $V^\sharp \to W$ of valuation rings of height 1 and a lifting $R' \to W$. After replacing W by an extension (5.34), we may assume that $W[1/\pi]$ is a perfectoid field over K with valuation ring W. By tilting correspondence 5.31 and 5.9, we get a lifting $R'^\flat \to W^\flat$ of $R^\flat \to V$.

Now, we apply 4.10 to the arc-covering $X \to Y$ of perfect affine $\mathbb{F}_p$-schemes. We get an exact augmented Čech complex

$$0 \to R^\flat \to R'^\flat \times R^\flat[\frac{1}{\pi^\flat}] \to (R'^\flat \times R^\flat[\frac{1}{\pi^\flat}]) \otimes_{R^\flat} (R'^\flat \times R^\flat[\frac{1}{\pi^\flat}]) \to \cdots . \tag{5.35.5}$$

Since each term is a perfect $\mathbb{F}_p$-algebra, the submodule of $\pi^\flat$-torsion elements is almost zero, in other words, each term is almost flat over $\mathcal{O}_{K^\flat}$. Modulo $\pi^\flat$, we get the almost exactness of Equation (5.35.4), which completes the proof. □

Definition 5.36. Let K be a pre-perfectoid field, $A \to B$ a morphism of $\mathcal{O}_K$-algebras.

(1) We say that $A \to B$ is *almost étale* if $A^{\mathrm{al}} \to B^{\mathrm{al}}$ is an étale morphism of $\mathcal{O}_K^{\mathrm{al}}$-algebras in the sense of [GR03, 3.1.1.(iv)].

(2) We say that $A \to B$ is *almost finite étale* if it is almost étale and if B^{al} is an almost finitely presented A^{al}-module in the sense of [GR03, 2.3.10] (cf. [Sch12, 4.13], [AGT16, V.7.1]).

We remark that in 5.36 if $A \to B$ is a morphism of K-algebras, then it is almost étale (resp. almost finite étale) if and only if it is étale (resp. finite étale).

Proposition 5.37. *Let K be a pre-perfectoid field, $\mathscr{C}$ the full subcategory of the category of $\mathcal{O}_K$-algebras formed by those $\mathcal{O}_K$-algebras which are almost pre-perfectoid.*

(1) *The subcategory $\mathscr{C}$ is stable under taking colimits and products.*
(2) *Let $A \to B$ be an almost étale morphism of $\mathcal{O}_K$-algebras. If $A \in \mathrm{Ob}(\mathscr{C})$, then $B \in \mathrm{Ob}(\mathscr{C})$.*

Proof. Let π be a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit (5.4).

(1) The subcategory $\mathscr{C}$ is stable under taking tensor products by 5.33. Let $(R_\lambda)_{\lambda\in\Lambda}$ be a directed system of objects in $\mathscr{C}$ and $R = \mathrm{colim}_{\lambda\in\Lambda} R_\lambda$. It is clear that the Frobenius induces an almost isomorphism $R/\pi_1 R \to R/\pi R$. On the other hand, $\widehat{R}$ is the π-adic completion of $\mathrm{colim}_{\lambda\in\Lambda} \widehat{R_\lambda}$. Since the latter is almost flat over $\mathcal{O}_{\widehat{K}}$ so is $\widehat{R}$ (5.20). Thus, $\mathscr{C}$ is stable under taking colimits.

Let $(R_\lambda)_{\lambda\in\Lambda}$ be a set of objects in $\mathscr{C}$. Since $R/\pi R = \prod_{\lambda\in\Lambda} R_\lambda/\pi R_\lambda$, the Frobenius induces an almost isomorphism $R/\pi_1 R \to R/\pi R$. Moreover, the submodule of π-torsion elements of $\widehat{R} = \prod_{\lambda\in\Lambda} \widehat{R_\lambda}$ is almost zero, which implies that $\widehat{R}$ is almost flat over $\mathcal{O}_{\widehat{K}}$ (5.12). We conclude that $\mathscr{C}$ is stable under taking products.

(2) Since B is almost flat over A, it is almost flat over $\mathcal{O}_K$ and thus $\widehat{B}$ is almost flat over $\mathcal{O}_{\widehat{K}}$ (5.20). Since B is almost étale over A, the map $B/\pi_1 B \to B/\pi B$ induced by the Frobenius is almost isomorphic to the base change of the map $A/\pi_1 A \to A/\pi A$ by $A \to B$ ([GR03, 3.5.13]), which completes the proof. □

Lemma 5.38. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost flat and almost pre-perfectoid, R' an R-algebra which is almost finite étale. Then, the integral closure of R in R' is almost isomorphic to both R' and the integral closure of R in $R'[1/\pi]$.*

Proof. Notice that R' is also almost flat and almost pre-perfectoid by 5.37. Since R' is almost finitely generated over R as an R-module, the elements of $\mathfrak{m}_K R'$ are integral over R (see [GR03, 2.3.10]). Thus, the integral closure of R in R' is almost isomorphic to R'. On the other hand, since R' is almost isomorphic to its integral closure in $R'[1/\pi]$ by 5.25, the integral closure of R in R' is almost isomorphic to the integral closure of R in $R'[1/\pi]$ by 5.28. □

5.39. We recall some basic definitions about affinoid algebras used in [Sch12] in order to prove the almost purity theorem 5.41 by reducing to *loc.cit*. Let K be a complete valuation field of height 1. A *Tate K-algebra* is a topological K-algebra $\mathcal{R}$ whose topology is generated by the open subsets $a\mathcal{R}_0$ for a subring $\mathcal{R}_0 \subset \mathcal{R}$ and any $a \in K^\times$. We denote by $\mathcal{R}^\circ$ the subring of power-bounded elements of $\mathcal{R}$, which is thus an $\mathcal{O}_K$-algebra. An *affinoid K-algebra* is a pair $(\mathcal{R}, \mathcal{R}^+)$ consisting of a Tate K-algebra $\mathcal{R}$ and a subring $\mathcal{R}^+$ of $\mathcal{R}^\circ$ which is open and integrally closed in $\mathcal{R}$. A morphism of affinoid K-algebras $(\mathcal{R}, \mathcal{R}^+) \to (\mathcal{R}', \mathcal{R}'^+)$ is a morphism of topological K-algebras $f : \mathcal{R} \to \mathcal{R}'$ with $f(\mathcal{R}^+) \subseteq \mathcal{R}'^+$. Such a morphism is called *finite étale* in the sense of [Sch12, 7.1. (i)] if $\mathcal{R}'$ is finite étale over $\mathcal{R}$ endowed with the canonical topology as a finitely generated $\mathcal{R}$-module and if $\mathcal{R}'^+$ is the integral closure of $\mathcal{R}^+$ in $\mathcal{R}'$.

For a perfectoid field K and an affinoid K-algebra $(\mathcal{R}, \mathcal{R}^+)$, the inclusion $\mathcal{R}^+ \subseteq \mathcal{R}^\circ$ is an almost isomorphism. Indeed, for any $\epsilon \in \mathfrak{m}_K$ and any power-bounded element $x \in \mathcal{R}^\circ$, we have $(\epsilon x)^n \in \mathcal{R}^+$ for $n \in \mathbb{N}$ large enough as $\mathcal{R}^+$ is open. Thus, $\epsilon x \in \mathcal{R}^+$ as $\mathcal{R}^+$ is integrally closed. We remark that $(\mathcal{R}, \mathcal{R}^+)$ is perfectoid in the sense of [Sch12, 6.1] if and only if $\mathcal{R}^\circ$ is bounded and almost perfectoid over $\mathcal{O}_K$ ([Sch12, 5.5, 5.6]).

5.40. There is a typical example for constructing affinoid algebras from commutative algebras (see [And18, Sorite 2.3.1]). Let K be a complete valuation field of height 1 with a pseudo-uniformizer π, R a flat $\mathcal{O}_K$-algebra. The K-algebra $R[1/\pi]$ endowed with the π-adic topology defined by R is a Tate K-algebra. Let $\overline{R}$ be the integral closure of R in $R[1/\pi]$. It is clear that any element of $\overline{R}$ is power-bounded. Thus, $(R[1/\pi], \overline{R})$ is an affinoid K-algebra.

Let S be a finite $R[1/\pi]$-algebra endowed with the canonical topology. More precisely, the topology can be defined as follows: We take a finite R-algebra R' contained in S which contains a family of generators of the $R[1/\pi]$-algebra S; then the canonical topology of $S = R'[1/\pi]$ is the π-adic topology defined by R' (which is independent of the choice of R'). Let $\overline{R'}$ be the integral closure of R' in $R'[1/\pi]$, which is also the integral closure of R in $R'[1/\pi]$. We remark that $(R[1/\pi], \overline{R}) \to (R'[1/\pi], \overline{R'})$ is a finite étale morphism of affinoid K-algebras if and only if $R[1/\pi] \to R'[1/\pi]$ is finite étale.

Theorem 5.41 (Almost purity, [Sch12, 7.9]). *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, R' the integral closure of R in a finite étale $R[1/\pi]$-algebra.*

(1) *The $\mathcal{O}_K$-algebra R' is almost pre-perfectoid and the π-adic completion $\widehat{R'}$ is almost finite étale over $\widehat{R}$.*
(2) *If R is π-torsion free, then R' is almost finite étale over R.*

Proof. (1) By 5.27, we can replace R by its image $R/R[\pi^\infty]$ in $R[1/\pi]$ (which does not change R') so that we may assume that R is π-torsion free (and thus so is $\widehat{R}$). Let S (resp. S') be the integral closure of $\widehat{R}$ in $\widehat{R}[1/\pi]$ (resp. of $R' \otimes_R \widehat{R}$ in $R' \otimes_R \widehat{R}[1/\pi]$). Then, we obtain a finite étale morphism of affinoid $\widehat{K}$-algebras $(\widehat{R}[1/\pi], S) \to (R' \otimes_R \widehat{R}[1/\pi], S')$ by 5.40.

Since $\widehat{R}$ is almost perfectoid, $\widehat{R} \to S$ is an almost isomorphism (5.25). Thus, S is bounded and almost perfectoid over $\mathcal{O}_{\widehat{K}}$. In other words, $(\widehat{R}[1/\pi], S)$ is a perfectoid affinoid $\widehat{K}$-algebra. Then, by almost purity ([Sch12, 7.9.(iii)]), the $\mathcal{O}_{\widehat{K}}$-algebra S' is almost perfectoid (thus $S' \to \widehat{S'}$ is an almost isomorphism by definition) and almost finite étale over S.

On the other hand, the two $\mathcal{O}_{\widehat{K}}$-algebras R' and $R' \otimes_R \widehat{R}$ have the same π-adic completion $\widehat{R'}$. Thus, the π-adic completions of the integral closures of R' and $R' \otimes_R \widehat{R}$ in $R'[1/\pi]$ and $R' \otimes_R \widehat{R}[1/\pi]$, respectively, are almost isomorphic to that of $\widehat{R'}$ in $\widehat{R'}[1/\pi]$ by 5.29. In other words, $\widehat{R'} \to \widehat{S'}$ is an almost isomorphism. In conclusion, R' is almost pre-perfectoid, and $\widehat{R'}$ is almost finite étale over $\widehat{R}$.

(2) As R is torsion free, we can proceed the same argument as above and use the same notation. We firstly consider a special case where $(R, \pi R)$ is a Henselian pair. Recall that the category of almost $\mathcal{O}_K$-algebras finite étale over R^{al} (resp. over $\widehat{R}^{\mathrm{al}}$) is equivalent to that over $(R/\pi R)^{\mathrm{al}}$ via the base change functor ([GR03, 5.5.7.(iii)]). Hence, there exists an R-algebra R'' which is almost finite étale over R such that $(R'' \otimes_R \widehat{R})^{\mathrm{al}}$ is isomorphic to $\widehat{R'}^{\mathrm{al}}$. On the other hand, recall that the category of finite étale $R[1/\pi]$-algebras is equivalent to the category of finite étale $\widehat{R}[1/\pi]$-algebras via the base change functor ([GR03, 5.4.53]). Notice that $R''[1/\pi] \otimes_R \widehat{R} \cong \widehat{R'}[1/\pi]$ by the construction of R'' and that $R'[1/\pi] \otimes_R \widehat{R} \cong \widehat{R'}[1/\pi]$ by the almost isomorphisms $\widehat{R'} \to \widehat{S'} \leftarrow S'$ (with the same notation as above). Hence, there is an isomorphism $R''[1/\pi] \cong R'[1/\pi]$. By 5.38, we see that R'' is almost isomorphic to R', which completes the proof in the special case.

In general, let $(T, \pi T)$ be the Henselization of the pair $(R, \pi R)$. Then, $T' = T \otimes_R R'$ is the integral closure of T in a finite étale $T[1/\pi]$-algebra (see 3.17, 3.18). By the special case above, we see that T' is almost finite étale over T. Notice that $R \to R[1/\pi] \times T$ is faithfully flat. By almost faithfully flat descent [AGT16, V.8.10], we see that R' is almost finite étale over R. □

6. Brief review on covanishing fibred sites

We give a brief review on covanishing fibred sites, which are developed by Abbes–Gros [AGT16, VI].

6.1. A *fibred site E/C* is a fibred category $\pi : E \to C$ whose fibres are sites such that for a cleavage and for any morphism $f : \beta \to \alpha$ in C, the inverse image functor $f^+ : E_\alpha \to E_\beta$ gives a morphism of sites (so that the same holds for any cleavage) (see [SGA 4_{II}, VI.7.2]).

Let x be an object of E over $\alpha \in \mathrm{Ob}(C)$. We denote by

$$\iota_\alpha^+ : E_\alpha \to E \tag{6.1.1}$$

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the inclusion functor of the fibre category E_α over α into the whole category E. A *vertical covering* of x is the image by ι_α^+ of a covering family $\{x_m \to x\}_{m\in M}$ in E_α. We call the topology generated by all vertical coverings the *total topology* on E (see [SGA 4_{II}, VI.7.4.2]).

Assume further that C is a site. A *Cartesian covering* of x is a family $\{x_n \to x\}_{n\in N}$ of morphisms of E such that there exists a covering family $\{\alpha_n \to \alpha\}_{n\in N}$ in C with x_n isomorphic to the pullback of x by $\alpha_n \to \alpha$ for each n.

Definition 6.2 [AGT16, VI.5.3]. A *covanishing fibred site* is a fibred category E/C endowed with a normalized cleavage ([SGA 1, VI.7.1]), a topology on C and a topology on the fibre category E_α for any $\alpha \in \mathrm{Ob}(C)$ satisfying the following conditions:

(1) Fibred products are representable in C.
(2) Finite limits are representable in the fibre category E_α for any $\alpha \in \mathrm{Ob}(C)$.
(3) For any morphism $f : \beta \to \alpha$ in C, the inverse image functor $f^+ : E_\alpha \to E_\beta$ is left exact and continuous (see 2.5).

We associate to E the *covanishing topology* which is generated by all vertical coverings and Cartesian coverings. We simply call a covering family for the covanishing topology a *covanishing covering*.

Definition 6.3. Let E/C be a covanishing fibred site. We call a composition of a Cartesian covering followed by vertical coverings a *standard covanishing covering*. More precisely, a standard covanishing covering is a family of morphisms of E

$$\{x_{nm} \to x\}_{n\in N, m\in M_n} \tag{6.3.1}$$

such that there is a Cartesian covering $\{x_n \to x\}_{n\in N}$ and for each $n \in N$ a vertical covering $\{x_{nm} \to x_n\}_{m\in M_n}$.

Proposition 6.4 [AGT16, VI.5.9]. *Let E/C be a covanishing fibred site. Assume that in each fibre any object is quasi-compact, then a family of morphisms $\{x_i \to x\}_{i\in I}$ of E is a covanishing covering if and only if it can be refined by a standard covanishing covering.*

6.5. Let E/C be a fibred category. Fixing a cleavage of E/C, to give a presheaf $\mathcal{F}$ on E is equivalent to give a presheaf $\mathcal{F}_\alpha$ on each fibre category E_α and transition morphisms $\mathcal{F}_\alpha \to f^{\mathrm{p}}\mathcal{F}_\beta$ for each morphism $f : \beta \to \alpha$ in C satisfying a cocycle relation (see [SGA 4_{II}, VI.7.4.7]). Thus, we simply denote a presheaf $\mathcal{F}$ on E by

$$\mathcal{F} = (\mathcal{F}_\alpha)_{\alpha\in \mathrm{Ob}(C)}, \tag{6.5.1}$$

where $\mathcal{F}_\alpha = \iota_\alpha^{\mathrm{p}}\mathcal{F}$ is the restriction of $\mathcal{F}$ on the fibre category E_α. If E/C is a fibred site, then $\mathcal{F}$ is a sheaf with respect to the total topology on E if and only if $\mathcal{F}_\alpha$ is a sheaf on E_α for each α ([SGA 4_{II}, VI.7.4.7]). Moreover, we have the following description of a covanishing sheaf.

Proposition 6.6 [AGT16, VI.5.10]. *Let E/C be a covanishing fibred site. Then, a presheaf $\mathcal{F}$ on E is a sheaf if and only if the following conditions hold:*

(v) *The presheaf $\mathcal{F}_\alpha = \iota_\alpha^{\mathrm{p}}\mathcal{F}$ on E_α is a sheaf for any $\alpha \in \mathrm{Ob}(C)$.*
(c) *For any covering family $\{f_i : \alpha_i \to \alpha\}_{i\in I}$ of C, if we set $\alpha_{ij} = \alpha_i \times_\alpha \alpha_j$ and $f_{ij} : \alpha_{ij} \to \alpha$, then the sequence of sheaves on E_α,*

$$\mathcal{F}_\alpha \to \prod_{i\in I} f_{i*}\mathcal{F}_{\alpha_i} \rightrightarrows \prod_{i,j\in I} f_{ij*}\mathcal{F}_{\alpha_{ij}}, \tag{6.6.1}$$

is exact.

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7. Faltings ringed sites

7.1. Let $Y \to X$ be a morphism of $\mathbb{U}$-small coherent schemes, and let $\mathbf{E}_{Y\to X}$ be the category of morphisms $V \to U$ of $\mathbb{U}$-small coherent schemes over the morphism $Y \to X$, namely, the category of commutative diagrams of coherent schemes

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Y \arrow[r] & X. \end{tikzcd} \tag{7.1.1}$$

Given a functor $I \to \mathbf{E}_{Y\to X}$ sending i to $V_i \to U_i$, if $\lim V_i$ and $\lim U_i$ are representable in the category of coherent schemes, then $\lim(V_i \to U_i)$ is representable by $\lim V_i \to \lim U_i$. We say that a morphism $(V' \to U') \to (V \to U)$ of objects of $\mathbf{E}_{Y\to X}$ is *Cartesian* if $V' \to V \times_U U'$ is an isomorphism. It is clear that the Cartesian morphisms in $\mathbf{E}_{Y\to X}$ are stable under base change.

Consider the functor

$$\phi^+ : \mathbf{E}_{Y\to X} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/X},\ (V \to U) \longmapsto U. \tag{7.1.2}$$

The fibre category over U is canonically equivalent to the category $\mathbf{Sch}^{\mathrm{coh}}_{/U_Y}$ of coherent U_Y-schemes, where $U_Y = Y \times_X U$. The base change by $U' \to U$ gives an inverse image functor $\mathbf{Sch}^{\mathrm{coh}}_{/U_Y} \to \mathbf{Sch}^{\mathrm{coh}}_{/U'_Y}$, which endows $\mathbf{E}_{Y\to X}/\mathbf{Sch}^{\mathrm{coh}}_{/X}$ with a structure of fibred category. We define a presheaf on $\mathbf{E}_{Y\to X}$ by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{7.1.3}$$

where U^V is the integral closure of U in V.

Definition 7.2. Let $Y \to X$ be a morphism of coherent schemes. A morphism $(V' \to U') \to (V \to U)$ in $\mathbf{E}_{Y\to X}$ is called *étale* if $U' \to U$ is étale and $V' \to V \times_U U'$ is finite étale.

Lemma 7.3. *Let $Y \to X$ be a morphism of coherent schemes, $(V'' \to U'') \xrightarrow{g} (V' \to U') \xrightarrow{f} (V \to U)$ morphisms in $\mathbf{E}_{Y\to X}$.*

(1) *If f is étale, then any base change of f is also étale.*
(2) *If f and g are étale, then $f \circ g$ is also étale.*
(3) *If f and $f \circ g$ are étale, then g is also étale.*

Proof. It follows directly from the definitions. □

7.4. Let $Y \to X$ be a morphism of coherent schemes. We still denote by $X_{\text{ét}}$ (resp. $X_{\text{fét}}$) the site formed by coherent étale (resp. finite étale) X-schemes endowed with the étale topology. Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the full subcategory of $\mathbf{E}_{Y\to X}$ formed by $V \to U$ étale over the final object $Y \to X$. It is clear that $\mathbf{E}^{\text{ét}}_{Y\to X}$ is stable under finite limits in $\mathbf{E}_{Y\to X}$. Then, the functor (7.1.2) induces a functor

$$\phi^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow X_{\text{ét}},\ (V \to U) \longmapsto U, \tag{7.4.1}$$

which endows $\mathbf{E}^{\text{ét}}_{Y\to X}/X_{\text{ét}}$ with a structure of fibred sites, whose fibre over U is the finite étale site $U_{Y,\text{fét}}$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the associated covanishing topology, that is, the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N} U_n \to U$ is surjective.

It is clear that any object of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is quasi-compact by 6.4. We still denote by $\overline{\mathscr{B}}$ the restriction of the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}_{Y\to X}$ to $\mathbf{E}^{\text{ét}}_{Y\to X}$ if there is no ambiguity.

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Lemma 7.5. *Let $Y \to X$ be a morphism of coherent schemes. Then, the presheaf on $\mathbf{Sch}^{\mathrm{coh}}_{/Y}$ sending Y' to $\Gamma(X^{Y'}, \mathcal{O}_{X^{Y'}})$ is a sheaf with respect to the fpqc topology ([Sta23, 022A]).*

Proof. We may regard $\mathcal{O}_{X^{Y'}}$ as a quasi-coherent $\mathcal{O}_X$-algebra over X. It suffices to show that for a finite family of morphisms $\{Y_i \to Y\}_{i\in I}$ with $Y' = \coprod_{i\in I} Y_i$ faithfully flat over Y, the sequence of quasi-coherent $\mathcal{O}_X$-algebras

$$0 \longrightarrow \mathcal{O}_{X^Y} \longrightarrow \mathcal{O}_{X^{Y'}} \rightrightarrows \mathcal{O}_{X^{Y'\times_Y Y'}} \tag{7.5.1}$$

is exact. Thus, we may assume that $X = \mathrm{Spec}(R)$ is affine. We set $A_0 = \Gamma(Y, \mathcal{O}_Y)$, $A_1 = \Gamma(Y', \mathcal{O}_{Y'})$, $A_2 = \Gamma(Y' \times_Y Y', \mathcal{O}_{Y'\times_Y Y'})$, $R_0 = \Gamma(X^Y, \mathcal{O}_{X^Y})$, $R_1 = \Gamma(X^{Y'}, \mathcal{O}_{X^{Y'}})$, $R_2 = \Gamma(X^{Y'\times_Y Y'}, \mathcal{O}_{X^{Y'\times_Y Y'}})$. Notice that R_i is the integral closure of R in A_i for $i = 0, 1, 2$ ([Sta23, 035F]). Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & R_0 \arrow[r] \arrow[d] & R_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & R_2 \arrow[d] \\ 0 \arrow[r] & A_0 \arrow[r] & A_1 \arrow[r, shift left] \arrow[r, shift right] & A_2. \end{tikzcd} \tag{7.5.2}$$

We see that the vertical arrows are injective and the second row is exact by faithfully flat descent. Notice that $R_0 = A_0 \cap R_1$ since they are both the integral closure of R in A_0 as $A_0 \subseteq A_1$. Thus, the first row is also exact, which completes the proof. □

Proposition 7.6. *Let $Y \to X$ be a morphism of coherent schemes. Then, the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$ is a sheaf.*

Proof. It follows directly from 6.6, whose first condition holds by 7.5 and whose second condition holds by 3.17 (cf. [AGT16, III.8.16]). □

Definition 7.7 [Fal02, page 214], [AGT16, VI.10.1]. We call $(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}})$ the *Faltings ringed site* of the morphism of coherent schemes $Y \to X$.

It is clear that the localization $(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X})_{/(V\to U)}$ of $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$ at an object $V \to U$ is canonically equivalent to the Faltings ringed site $\mathbf{E}^{\mathrm{\acute{e}t}}_{V\to U}$ of the morphism $V \to U$ by 6.4 (cf. [AGT16, VI.10.14]).

7.8. Let $Y \to X$ be a morphism of coherent schemes. Consider the natural functors

$$\psi^+ : \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X} \longrightarrow Y_{\mathrm{\acute{e}t}},\ (V \to U) \longmapsto V, \tag{7.8.1}$$

$$\beta^+ : Y_{\mathrm{f\acute{e}t}} \longrightarrow \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X},\ V \longmapsto (V \to X), \tag{7.8.2}$$

$$\sigma^+ : X_{\mathrm{\acute{e}t}} \longrightarrow \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X},\ U \longmapsto (Y \times_X U \to U). \tag{7.8.3}$$

They are left exact and continuous (cf. [AGT16, VI 10.6, 10.7]). Then, we obtain a commutative diagram of sites associated functorially to the morphism $Y \to X$ by 2.5,

$$\begin{tikzcd} & Y_{\mathrm{\acute{e}t}} \arrow[dl, "\rho"'] \arrow[d, "\psi"] \arrow[dr] & \\ Y_{\mathrm{f\acute{e}t}} & \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X} \arrow[l, "\beta"] \arrow[r, "\sigma"'] & X_{\mathrm{\acute{e}t}}, \end{tikzcd} \tag{7.8.4}$$

where $\rho : Y_{\mathrm{\acute{e}t}} \to Y_{\mathrm{f\acute{e}t}}$ is defined by the inclusion functor, and the unlabelled arrow $Y_{\mathrm{\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ is induced by the morphism $Y \to X$. Moreover, if $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}$ denotes the structural sheaf on $X_{\mathrm{\acute{e}t}}$ sending U to $\Gamma(U, \mathcal{O}_U)$, then σ^+ actually defines a morphism of ringed sites

$$\sigma : (\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}) \longrightarrow (X_{\mathrm{\acute{e}t}}, \mathcal{O}_{X_{\mathrm{\acute{e}t}}}). \tag{7.8.5}$$

We will study more properties of these morphisms in the remaining sections.

Lemma 7.9. *Let X be the spectrum of an absolutely integrally closed valuation ring, Y a quasi-compact open subscheme of X. Then, for any presheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, we have $\mathcal{F}^{\text{a}}(Y\to X)=\mathcal{F}(Y\to X)$. In particular, the associated topos of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is local ([SGA 4_{II}, VI.8.4.6]).*

Proof. Notice that Y is also the spectrum of an absolutely integrally closed valuation ring by 3.11.(1) and that absolutely integrally closed valuation rings are strictly Henselian. Thus, any covering of $Y\to X$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ can be refined by the identity covering by 6.4. We see that $\mathcal{F}^{\text{a}}(Y\to X)=\mathcal{F}(Y\to X)$ for any presheaf $\mathcal{F}$. For the last assertion, it suffices to show that the section functor $\Gamma(Y\to X,-)$ commutes with colimits of sheaves. For a colimit of sheaves $\mathcal{F}=\operatorname{colim}\mathcal{F}_i$, let $\mathcal{G}$ be the colimit of presheaves $\mathcal{G}=\operatorname{colim}\mathcal{F}_i$. Then, we have $\mathcal{F}=\mathcal{G}^{\text{a}}$ and $\Gamma(Y\to X,\mathcal{F})=\Gamma(Y\to X,\mathcal{G})=\operatorname{colim}\Gamma(Y\to X,\mathcal{F}_i)$. □

7.10. Let $(Y_\lambda\to X_\lambda)_{\lambda\in\Lambda}$ be a $\mathbb{U}$-small directed inverse system of morphisms of $\mathbb{U}$-small coherent schemes with affine transition morphisms $Y_{\lambda'}\to Y_\lambda$ and $X_{\lambda'}\to X_\lambda$ ($\lambda'\geq\lambda$). We set $(Y\to X)=\lim_{\lambda\in\Lambda}(Y_\lambda\to X_\lambda)$. We regard the directed set Λ as a filtered category and regard the inverse system $(Y_\lambda\to X_\lambda)_{\lambda\in\Lambda}$ as a functor $\varphi:\Lambda^{\text{op}}\to\mathbf{E}$ from the opposite category of Λ to the category of morphisms of $\mathbb{U}$-small coherent schemes. Consider the fibred category $\mathbf{E}^{\text{ét}}_{\varphi}\to\Lambda^{\text{op}}$ defined by φ whose fibre category over λ is $\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ and whose inverse image functor $\varphi^+_{\lambda'\lambda}:\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}\to\mathbf{E}^{\text{ét}}_{Y'_\lambda\to X'_\lambda}$ associated to a morphism $\lambda'\to\lambda$ in Λ^{op} is given by the base change by the transition morphism $(Y_{\lambda'}\to X_{\lambda'})\to(Y_\lambda\to X_\lambda)$ (cf. [AGT16, VI.11.2]). Let $\varphi^+_\lambda:\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}\to\mathbf{E}^{\text{ét}}_{Y\to X}$ be the functor defined by the base change by the transition morphism $(Y\to X)\to(Y_\lambda\to X_\lambda)$.

Recall that the filtered colimit of categories $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$ is representable by the category $\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}$ whose objects are those of $\mathbf{E}^{\text{ét}}_{\varphi}$ and whose morphisms are given by ([SGA 4_{II}, VI 6.3, 6.5])

$$\operatorname{Hom}_{\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}}((V\to U),(V'\to U'))=\underset{\substack{(V''\to U'')\to(V\to U)\\ \text{Cartesian}}}{\operatorname{colim}}\operatorname{Hom}_{\mathbf{E}^{\text{ét}}_{\varphi}}((V''\to U''),(V'\to U')),\tag{7.10.1}$$

where the colimit is taken over the opposite category of the cofiltered category of Cartesian morphisms with target $V\to U$ of the fibred category $\mathbf{E}^{\text{ét}}_{\varphi}$ over Λ^{op} (distinguish with the Cartesian morphisms defined in 7.1). We see that the functors φ^+_λ induces an equivalence of categories by [EGA IV_3, 8.8.2, 8.10.5] and [EGA IV_4, 17.7.8]

$$\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}\xrightarrow{\sim}\mathbf{E}^{\text{ét}}_{Y\to X}.\tag{7.10.2}$$

Recall that the cofiltered limit of sites $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$ is representable by $\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}$ endowed with the coarsest topology such that the natural functors $\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}\to\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}$ are continuous ([SGA 4_{II}, VI.8.2.3]).

Lemma 7.11. *With the notation in 7.10, for any covering family $\mathfrak{U}=\{f_k:(V_k\to U_k)\to(V\to U)\}_{k\in K}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ with K finite, there exists an index $\lambda_0\in\Lambda$ and a covering family $\mathfrak{U}_{\lambda_0}=\{f_{k\lambda_0}:(V_{k\lambda_0}\to U_{k\lambda_0})\to(V_{\lambda_0}\to U_{\lambda_0})\}_{k\in K}$ in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$ such that f_k is the base change of $f_{k\lambda_0}$ by the transition morphism $(Y\to X)\to(Y_{\lambda_0}\to X_{\lambda_0})$.*

Proof. There is a standard covanishing covering $\mathfrak{U}'=\{g_{nm}:(V'_{nm}\to U'_n)\to(V\to U)\}_{n\in N,m\in M_n}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ with N, M_n finite, which refines $\mathfrak{U}$ by 6.4. The equivalence (7.10.2) implies that there exists an index $\lambda_1\in\Lambda$ and families of morphisms $\mathfrak{U}'_{\lambda_1}=\{g_{nm\lambda_1}:(V'_{nm\lambda_1}\to U'_{n\lambda_1})\to(V_{\lambda_1}\to U_{\lambda_1})\}_{n\in N,m\in M_n}$ (resp. $\mathfrak{U}_{\lambda_1}=\{f_{k\lambda_1}:(V_{k\lambda_1}\to U_{k\lambda_1})\to(V_{\lambda_1}\to U_{\lambda_1})\}_{k\in K}$) in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_1}\to X_{\lambda_1}}$ such that g_{nm} (resp. f_k) is the base change of $g_{nm\lambda_1}$ (resp. $f_{k\lambda_1}$) by the transition morphism $(Y\to X)\to(Y_{\lambda_1}\to X_{\lambda_1})$ and that $\mathfrak{U}'_{\lambda_1}$ refines $\mathfrak{U}_{\lambda_1}$. For each $\lambda\geq\lambda_1$, let $g_{nm\lambda}:(V'_{nm\lambda}\to U'_{n\lambda})\to(V_\lambda\to U_\lambda)$ (resp. $f_{k\lambda}:(V_{k\lambda}\to U_{k\lambda})\to(V_\lambda\to U_\lambda)$) be the base change of $g_{nm\lambda_1}$ (resp. $f_{k\lambda_1}$) by the transition morphism $(Y_\lambda\to X_\lambda)\to(Y_{\lambda_1}\to X_{\lambda_1})$. Since the morphisms $\coprod_{n\in N}U'_n\to U$ and $\coprod_{m\in M_n}V'_{nm}\to V\times_U U'_n$ are surjective, there exists an index $\lambda_0\geq\lambda_1$ such that the morphisms $\coprod_{n\in N}U'_{n\lambda_0}\to U_{\lambda_0}$ and $\coprod_{m\in M_n}V'_{nm\lambda_0}\to V_{\lambda_0}\times_{U_{\lambda_0}}U'_{n\lambda_0}$ are also surjective by [EGA IV_3, 8.10.5], that is, $\mathfrak{U}'_{\lambda_0}=\{g_{nm\lambda_0}\}_{n\in N,m\in M_n}$ is a standard covanishing covering in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$. Thus, $\mathfrak{U}_{\lambda_0}=\{f_{k\lambda_0}\}_{k\in K}$ is a covering family in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$. □

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Proposition 7.12 [AGT16, VI.11]. *With the notation in 7.10, $\mathbf{E}^{\text{ét}}_{Y\to X}$ represents the limit of sites $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$, and $\overline{\mathscr{B}} = \operatorname{colim}_{\lambda\in\Lambda}\varphi_\lambda^{-1}\overline{\mathscr{B}}$.*

Proof. The first statement is proved in [AGT16, VI.11.3]. It also follows directly from the discussion in 7.10 and 7.11. For the second statement, notice that $\operatorname{colim}_{\lambda\in\Lambda}\varphi_\lambda^{-1}\overline{\mathscr{B}} = (\operatorname{colim}_{\lambda\in\Lambda}\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})^{\mathrm{a}}$ ([Sta23, 00WI]). It suffices to show that $\overline{\mathscr{B}}(V\to U) = \operatorname{colim}_{\lambda\in\Lambda}(\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})(V\to U)$ for each object $V\to U$ of $\mathbf{E}^{\text{ét}}_{Y\to X}$. It follows from the equivalence (7.10.2) that there exists an index $\lambda_0\in\Lambda$ and an object $V_{\lambda_0}\to U_{\lambda_0}$ of $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$ such that $V\to U$ is the base change of $V_{\lambda_0}\to U_{\lambda_0}$ by the transition morphism. For each $\lambda\geq\lambda_0$, let $V_\lambda\to U_\lambda$ be the base change of $V_{\lambda_0}\to U_{\lambda_0}$ by the transition morphism $(Y_\lambda\to X_\lambda)\to(Y_{\lambda_0}\to X_{\lambda_0})$. Then, we have $\operatorname{colim}_{\lambda\in\Lambda}(\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})(V\to U) = \operatorname{colim}_{\lambda\in\Lambda}\overline{\mathscr{B}}(V_\lambda\to U_\lambda)$ by [SGA 4_{II}, VI 8.5.2, 8.5.7]. The conclusion follows from $\overline{\mathscr{B}}(V\to U) = \operatorname{colim}_{\lambda\in\Lambda}\overline{\mathscr{B}}(V_\lambda\to U_\lambda)$ by 3.18. □

Definition 7.13. A morphism $X\to S$ of coherent schemes is called *pro-étale* (resp. *pro-finite étale*) if there is a directed inverse system of étale (resp. finite étale) S-schemes $(X_\lambda)_{\lambda\in\Lambda}$ with affine transition morphisms such that there is an isomorphism of S-schemes $X\cong\lim_{\lambda\in\Lambda}X_\lambda$. We call such an inverse system $(X_\lambda)_{\lambda\in\Lambda}$ a *pro-étale presentation* (resp. *pro-finite étale presentation*) of X over S.

Lemma 7.14. *Let $X\xrightarrow{g}Y\xrightarrow{f}S$ be morphisms of coherent schemes.*

(1) *If f is pro-étale (resp. pro-finite étale), then f is flat (resp. flat and integral).*
(2) *Any base change of a pro-étale (resp. pro-finite étale) morphism is pro-étale (resp. pro-finite étale).*
(3) *If f and g are pro-étale (resp. pro-finite étale), then so is $f\circ g$.*
(4) *If f and $f\circ g$ are pro-étale (resp. pro-finite étale), then so is g.*
(5) *If f is pro-étale with a pro-étale presentation $Y=\lim Y_\beta$, and if g is étale (resp. finite étale), then there is an index β_0 and an étale (resp. finite étale) S-morphism $g_{\beta_0}: X_{\beta_0}\to Y_{\beta_0}$ such that g is the base change of g_{β_0} by $Y\to Y_{\beta_0}$.*
(6) *Let Z and Z' be coherent schemes pro-étale over S with pro-étale presentations $Z=\lim Z_\alpha$, $Z'=\lim Z'_\beta$, then*

$$\operatorname{Hom}_S(Z,Z') = \lim_\beta\operatorname{colim}_\alpha\operatorname{Hom}_S(Z_\alpha,Z'_\beta). \tag{7.14.1}$$

Proof. (1) and (2) follow directly from the definition.

(3) We follow closely the proof of 3.6. Let $X=\lim X_\alpha$ and $Y=\lim Y_\beta$ be pro-étale (resp. pro-finite étale) presentations over Y and over S, respectively. As Y_β are coherent, for each α, there is an index β_α and an étale (resp. finite étale) Y_{β_α}-scheme $X_{\alpha\beta_\alpha}$ such that $X_\alpha\to Y$ is the base change of $X_{\alpha\beta_\alpha}\to Y_{\beta_\alpha}$ ([EGA IV_3, 8.8.2, 8.10.5], [EGA IV_4, 17.7.8]). For each $\beta\geq\beta_\alpha$, let $X_{\alpha\beta}\to Y_\beta$ be the base change of $X_{\alpha\beta_\alpha}\to Y_{\beta_\alpha}$ by $Y_\beta\to Y_{\beta_\alpha}$. Then, we have $X=\lim_{\alpha,\beta\geq\beta_\alpha}X_{\alpha\beta}$ by [EGA IV_3, 8.8.2] (cf. 3.6), which is pro-finite étale over S. For (5), one can take $X=X_\alpha$.

(6) We have

$$\operatorname{Hom}_S(Z,Z') = \lim_\beta\operatorname{Hom}_S(Z,Z'_\beta) = \lim_\beta\operatorname{colim}_\alpha\operatorname{Hom}_S(Z_\alpha,Z'_\beta) \tag{7.14.2}$$

where the first equality follows from the universal property of limits of schemes, and the second follows from the fact that $Z'_\beta\to S$ is locally of finite presentation ([EGA IV_3, 8.14.2]). For (4), we take $Z=X$ and $Z'=Y$. Then, for each index β, we have an S-morphism $X_\alpha\to Y_\beta$ for α big enough, which is also étale (resp. finite étale). Then, $X=\lim_\alpha X_\alpha=\lim_{\alpha,\beta}X_\alpha\times_{Y_\beta}Y$ is pro-étale (resp. pro-finite étale) over Y. □

Remark 7.15. A pro-étale (resp. pro-finite étale) morphism of $\mathbb{U}$-small coherent schemes $X\to S$ admits a $\mathbb{U}$-small pro-étale (resp. pro-finite étale) presentation. Indeed, let $X=\lim_{\lambda\in\Lambda}X_\lambda$ be a presentation of $X\to S$. We may regard Λ as a filtered category with an initial object 0. Consider the category

$\mathscr{C} = {}_{X\backslash}X_{0,\text{ét,aff}}$ (resp. $\mathscr{C} = {}_{X\backslash}X_{0,\text{fét}}$) of affine (resp. finite) étale X_0-schemes which are under X. Notice that $\mathscr{C}$ is essentially $\mathbb{U}$-small and that the natural functor $\Lambda \to \mathscr{C}^{\text{op}}$ is cofinal by 7.14.(6) ([SGA 4_{I}, I.8.1.3]). Therefore, after replacing $\mathscr{C}^{\text{op}}$ by a $\mathbb{U}$-small directed set Λ', we obtain a $\mathbb{U}$-small presentation $X = \lim_{X' \in \Lambda'} X'$ ([SGA 4_{I}, I.8.1.6]).

Definition 7.16. For any $\mathbb{U}$-small coherent scheme X, we endow the category of $\mathbb{U}$-small coherent pro-étale (resp. pro-finite étale) X-schemes with the topology generated by the pretopology formed by families of morphisms

$$\{f_i : U_i \to U\}_{i \in I} \tag{7.16.1}$$

such that I is finite and that $U = \bigcup f_i(U_i)$. This defines a site $X_{\text{proét}}$ (resp. $X_{\text{profét}}$), called the *pro-étale site* (resp. *pro-finite étale site*) of X.

It is clear that the localization $X_{\text{proét}/U}$ (resp. $X_{\text{profét}/U}$) of $X_{\text{proét}}$ (resp. $X_{\text{profét}}$) at an object U is canonically equivalent to the pro-étale (resp. pro-finite étale) site $U_{\text{proét}}$ (resp. $U_{\text{profét}}$) of U. By definition, any object in $X_{\text{proét}}$ (resp. $X_{\text{profét}}$) is quasi-compact.

7.17. We compare our definitions of pro-étale site and pro-finite étale site with some other definitions existing in the literature. But we don't use the comparison result in this paper.

Let X be a $\mathbb{U}$-small Noetherian scheme. Consider the category of pro-objects pro-$X_{\text{fét}}$ of $X_{\text{fét}}$, that is, the category whose objects are functors $F : \mathcal{A} \to X_{\text{fét}}$ with $\mathcal{A}$ a $\mathbb{U}$-small cofiltered category and whose morphisms are given by $\operatorname{Hom}(F, G) = \lim_{\beta \in \mathcal{B}} \operatorname{colim}_{\alpha \in \mathcal{A}} \operatorname{Hom}(F(\alpha), G(\beta))$ for any $F : \mathcal{A} \to X_{\text{fét}}$ and $G : \mathcal{B} \to X_{\text{fét}}$ ([Sch13, 3.2]). We may simply denote such a functor F by $(X_\alpha)_{\alpha \in A}$. Remark that $\lim_{\alpha \in \mathcal{A}} X_\alpha$ exists which is pro-finite étale over X. Consider the functor

$$\text{pro-}X_{\text{fét}} \longrightarrow X_{\text{profét}}, \ (X_\alpha)_{\alpha \in A} \mapsto \lim_{\alpha \in \mathcal{A}} X_\alpha, \tag{7.17.1}$$

which is well defined and fully faithful by 7.14.(6) and essentially surjective by 7.15. Thus, according to [Sch13, 3.3] and its corrigendum [Sch16], Scholze's pro-finite étale site $X^{\text{S}}_{\text{profét}}$ has the underlying category $X_{\text{profét}}$ and its topology is generated by the families of morphisms

$$\{U_i \xrightarrow{f_i} U' \xrightarrow{f} U\}_{i \in I}, \tag{7.17.2}$$

where f is a morphism in the category $X_{\text{profét}}$, I is finite and $\coprod_{i \in I} f_i : \coprod_{i \in I} U_i \to U'$ is finite étale surjective, and there exists a $\mathbb{U}$-small well-ordered directed set Λ with a least index 0 and a directed inverse system of $\mathbb{U}$-small coherent pro-finite étale X-schemes $(U'_\lambda)_{\lambda \in \Lambda}$ such that $U = U'_0$, $U' = \lim_{\lambda \in \Lambda} U'_\lambda$ and that for each $\lambda \in \Lambda$ the natural morphism $U'_\lambda \to \lim_{\mu < \lambda} U'_\mu$ is finite étale surjective (cf. [Ker16, 5.5], 7.14 and [EGA IV_3, 8.10.5.(vi)]). It is clear that the topology of our pro-finite étale site $X_{\text{profét}}$ is finer than that of $X^{\text{S}}_{\text{profét}}$. We remark that if X is connected, then $X^{\text{S}}_{\text{profét}}$ gives a site-theoretic interpretation of the continuous group cohomology of the fundamental group of X ([Sch13, 3.7]). For simplicity, we don't consider $X^{\text{S}}_{\text{profét}}$ in the rest of the paper, but we can replace $X_{\text{profét}}$ by $X^{\text{S}}_{\text{profét}}$ for most of the statements in this paper (see [Ker16, 6]).

7.18. Let X be a $\mathbb{U}$-small scheme. Bhatt–Scholze's pro-étale site $X^{\text{BS}}_{\text{proét}}$ has the underlying category of $\mathbb{U}$-small weakly étale X-schemes and a family of morphisms $\{f_i : Y_i \to Y\}_{i \in I}$ in $X^{\text{BS}}_{\text{proét}}$ is a covering if and only if for any affine open subscheme U of Y, there exists a map $a : \{1, \dots, n\} \to I$ and affine open subschemes U_j of $Y_{a(j)}$ $(j = 1, \dots, n)$ such that $U = \bigcup_{j=1}^n f_{a(j)}(U_j)$ ([BS15, 4.1.1], cf. [Sta23, 0989]). Remark that a pro-étale morphism of coherent schemes is weakly étale by [BS15, 2.3.3.1]. Thus, for a coherent scheme X, $X_{\text{proét}}$ is a full subcategory of $X^{\text{BS}}_{\text{proét}}$.

Lemma 7.19. *Let X be a coherent scheme. The full subcategory $X_{\text{proét}}$ of $X^{\text{BS}}_{\text{proét}}$ is a topologically generating family, and the induced topology on $X_{\text{proét}}$ coincides the topology defined in 7.16. In particular, the topoi of sheaves of $\mathbb{V}$-small sets associated to the two sites are naturally equivalent.*

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Proof. For a weakly étale X-scheme Y, we show that it can be covered by pro-étale X-schemes. After replacing X by a finite affine open covering and replacing Y by an affine open covering, we may assume that X and Y are affine. Then, the result follows from the fact that for any weakly étale morphism of rings $A \to B$ there exists a faithfully flat ind-étale morphism $B \to C$ such that $A \to C$ is ind-étale by [BS15, 2.3.4] (cf. [BS15, 4.1.3]). Thus, $X_{\text{proét}}$ is a topologically generating family of $X^{\text{BS}}_{\text{proét}}$. A family of morphisms $\{f_i : Y_i \to Y\}_{i\in I}$ in $X_{\text{proét}}$ is a covering with respect to the induced topology if and only if for any affine open subscheme U of Y, there exists a map $a : \{1,\dots,n\} \to I$ and affine open subschemes U_j of $Y_{a(j)}$ $(j = 1,\dots,n)$ such that $U = \bigcup_{j=1}^n f_{a(j)}(U_j)$ ([SGA 4_{I}, III.3.3]). Notice that Y_i and Y are coherent, thus $\{f_i\}_{i\in I}$ is a covering if and only if there exists a finite subset $I_0 \subseteq I$ such that $Y = \bigcup_{i\in I_0} f_i(Y_i)$, which shows that the induced topology on $X_{\text{proét}}$ coincides the topology defined in 7.16. Finally, the 'in particular' part follows from [SGA 4_{I}, III.4.1]. □

Definition 7.20. Let $Y \to X$ be a morphism of coherent schemes. A morphism $(V' \to U') \to (V \to U)$ in $\mathbf{E}_{Y\to X}$ is called *pro-étale* if $U' \to U$ is pro-étale and $V' \to V \times_U U'$ is pro-finite étale. A *pro-étale presentation* of such a morphism is a directed inverse system $(V_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ étale over $V \to U$ with affine transition morphisms $U_{\lambda'} \to U_\lambda$ and $V_{\lambda'} \to V_\lambda$ $(\lambda' \geq \lambda)$ such that $(V' \to U') = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$.

Lemma 7.21. *Let $Y \to X$ be a morphism of coherent schemes, $(V'' \to U'') \xrightarrow{g} (V' \to U') \xrightarrow{f} (V \to U)$ morphisms in $\mathbf{E}_{Y\to X}$.*

(1) *If f is pro-étale, then it admits a pro-étale presentation.*
(2) *If f is pro-étale, then any base change of f is also pro-étale.*
(3) *If f and g are pro-étale, then $f \circ g$ is also pro-étale.*
(4) *If f and $f \circ g$ are pro-étale, then g is also pro-étale.*

Proof. It follows directly from 7.14 and its arguments. □

Remark 7.22. Similar to 7.15, a pro-étale morphism in $\mathbf{E}_{Y\to X}$ admits a $\mathbb{U}$-small presentation.

7.23. Let $Y \to X$ be a morphism of coherent schemes, $\mathbf{E}^{\text{proét}}_{Y\to X}$ the full subcategory of $\mathbf{E}_{Y\to X}$ formed by objects which are pro-étale over the final object $Y \to X$. It is clear that $\mathbf{E}^{\text{proét}}_{Y\to X}$ is stable under finite limits in $\mathbf{E}_{Y\to X}$. Then, the functor (7.1.2) induces a functor

$$\phi^+ : \mathbf{E}^{\text{proét}}_{Y\to X} \longrightarrow X_{\text{proét}},\ (V \to U) \longmapsto U, \tag{7.23.1}$$

which endows $\mathbf{E}^{\text{proét}}_{Y\to X}/X_{\text{proét}}$ with a structure of fibred sites, whose fibre over U is the pro-finite étale site $U_{Y,\text{profét}}$. We endow $\mathbf{E}^{\text{proét}}_{Y\to X}$ with the associated covanishing topology, that is, the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N} U_n \to U$ is surjective.

It is clear that any object in $\mathbf{E}^{\text{proét}}_{Y\to X}$ is quasi-compact by 6.4. We still denote by $\overline{\mathscr{B}}$ the restriction of the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}_{Y\to X}$ to $\mathbf{E}^{\text{proét}}_{Y\to X}$ if there is no ambiguity. We will show in 7.32 that $\overline{\mathscr{B}}$ is a sheaf on $\mathbf{E}^{\text{proét}}_{Y\to X}$.

Definition 7.24. We call $(\mathbf{E}^{\text{proét}}_{Y\to X}, \overline{\mathscr{B}})$ the *pro-étale Faltings ringed site* of the morphism of coherent schemes $Y \to X$.

It is clear that the localization $(\mathbf{E}^{\text{proét}}_{Y\to X})_{/(V\to U)}$ of $\mathbf{E}^{\text{proét}}_{Y\to X}$ at an object $V \to U$ is canonically equivalent to the pro-étale Faltings ringed site $\mathbf{E}^{\text{proét}}_{V\to U}$ of the morphism $V \to U$ by 6.4.

Remark 7.25. The categories $X_{\text{proét}}$, $X_{\text{profét}}$ and $\mathbf{E}^{\text{proét}}_{Y\to X}$ are essentially $\mathbb{V}$-small categories.

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7.26. Let $Y \to X$ be a morphism of coherent schemes. Consider the natural functors

$$\psi^+ : \mathbf{E}^{\text{proét}}_{Y\to X} \longrightarrow Y_{\text{proét}},\ (V \to U) \longmapsto V, \tag{7.26.1}$$

$$\beta^+ : Y_{\text{profét}} \longrightarrow \mathbf{E}^{\text{proét}}_{Y\to X},\ V \longmapsto (V \to X), \tag{7.26.2}$$

$$\sigma^+ : X_{\text{proét}} \longrightarrow \mathbf{E}^{\text{proét}}_{Y\to X},\ U \longmapsto (Y \times_X U \to U). \tag{7.26.3}$$

They are left exact and continuous (cf. 7.8). Then, we obtain a commutative diagram of sites associated functorially to the morphism $Y \to X$ by 2.5,

$$\begin{tikzcd} & Y_{\text{proét}} \arrow[dl, "\rho"'] \arrow[d, "\psi"] \arrow[dr] & \\ Y_{\text{profét}} & \mathbf{E}^{\text{proét}}_{Y\to X} \arrow[l, "\beta"] \arrow[r, "\sigma"'] & X_{\text{proét}}, \end{tikzcd} \tag{7.26.4}$$

where $\rho : Y_{\text{proét}} \to Y_{\text{profét}}$ is defined by the inclusion functor, and the unlabelled arrow $Y_{\text{proét}} \to X_{\text{proét}}$ is induced by the morphism $Y \to X$. Moreover, if $\mathcal{O}_{X_{\text{proét}}}$ denotes the structural sheaf on $X_{\text{proét}}$ sending U to $\Gamma(U, \mathcal{O}_U)$, then σ^+ actually defines a morphism of ringed sites

$$\sigma : (\mathbf{E}^{\text{proét}}_{Y\to X}, \overline{\mathscr{B}}) \longrightarrow (X_{\text{proét}}, \mathcal{O}_{X_{\text{proét}}}). \tag{7.26.5}$$

Lemma 7.27. *Let $Y \to X$ be a morphism of coherent schemes. Then, the inclusion functor*

$$\nu^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow \mathbf{E}^{\text{proét}}_{Y\to X},\ (V \to U) \longmapsto (V \to U) \tag{7.27.1}$$

defines a morphism of sites $\nu : \mathbf{E}^{\text{proét}}_{Y\to X} \to \mathbf{E}^{\text{ét}}_{Y\to X}$.

Proof. It is clear that ν^+ commutes with finite limits and sends a standard covanishing covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$ to a standard covanishing covering in $\mathbf{E}^{\text{proét}}_{Y\to X}$ (6.3). Therefore, ν^+ is continuous by 6.4 and defines a morphism of sites. □

Lemma 7.28. *Let $Y \to X$ be a morphism of coherent schemes. Then, the topology on $\mathbf{E}^{\text{ét}}_{Y\to X}$ is the topology induced from $\mathbf{E}^{\text{proét}}_{Y\to X}$.*

Proof. After 7.27, it suffices to show that for a family of morphisms $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$, if $\nu^+(\mathfrak{U})$ is a covering in $\mathbf{E}^{\text{proét}}_{Y\to X}$, then $\mathfrak{U}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$. We may assume that K is finite. There is a standard covanishing covering $\mathfrak{U}' = \{(V'_{nm} \to U'_n) \to (V \to U)\}_{n\in N, m\in M_n}$ in $\mathbf{E}^{\text{proét}}_{Y\to X}$ with N, M_n finite, which refines $\nu^+(\mathfrak{U})$ by 6.4. We take a directed set Ξ such that for each $n \in N$, we can take a pro-étale presentation $U'_n = \lim_{\xi\in\Xi} U'_{n\xi}$ over U, and we take a directed set Σ such that for each $n \in N$ and $m \in M_n$, we can take a pro-finite étale presentation $V'_{nm} = \lim_{\sigma\in\Sigma} V'_{nm\sigma}$ over $V \times_U U'_n$. By 7.14 (5), for each $\sigma \in \Sigma$, there exists an index $\xi_\sigma \in \Xi$ and a finite étale morphism $V'_{nm\sigma\xi_\sigma} \to V \times_U U'_{n\xi_\sigma}$ for each n and m, whose base change by $U'_n \to U'_{n\xi_\sigma}$ is $V'_{nm\sigma} \to V \times_U U'_n$. Let $V'_{nm\sigma\xi} \to V \times_U U'_{n\xi}$ be the base change of $V'_{nm\sigma\xi_\sigma} \to V \times_U U'_{n\xi_\sigma}$ by the transition morphism $U'_{n\xi} \to U'_{n\xi_\sigma}$ for each $\xi \geq \xi_\sigma$. Since $\coprod_{m\in M_n} V'_{nm\sigma} \to V \times_U U'_n$ is surjective, after enlarging ξ_σ, we may assume that $\coprod_{m\in M_n} V'_{nm\sigma\xi} \to V \times_U U'_{n\xi}$ is also surjective for $\xi \geq \xi_\sigma$ by [EGA IV$_3$, 8.10.5.(vi)]. It is clear that $\coprod_{n\in N} U'_{n\xi} \to U$ is surjective for each $\xi \in \Xi$. Therefore, $\mathfrak{U}'_{\sigma\xi} = \{(V'_{nm\sigma\xi} \to U'_{n\xi}) \to (V \to U)\}_{n\in N, m\in M_n}$ is a standard covanishing covering in $\mathbf{E}^{\text{ét}}_{V\to U}$ for each $\sigma \in \Sigma$ and $\xi \geq \xi_\sigma$. Notice that for each $n \in N$ and $m \in M_n$, there exists $k \in K$ such that the morphism $(V'_{nm\sigma\xi} \to U'_{n\xi}) \to (V \to U)$ factors through $(V_k \to U_k)$ for σ, ξ big enough by 7.14 (6), which shows that $\mathfrak{U}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$. □

Lemma 7.29. *Let $Y \to X$ be a morphism of coherent schemes, $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ a covering in $\mathbf{E}^{\text{proét}}_{Y\to X}$ with K finite. Then, there exist pro-étale presentations $(V \to U) = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$,*

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$(V_k \to U_k) = \lim_{\lambda\in\Lambda}(V_{k\lambda} \to U_{k\lambda})$ *over* $Y \to X$ *and compatible étale morphisms* $(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)$ *such that the family* $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ *is a covering in* $\mathbf{E}^{\text{ét}}_{Y\to X}$.

Proof. We follow closely the proof of 3.6. We take a directed set A such that for each $k \in K$ we can take a pro-étale presentation $(V_k \to U_k) = \lim_{\alpha\in A}(V_{k\alpha} \to U_{k\alpha})$ over $(V \to U)$. Then, $\mathfrak{U}_\alpha = \{(f_{k\alpha} : V_{k\alpha} \to U_{k\alpha}) \to (V \to U)\}_{k\in K}$ is a covering family in $\mathbf{E}^{\text{ét}}_{V\to U}$ for each $\alpha \in A$ by 7.28.

Let $(V \to U) = \lim_{\beta\in B}(V_\beta \to U_\beta)$ be a pro-étale presentation over $Y \to X$. For each $\alpha \in A$, there exists an index $\beta_\alpha \in B$ and a covering family $\mathfrak{U}_{\alpha\beta_\alpha} = \{f_{k\alpha\beta_\alpha} : (V_{k\alpha\beta_\alpha} \to U_{k\alpha\beta_\alpha}) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})\}_{k\in K}$ such that $f_{k\alpha}$ is the base change of $f_{k\alpha\beta_\alpha}$ by the transition morphism $(V \to U) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})$ (7.11). For each $\beta \geq \beta_\alpha$, let $f_{k\alpha\beta} : (V_{k\alpha\beta} \to U_{k\alpha\beta}) \to (V_\beta \to U_\beta)$ be the base change of $f_{k\alpha\beta_\alpha}$ by the transition morphism $(V_\beta \to U_\beta) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})$. We take $\Lambda = \{(\alpha,\beta) \in A \times B \mid \beta \geq \beta_\alpha\}$, $(V_\lambda \to U_\lambda) = (V_\beta \to U_\beta)$ and $(V_{k\lambda} \to U_{k\lambda}) = (V_{k\alpha\beta} \to U_{k\alpha\beta})$ for each $\lambda = (\alpha,\beta) \in \Lambda$. Then, the families $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ meet the requirements in the lemma (cf. 3.6). □

Lemma 7.30. *Let* $Y \to X$ *be a morphism of coherent schemes,* $\mathcal{F}$ *a presheaf on* $\mathbf{E}^{\text{ét}}_{Y\to X}$, $V \to U$ *an object of* $\mathbf{E}^{\text{proét}}_{Y\to X}$ *with a pro-étale presentation* $(V \to U) = \lim(V_\lambda \to U_\lambda)$. *Then, we have* $\nu_{\mathrm{p}}\mathcal{F}(V \to U) = \operatorname{colim} \mathcal{F}(V_\lambda \to U_\lambda)$, *where* $\nu^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{E}^{\text{proét}}_{Y\to X}$ *is the inclusion functor.*

Proof. Notice that the presheaf $\mathcal{F}$ is a filtered colimit of representable presheaves by [SGA 4_{I}, I.3.4]

$$\mathcal{F} = \operatorname*{colim}_{(V'\to U')\in(\mathbf{E}^{\text{ét}}_{Y\to X})_{/\mathcal{F}}} h^{\text{ét}}_{V'\to U'}. \tag{7.30.1}$$

Thus, we may assume that $\mathcal{F}$ is representable by $V' \to U'$ since the section functor $\Gamma(V \to U, -)$ commutes with colimits of presheaves ([Sta23, 00VB]). Then, we have

$$\begin{aligned}\nu_{\mathrm{p}} h^{\text{ét}}_{V'\to U'}(V \to U) &= h^{\text{proét}}_{V'\to U'}(V \to U) \\ &= \operatorname{Hom}_{\mathbf{E}^{\text{proét}}_{Y\to X}}((V \to U), (V' \to U')) \\ &= \operatorname{colim} \operatorname{Hom}_{\mathbf{E}^{\text{ét}}_{Y\to X}}((V_\lambda \to U_\lambda), (V' \to U')) \\ &= \operatorname{colim} h^{\text{ét}}_{V'\to U'}(V_\lambda \to U_\lambda),\end{aligned} \tag{7.30.2}$$

where the first equality follows from [Sta23, 04D2], and the third equality follows from [EGA IV_3, 8.14.2] since U' and V' are locally of finite presentation over X and $Y \times_X U'$, respectively. □

Proposition 7.31. *Let* $Y \to X$ *be a morphism of coherent schemes,* $\mathcal{F}$ *an abelian sheaf on* $\mathbf{E}^{\text{ét}}_{Y\to X}$, $V \to U$ *an object of* $\mathbf{E}^{\text{proét}}_{Y\to X}$ *with a pro-étale presentation* $(V \to U) = \lim(V_\lambda \to U_\lambda)$. *Then, for any integer* q, *we have*

$$H^q(\mathbf{E}^{\text{proét}}_{V\to U}, \nu^{-1}\mathcal{F}) = \operatorname{colim} H^q(\mathbf{E}^{\text{ét}}_{V_\lambda\to U_\lambda}, \mathcal{F}), \tag{7.31.1}$$

where $\nu : \mathbf{E}^{\text{proét}}_{Y\to X} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ *is the morphism of sites defined by the inclusion functor (7.27). In particular, the canonical morphism* $\mathcal{F} \longrightarrow \mathrm{R}\nu_*\nu^{-1}\mathcal{F}$ *is an isomorphism.*

Proof. We follow closely the proof of 3.8. For the second assertion, since $\mathrm{R}^q\nu_*\nu^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $(V \to U) \mapsto H^q(\mathbf{E}^{\text{proét}}_{V\to U}, \nu^{-1}\mathcal{F}) = H^q(\mathbf{E}^{\text{ét}}_{V\to U}, \mathcal{F})$ by the first assertion, which is $\mathcal{F}$ if $q = 0$ and vanishes otherwise.

For the first assertion, we may assume that $\mathcal{F} = \mathcal{I}$ is an abelian injective sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. 3.8). We claim that for any covering in $\mathbf{E}^{\text{proét}}_{Y\to X}$, $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ with K finite, the augmented Čech complex associated to the presheaf $\nu_{\mathrm{p}}\mathcal{I}$,

$$0 \to \nu_{\mathrm{p}}\mathcal{I}(V \to U) \to \prod_k \nu_{\mathrm{p}}\mathcal{I}(V_k \to U_k) \to \prod_{k,k'} \nu_{\mathrm{p}}\mathcal{I}(V_k \times_V V_{k'} \to U_k \times_U U_{k'}) \to \cdots \tag{7.31.2}$$

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is exact. Admitting this claim, we see that $\nu_{\mathrm{p}}\mathcal{I}$ is indeed a sheaf, that is, $\nu^{-1}\mathcal{I} = \nu_{\mathrm{p}}\mathcal{I}$, and the vanishing of higher Čech cohomologies implies that $H^q(\mathbf{E}^{\mathrm{proét}}_{V\to U}, \nu^{-1}\mathcal{I}) = 0$ for any $q > 0$, which completes the proof together with 7.30. For the claim, let $(V \to U) = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$ and $(V_k \to U_k) = \lim_{\lambda\in\Lambda}(V_{k\lambda} \to U_{k\lambda})$ be the pro-étale presentations constructed in 7.29. The family $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ is a covering in $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$. By 7.30, the sequence (7.31.2) is the filtered colimit of the augmented Čech complexes

$$0 \to \mathcal{I}(V_\lambda \to U_\lambda) \to \prod_k \mathcal{I}(V_{k\lambda} \to U_{k\lambda}) \to \prod_{k,k'} \mathcal{I}(V_{k\lambda}\times_{V_\lambda} V_{k'\lambda} \to U_{k\lambda}\times_{U_\lambda} U_{k'\lambda}) \to \cdots \tag{7.31.3}$$

which are exact since $\mathcal{I}$ is an injective abelian sheaf on $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$. □

Corollary 7.32. *With the notation in* 7.31, *the presheaf* $\overline{\mathscr{B}}$ *on* $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ *is a sheaf, and the canonical morphisms* $\nu^{-1}\overline{\mathscr{B}} \to \overline{\mathscr{B}}$ *and* $\overline{\mathscr{B}} \to \mathrm{R}\nu_*\overline{\mathscr{B}}$ *are isomorphisms. If moreover* p *is invertible on* Y*, then for each integer* $n \geq 0$*, the canonical morphisms* $\nu^{-1}(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \to \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}$ *and* $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \mathrm{R}\nu_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$ *are isomorphisms.*

Proof. For any pro-étale presentation $(V \to U) = \lim(V_\lambda \to U_\lambda)$, we have $\nu^{-1}\overline{\mathscr{B}}(V \to U) = \operatorname{colim}\overline{\mathscr{B}}(V_\lambda \to U_\lambda) = \overline{\mathscr{B}}(V \to U)$ by 7.30 and 3.18. This verifies that $\overline{\mathscr{B}}$ is a sheaf on $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ and that $\nu^{-1}\overline{\mathscr{B}} \to \overline{\mathscr{B}}$ is an isomorphism. The second isomorphism follows from the first and 7.31. For the last assertion, notice that the multiplication by p^n is injective on $\overline{\mathscr{B}}$ so that the conclusion follows from the exact sequence

$$0 \longrightarrow \overline{\mathscr{B}} \xrightarrow{\cdot p^n} \overline{\mathscr{B}} \longrightarrow \overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \longrightarrow 0. \tag{7.32.1}$$

□

8. Cohomological descent of the structural sheaves

Definition 8.1. Let K be a pre-perfectoid field of mixed characteristic $(0, p)$, $Y \to X$ a morphism of coherent schemes such that $Y \to X^Y$ is over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$, where X^Y denotes the integral closure of X in Y. We say that $Y \to X$ is *Faltings acyclic* if X is affine and if the canonical morphism

$$A/pA \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{proét}}_{Y\to X}, \overline{\mathscr{B}}/p\overline{\mathscr{B}}) \tag{8.1.1}$$

is an almost isomorphism (see 5.7), where A denotes the $\mathcal{O}_K$-algebra $\overline{\mathscr{B}}(Y \to X)$ (i.e., $X^Y = \mathrm{Spec}(A)$).

Remark 8.2. In 8.1, the canonical morphism $\mathrm{R}\Gamma(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p\overline{\mathscr{B}}) \to \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{proét}}_{Y\to X}, \overline{\mathscr{B}}/p\overline{\mathscr{B}})$ is an isomorphism by 7.32.

Lemma 8.3. *Let* K *be a pre-perfectoid field of mixed characteristic* $(0, p)$*,* C *a site,* $\mathcal{O}$ *a sheaf of flat* $\mathcal{O}_K$*-algebras over* C*,* $A = \Gamma(C, \mathcal{O})$*. Then, the following conditions are equivalent:*

(1) *For a pseudo-uniformizer* π *of* K*, the canonical morphism* $A/\pi A \to \mathrm{R}\Gamma(C, \mathcal{O}/\pi\mathcal{O})$ *is an (resp. almost) isomorphism.*
(2) *For any pseudo-uniformizer* π' *of* K*, the canonical morphism* $A/\pi' A \to \mathrm{R}\Gamma(C, \mathcal{O}/\pi'\mathcal{O})$ *is an (resp. almost) isomorphism.*

Proof. It suffices to show (1) ⇒ (2). As A and $\mathcal{O}$ are flat over $\mathcal{O}_K$, for any integer $n > 0$, we have canonical short exact sequences $0 \to A/\pi A \to A/\pi^n A \to A/\pi^{n-1}A \to 0$ and $0 \to \mathcal{O}/\pi\mathcal{O} \to \mathcal{O}/\pi^n\mathcal{O} \to \mathcal{O}/\pi^{n-1}\mathcal{O} \to 0$. Thus, we deduce easily the statement for $\pi' = \pi^n$ by dévissage.

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For a general pseudo-uniformizer π', we take an integer $n > 0$ such that $\pi'' = \pi^n/\pi'$ is a pseudo-uniformizer of K. As A and $\mathcal{O}$ are flat over $\mathcal{O}_K$, we have a natural morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A/\pi'' A \arrow[r] \arrow[d, "\alpha_1"] & A/\pi^n A \arrow[r] \arrow[d, "\alpha_2"] & A/\pi' A \arrow[r] \arrow[d, "\alpha_3"] & 0 \\ 0 \arrow[r] & H^0(C, \mathcal{O}/\pi''\mathcal{O}) \arrow[r] & H^0(C, \mathcal{O}/\pi^n\mathcal{O}) \arrow[r] & H^0(C, \mathcal{O}/\pi'\mathcal{O}). \end{tikzcd} \tag{8.3.1}$$

By the discussion above, α_2 is an (resp. almost) isomorphism. Thus, α_1 is (resp. almost) injective. Since any pseudo-uniformizer of K is of the form $\pi'' = \pi^n/\pi'$ for some pseudo-uniformizer π' of K and some integer $n > 0$, α_3 is (resp. almost) injective. By diagram chasing, we see that α_1 is an (resp. almost) isomorphism (and thus so is α_3). It remains to show that $H^q(C, \mathcal{O}/\pi'\mathcal{O})$ is (resp. almost) zero for $q > 0$. Recall that $H^q(C, \mathcal{O}/\pi^n\mathcal{O})$ is (resp. almost) zero. By the long exact sequence associated to the short exact sequence $0 \to \mathcal{O}/\pi''\mathcal{O} \to \mathcal{O}/\pi^n\mathcal{O} \to \mathcal{O}/\pi'\mathcal{O} \to 0$, we see that $H^1(C, \mathcal{O}/\pi''\mathcal{O})$ is (resp. almost) zero and that $H^q(C, \mathcal{O}/\pi'\mathcal{O}) \to H^{q+1}(C, \mathcal{O}/\pi''\mathcal{O})$ is an (resp. almost) isomorphism. Hence, we complete the proof by induction. □

Remark 8.4. With the notation in 8.1, we deduce from 8.3 that $Y \to X$ is Faltings acyclic if and only if X is affine and $A/\pi A \to R\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi\overline{\mathscr{B}})$ is an almost isomorphism for any pseudo-uniformizer π of K. We will give a criterion for being 'Faltings acyclic' in terms of 'almost pre-perfectoid' in 8.24.

Lemma 8.5. *Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. Then, the functor*

$$\epsilon^+ : \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X} \longrightarrow \mathbf{I}_{Y\to X^Y},\ (V \to U) \longmapsto U^V, \tag{8.5.1}$$

is well defined, left exact and continuous. Moreover, we have $Y \times_{X^Y} U^V = V$.

Proof. Since $U' = X^Y \times_X U$ is integral over U, we have $U^V = U'^V$. Applying 3.19.(4) to $V \to U'$ over $Y \to X^Y$, we see that the X^Y-scheme U^V is Y-integrally closed with $Y \times_{X^Y} U^V = V$, and thus the functor ϵ^+ is well defined. Let $(V_1 \to U_1) \to (V_0 \to U_0) \leftarrow (V_2 \to U_2)$ be a diagram in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$. By 3.21, $U_1^{V_1}\overline{\times}_{U_0^{V_0}} U_2^{V_2} = (U_1^{V_1} \times_{U_0^{V_0}} U_2^{V_2})^{V_1\times_{V_0}V_2} = (U_1 \times_{U_0} U_2)^{V_1\times_{V_0}V_2}$ which shows the left exactness of ϵ^+. For the continuity, notice that any covering in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ can be refined by a standard covanishing covering (6.4). For a Cartesian covering family $\mathfrak{U} = \{(V\times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$ with N finite, we apply 3.15 to the commutative diagram

$$\begin{tikzcd} \coprod_{n\in N} V\times_U U_n \arrow[r] \arrow[d] & \coprod_{n\in N} U_n^{V\times_U U_n} \arrow[r] \arrow[d] & \coprod_{n\in N} U_n \arrow[d] \\ V \arrow[r] & U^V \arrow[r] & U \end{tikzcd} \tag{8.5.2}$$

then we see that $\epsilon^+(\mathfrak{U})$ is a covering family in $\mathbf{I}_{Y\to X^Y}$. For a vertical covering family $\mathfrak{U} = \{(V_m \to U) \to (V \to U)\}_{m\in M}$ with M finite, we apply 3.15 to the commutative diagram

$$\begin{tikzcd} \coprod_{m\in M} V_m \arrow[r] \arrow[d] & \coprod_{m\in M} U^{V_m} \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U^V \arrow[r] & U \end{tikzcd} \tag{8.5.3}$$

then we see that $\epsilon^+(\mathfrak{U})$ is also a covering family in $\mathbf{I}_{Y\to X^Y}$. □

8.6. Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. Then, there are morphisms of sites

$$\epsilon : \mathbf{I}_{Y\to X^Y} \longrightarrow \mathbf{E}^{\text{proét}}_{Y\to X}, \tag{8.6.1}$$

$$\varepsilon : \mathbf{I}_{Y\to X^Y} \longrightarrow \mathbf{E}^{\text{ét}}_{Y\to X} \tag{8.6.2}$$

defined by Equation (8.5.1) and the composition of Equation (8.5.1) with Equation (7.27.1), respectively. We temporarily denote by $\mathscr{O}^{\text{pre}}$ the presheaf on $\mathbf{I}_{Y\to X^Y}$ sending W to $\Gamma(W, \mathcal{O}_W)$ (thus $\mathscr{O} = (\mathscr{O}^{\text{pre}})^{\text{a}}$). Notice that we have $\overline{\mathscr{B}} = \epsilon^{\text{p}}\mathscr{O}^{\text{pre}}$ (resp. $\overline{\mathscr{B}} = \varepsilon^{\text{p}}\mathscr{O}^{\text{pre}}$). The canonical morphism $\epsilon^{\text{p}}\mathscr{O}^{\text{pre}} \to \epsilon^{\text{p}}\mathscr{O}$ (resp. $\varepsilon^{\text{p}}\mathscr{O}^{\text{pre}} \to \varepsilon^{\text{p}}\mathscr{O}$) induces a canonical morphism $\overline{\mathscr{B}} \to \epsilon_*\mathscr{O}$ (resp. $\overline{\mathscr{B}} \to \varepsilon_*\mathscr{O}$).

8.7. Let K be a pre-perfectoid field (5.1) of mixed characteristic $(0, p)$, $\eta = \operatorname{Spec}(K)$, $S = \operatorname{Spec}(\mathcal{O}_K)$, $Y \to X$ a morphism of coherent schemes such that X^Y is an S-scheme with generic fibre $(X^Y)_\eta = Y$. In particular, X^Y is an object of $\mathbf{I}_{\eta\to S}$.

Lemma 8.8. *For any ring R, there is an R-algebra R_∞ satisfying the following conditions:*

(i) *The scheme* $\operatorname{Spec}(R_\infty[1/p])$ *is pro-finite étale and faithfully flat over* $\operatorname{Spec}(R[1/p])$.
(ii) *The R-algebra R_∞ is the integral closure of R in $R_\infty[1/p]$.*
(iii) *Any unit t of R_∞ admits a p-th root $t^{1/p}$ in R_∞.*

Moreover, if p lies in the Jacobson radical $J(R)$ of R, and if there is a p^2-th root $p_2 \in R$ of p up to a unit (cf. 5.4) and we write $p_1 = p_2^p$, then we may require further that

(iv) *the Frobenius of R_∞/pR_∞ induces an isomorphism $R_\infty/p_1R_\infty \to R_\infty/pR_\infty$.*

Proof. Setting $B_0 = R[1/p]$, we construct inductively a ring B_{n+1} ind-finite étale over B_n and we denote by R_n the integral closure of R in B_n. For $n \geq 0$, we set

$$B_{n+1} = \operatorname*{colim}_{T\subseteq R_n^\times} \bigotimes\nolimits_{B_n}^{t\in T} B_n[X]/(X^p - t), \tag{8.8.1}$$

where the colimit runs through all finite subsets T of the subset $R_n^\times$ of units of R_n and the transition maps are given by the inclusion relation of these finite subsets T. Notice that $B_n[X]/(X^p - t)$ is finite étale and faithfully flat over B_n, thus B_{n+1} is ind-finite étale and faithfully flat over B_n. Now, we take $B_\infty = \operatorname{colim}_n B_n$. The integral closure R_∞ of R in B_∞ is equal to $\operatorname{colim}_n R_n$ by 3.18. We claim that R_∞ satisfies the first three conditions. Firstly, we see inductively that $B_n = R_n[1/p]$ $(0 \leq n \leq \infty)$ by 3.17. Thus, (i), (ii) follow immediately. For (iii), notice that we have $R_\infty^\times = \operatorname{colim}_n R_n^\times$. For an unit $t \in R_\infty^\times$, we suppose that it is the image of $t_n \in R_n^\times$. By construction, there exists an element $x_{n+1} \in R_{n+1}$ such that $x_{n+1}^p = t_n$. Thus, t admits a p-th root in R_∞.

For (iv), the injectivity follows from the fact that R_∞ is integrally closed in $R_\infty[1/p]$ (see 5.21). For the surjectivity, let $a \in R_\infty$. Firstly, since R_∞ is integral over R, p also lies in the Jacobson radical $J(R_\infty)$ of R_∞. Thus, $1 + p_1a \in R_\infty^\times$ and then by (iii) there is $b \in R_\infty$ such that $b^p = 1 + p_1a$. We write $(b-1)^p = p_1a'$ for some $a' \in a + p_1R_\infty$. Thus, $1 + a' - a \in R_\infty^\times$, and then by (iii) there is $c \in R_\infty$ such that $c^p = 1 + a' - a$. On the other hand, since R_∞ is integrally closed in $R_\infty[1/p]$, we have $x = (b-1)/p_2 \in R_\infty$. Now, we have $(x - c + 1)^p \equiv x^p - c^p + 1 \equiv a \pmod{pR_\infty}$, which completes the proof. □

Remark 8.9. In 8.8, it follows from the construction that $\operatorname{Spec}(R_\infty[1/p]) \to \operatorname{Spec}(R[1/p])$ is a covering in $\operatorname{Spec}(R[1/p])^{\text{S}}_{\text{profét}}$ (7.17).

Proposition 8.10. *With the notation in* 8.7, for any object $V \to U$ in $\mathbf{E}^{\text{proét}}_{Y\to X}$, there exists a covering $\{(V_i \to U_i) \to (V \to U)\}_{i\in I}$ with I finite such that for each $i \in I$, $U_i^{V_i}$ is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid (5.19).

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Proof. After replacing U by an affine open covering, we may assume that $U = \mathrm{Spec}(A)$. Consider the category $\mathscr{C}$ of étale A-algebras B such that $A/pA \to B/pB$ is an isomorphism, and the colimit $A^{\mathrm{h}} = \mathrm{colim}\, B$ over $\mathscr{C}$. In fact, $\mathscr{C}$ is filtered and $(A^{\mathrm{h}}, pA^{\mathrm{h}})$ is the Henselization of the pair (A, pA) (see [Sta23, 0A02]). It is clear that $\mathrm{Spec}(A^{\mathrm{h}}) \coprod \mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A)$ is a covering in $U_{\text{proét}}$. So we reduce to the situation where $p \in J(A)$ or $p \in A^{\times}$. The latter case is trivial since the p-adic completion of $R = \Gamma(U^V, \mathcal{O}_{U^V})$ is zero if p is invertible in A. Therefore, we may assume that $p \in J(A)$ in the following.

Since $R = \Gamma(U^V, \mathcal{O}_{U^V})$ is integral over A, we also have $p \in J(R)$. Applying 8.8 to the $\mathcal{O}_K$-algebra R, we obtain a covering $V_\infty = \mathrm{Spec}(R_\infty[1/p]) \to V = \mathrm{Spec}(R[1/p])$ in $V_{\text{profét}}$ such that $R_\infty = \Gamma(U^{V_\infty}, \mathcal{O}_{U^{V_\infty}})$ is an $\mathcal{O}_K$-algebra which is almost pre-perfectoid by 5.4 and 5.20. □

Proposition 8.11. *With the notation in 8.7, if W is an object of $\mathbf{I}_{\eta\to S}$ such that W is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, then for any pseudo-uniformizer π of K, the canonical morphism*

$$\Gamma(W, \mathcal{O}_W)/\pi\Gamma(W, \mathcal{O}_W) \to \mathrm{R}\Gamma(\mathbf{I}_{W_\eta \to W}, \mathscr{O}/\pi\mathscr{O}) \tag{8.11.1}$$

is an almost isomorphism (5.7).

Proof. It suffices to prove the case $\pi = p$ by 8.3. Let $\mathscr{C}$ be the full-subcategory of $\mathbf{I}_{\eta\to S}$ formed by the spectrums of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. It is stable under fibred product by 5.33, 5.30 and 3.21, and it forms a topologically generating family for the site $\mathbf{I}_{\eta\to S}$ by 8.5 and 8.10. It suffices to show that for any covering in $\mathbf{I}_{\eta\to S}$, $\mathfrak{U} = \{W_i \to W\}_{i\in I}$ consisting of objects of $\mathscr{C}$ with I finite, the augmented Čech complex associated to the presheaf $W \mapsto \Gamma(W, \mathcal{O}_W)/p\Gamma(W, \mathcal{O}_W)$ on $\mathbf{I}_{\eta\to S}$ (whose associated sheaf is just $\mathscr{O}/p\mathscr{O}$),

$$0 \to \Gamma(W, \mathcal{O}_W)/p \to \prod_{i\in I} \Gamma(W_i, \mathcal{O}_{W_i})/p \to \prod_{i,j\in I} \Gamma(W_i \overline{\times}_W W_j, \mathcal{O}_{W_i \overline{\times}_W W_j})/p \to \cdots \tag{8.11.2}$$

is almost exact. Indeed, the almost exactness shows firstly that $\Gamma(W, \mathcal{O}_W)/p \to H^0(\mathbf{I}_{W_\eta\to W}, \mathscr{O}/p\mathscr{O})$ is an almost isomorphism (cf. [Sta23, 00W1]), so that the augmented Čech complex associated to the sheaf $\mathscr{O}/p\mathscr{O}$ is also almost exact. Then, the conclusion follows from the almost vanishing of the higher Čech cohomologies of $\mathscr{O}/p\mathscr{O}$ by [Sta23, 03F9].

We set $R = \Gamma(W, \mathcal{O}_W)$ and $R' = \prod_{i\in I} \Gamma(W_i, \mathcal{O}_{W_i})$. They are almost pre-perfectoid, and $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a v-covering by definition. Thus, the almost exactness of Equation (8.11.2) follows from 5.33, 5.30 and 5.35. □

Theorem 8.12. *With the notation in* 8.7, let $\epsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\text{proét}}_{Y\to X}$ be the morphism of sites defined in 8.6. Then, for any pseudo-uniformizer π of K, the canonical morphism

$$\overline{\mathscr{B}}/\pi\overline{\mathscr{B}} \to \mathrm{R}\epsilon_*(\mathscr{O}/\pi\mathscr{O}) \tag{8.12.1}$$

is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{proét}}_{Y\to X}})$ (5.7).

Proof. Since $\mathrm{R}^q\epsilon_*(\mathscr{O}/\pi\mathscr{O})$ is the sheaf associated to the presheaf $(V \to U) \mapsto H^q(\mathbf{I}_{V\to U^V}, \mathscr{O}/\pi\mathscr{O})$ and any object in $\mathbf{E}^{\text{proét}}_{Y\to X}$ can be covered by those objects whose image under ϵ^+ are the spectrums of $\mathcal{O}_K$-algebras which are almost pre-perfectoid by 8.10, the conclusion follows from 8.11. □

Corollary 8.13. *With the notation in 8.7, let $V \to U$ be an object of $\mathbf{E}^{\text{proét}}_{Y\to X}$ such that U is affine and that the integral closure $U^V = \mathrm{Spec}(A)$ is the spectrum of an $\mathcal{O}_K$-algebra A which is almost pre-perfectoid. Then, $V \to U$ is Faltings acyclic.*

Proof. It follows directly from 8.12 and 8.11. □

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Corollary 8.14. *With the notation in 8.7, let* $\varepsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ *be the morphism of sites defined in 8.6. Then, for any finite locally constant abelian sheaf* $\mathbb{L}$ *on* $\mathbf{E}^{\text{ét}}_{Y\to X}$*, the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{8.14.1}$$

is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ *(5.7).*

Proof. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$, thus we may assume that $\mathbb{L}$ is the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$. Then, the conclusion follows from 8.12 and 7.32. □

Remark 8.15. In 8.14, if $\mathbb{L}$ is a bounded complex of abelian sheaves on $\mathbf{E}^{\text{ét}}_{Y\to X}$ with finite locally constant cohomology sheaves, then the canonical morphism $\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O})$ is also an almost isomorphism. Indeed, after replacing $\mathbb{L}$ by $\mathbb{L}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}_p$, we may assume that $\mathbb{L}$ is a complex of $\mathbb{Z}/p^n\mathbb{Z}$-modules for some integer n ([Sta23, 0DD7]). Then, there exists a covering family $\{(Y_i \to X_i) \to (Y \to X)\}_{i\in I}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ such that the restriction of $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y_i\to X_i}$ is represented by a bounded complex of finite locally constant $\mathbb{Z}/p^n\mathbb{Z}$-modules ([Sta23, 094G]). Then, the conclusion follows directly from 8.14.

Corollary 8.16. *With the notation in 8.7, let* $Y \to X_i$ $(i = 1, 2)$ *be a morphism of coherent schemes such that* X_i^Y *is an S-scheme with generic fibre* $(X_i^Y)_\eta = Y$*,* $\mathbb{L}$ *a finite locally constant abelian sheaf on* $\mathbf{E}^{\text{ét}}_{Y\to X_2}$*. If there is a morphism* $f : X_1 \to X_2$ *under* Y *such that the natural morphism* $g : X_1^Y \to X_2^Y$ *is a separated v-covering and that* $g^{-1}(Y) = Y$*, and if we denote by* $u : \mathbf{E}^{\text{ét}}_{Y\to X_1} \to \mathbf{E}^{\text{ét}}_{Y\to X_2}$ *the corresponding morphism of sites, then the natural morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}u_*(u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \tag{8.16.1}$$

is an almost isomorphism.

Proof. The morphism u is defined by the functor $u^+ : \mathbf{E}^{\text{ét}}_{Y\to X_2} \to \mathbf{E}^{\text{ét}}_{Y\to X_1}$ sending $(V \to U_2)$ to $(V \to U_1) = (V \to X_1 \times_{X_2} U_2)$. We set $V_0 = Y \times_{X_1} U_1 = Y \times_{X_2} U_2$. According to 3.17, $U_1^{V_0} \to U_2^{V_0}$ is the base change of $X_1^Y \to X_2^Y$ by $U_2 \to X_2$, and thus it is a separated v-covering. Notice that V_0 is an open subscheme in both $U_1^{V_0}$ and $U_2^{V_0}$, and moreover $V_0 = V_0 \times_{U_2^{V_0}} U_1^{V_0}$. Applying 3.15 to the commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U_1^V \arrow[r] \arrow[d] & U_1^{V_0} \arrow[d] \\ V \arrow[r] & U_2^V \arrow[r] & U_2^{V_0} \end{tikzcd} \tag{8.16.2}$$

it follows that $U_1^V \to U_2^V$ is also a separated v-covering. Let $\varepsilon_i : \mathbf{I}_{Y\to X_i^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X_i}$ $(i = 1, 2)$ be the morphisms of sites defined in 8.6. The sheaf $\mathrm{R}^q u_*(u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$ is associated to the presheaf $(V \to U_2) \mapsto H^q(\mathbf{E}^{\text{ét}}_{V\to U_1}, u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$. We have

$$\begin{aligned} H^q(\mathbf{E}^{\text{ét}}_{V\to U_1}, u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) &\to H^q(\mathbf{I}_{V\to U_1^V}, \varepsilon_1^{-1}u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \\ &= H^q(\mathbf{I}_{V\to U_2^V}, \varepsilon_2^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \leftarrow H^q(\mathbf{E}^{\text{ét}}_{V\to U_2}, \mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \end{aligned} \tag{8.16.3}$$

where the equality follows from the fact that the morphism of representable sheaves associated to $U_1^V \to U_2^V$ on $\mathbf{I}_{\eta\to S}$ is an isomorphism by 3.24, and where the two arrows are almost isomorphisms by 8.14, which completes the proof. □

8.17. Let Δ be the category formed by finite ordered sets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ with nondecreasing maps ([Sta23, 0164]). For a functor from its opposite category Δ^{op} to the category $\mathbf{E}$ of morphisms

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of coherent schemes sending $[n]$ to $Y_n \to X_n$, we simply denote it by $Y_\bullet \to X_\bullet$. Then, we obtain a fibred site $\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}$ over Δ^{op} whose fibre category over $[n]$ is $\mathbf{E}^{\text{ét}}_{Y_n\to X_n}$ and the inverse image functor $f^+ : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y_m\to X_m}$ associated to a morphism $f : [m] \to [n]$ in Δ^{op} is induced the base change by the morphism $(Y_m \to X_m) \to (Y_n \to X_n)$ associated to f. We endow $\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}$ with the total topology (6.1) and call it the *simplicial Faltings site* associated to $Y_\bullet \to X_\bullet$ ([Sta23, 09WE.(A)]). The sheaf $\overline{\mathscr{B}}$ on each $\mathbf{E}^{\text{ét}}_{Y_n\to X_n}$ induces a sheaf $\overline{\mathscr{B}}_\bullet = (\overline{\mathscr{B}})_{[n]\in\text{Ob}(\Delta)}$ on $\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}$ with the notation in 6.5.

For an augmentation $(Y_\bullet \to X_\bullet) \to (Y \to X)$ in $\mathbf{E}$ ([Sta23, 018F]), we obtain an augmentation of simplicial site $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ ([Sta23, 0D6Z.(A)]). We denote by $a_n : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the natural morphism induced by $(Y_n \to X_n) \to (Y \to X)$. Notice that for any sheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, we have $a^{-1}\mathcal{F} = (a_n^{-1}\mathcal{F})_{[n]\in\text{Ob}(\Delta)}$ with the notation in 6.5 ([Sta23, 0D70]).

Corollary 8.18. *With the notation in 8.7, let $\mathbb{L}$ a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$, $X_\bullet \to X$ an augmentation of simplicial coherent scheme. If we set $Y_\bullet = Y \times_X X_\bullet$ and denote by $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the augmentation of simplicial site, assuming that $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\eta\to S}$, then the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{8.18.1}$$

is an almost isomorphism.

Proof. Let $b : \mathbf{I}_{Y_\bullet\to X_\bullet^{Y_\bullet}} \to \mathbf{I}_{Y\to X^Y}$ be the augmentation of simplicial site associated to the augmentation of simplicial object $X_\bullet^{Y_\bullet} \to X^Y$ of $\mathbf{I}_{\eta\to S}$ ([Sta23, 09X8]). The functorial morphism of sites $\varepsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ defined in 8.6 induces a commutative diagram of topoi ([Sta23, 0D99])

$$\begin{array}{ccc} \mathbf{I}^{\sim}_{Y_\bullet\to X_\bullet^{Y_\bullet}} & \xrightarrow{\varepsilon_\bullet} & \mathbf{E}^{\text{ét}\sim}_{Y_\bullet\to X_\bullet} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ \mathbf{I}^{\sim}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}\sim}_{Y\to X}. \end{array} \tag{8.18.2}$$

We denote by $a_n : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ and $b_n : \mathbf{I}_{Y_n\to X_n^{Y_n}} \to \mathbf{I}_{Y\to X^Y}$ the natural morphisms of sites. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{R}a_*(a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet) & \xleftarrow{\alpha_1} & \mathbb{L}\otimes\overline{\mathscr{B}} & \xrightarrow{\alpha_2} & \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L}\otimes\mathscr{O}) \\ {\scriptstyle\alpha_3}\downarrow & & & & \downarrow{\scriptstyle\alpha_4} \\ \mathrm{R}a_*\mathrm{R}\varepsilon_{\bullet*}\varepsilon_\bullet^{-1}(a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet) & \xrightarrow{\alpha_5} & \mathrm{R}c_*(c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet) & = & \mathrm{R}\varepsilon_*\mathrm{R}b_*b^{-1}(\varepsilon^{-1}\mathbb{L}\otimes\mathscr{O}) \end{array}, \tag{8.18.3}$$

where $c = a\circ\varepsilon_\bullet = \varepsilon\circ b$, and α_2 (resp. α_5) is induced by the canonical morphism $\varepsilon^{-1}\overline{\mathscr{B}} \to \mathscr{O}$ (resp. $\varepsilon_\bullet^{-1}\overline{\mathscr{B}}_\bullet \to \mathscr{O}_\bullet$), and other arrows are the canonical morphisms.

Notice that α_2 is an almost isomorphism by 8.14 and that α_4 is an isomorphism by [Sta23, 0D8N] as $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\eta\to S}$. It remains to show that $\alpha_5\circ\alpha_3$ is an almost isomorphism. With the notation in 6.5, we have

$$a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet = (a_n^{-1}\mathbb{L}\otimes\overline{\mathscr{B}})_{[n]\in\text{Ob}(\Delta)} \text{ and} \tag{8.18.4}$$

$$c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet = (\varepsilon_n^{-1}a_n^{-1}\mathbb{L}\otimes\mathscr{O})_{[n]\in\text{Ob}(\Delta)}. \tag{8.18.5}$$

Moreover, by [Sta23, 0D97] we have

$$\mathrm{R}^q\varepsilon_{\bullet*}(c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet) = (\mathrm{R}^q\varepsilon_{n*}(\varepsilon_n^{-1}a_n^{-1}\mathbb{L}\otimes\mathscr{O}))_{[n]\in\text{Ob}(\Delta)} \tag{8.18.6}$$

for each integer q. Therefore, $a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet \to \mathrm{R}\varepsilon_{\bullet *}(c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet)$ is an almost isomorphism by 8.14 and so is $\alpha_5\circ\alpha_3$. □

Lemma 8.19. *With the notation in 8.7, assume that X^Y is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid. Let $V\to U$ be an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ with U affine. Then, U^V is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & U^V \arrow[d] & \\
Y\times_X U \arrow[r] \arrow[d] & U^{Y\times_X U} \arrow[r] \arrow[d] & U \arrow[d] \\
Y \arrow[r] \arrow[d] & X^Y \arrow[r] \arrow[d] & X \\
\mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(\mathcal{O}_K). &
\end{tikzcd} \tag{8.19.1}$$

Since taking integral closures commutes with étale base change and filtered colimits (3.17, 3.18), all the squares in Equation (8.19.1) are Cartesian (3.19). Notice that $U^{Y\times_X U}$ is integral over U and thus affine. Since $U^{Y\times_X U}$ is pro-étale over X^Y, it is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid by 5.37. As V is pro-finite étale over $Y\times_X U$, by almost purity 5.41 and 5.37, we see that U^V is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid. □

8.20. Let K be a pre-perfectoid field of mixed characteristic $(0,p)$, $\eta=\mathrm{Spec}(K)$, $S=\mathrm{Spec}(\mathcal{O}_K)$, $Y\to X$ a morphism of coherent schemes such that X^Y is an S-scheme and that the induced morphism $Y\to X^Y$ is an open immersion over $\eta\to S$. Remark that the morphism $X^Y_\eta\to X$ over $\eta\to S$ is in the situation 8.7. We assume further that there exist finitely many nonzero divisors $f_1,\dots,f_r$ of $\Gamma(X^Y_\eta,\mathcal{O}_{X^Y_\eta})$ such that the divisor $D=\sum_{i=1}^r \mathrm{div}(f_i)$ on X^Y_η has support $X^Y_\eta\setminus Y$ and that at each strict Henselization of X^Y_η those elements f_i contained in the maximal ideal form a subset of a regular system of parameters (in particular, D is a normal crossings divisor on X^Y_η, and we allow D to be empty, i.e., $r=0$). We set

$$Y_\infty=\lim_n \mathrm{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r)), \tag{8.20.1}$$

where the limit is taken over $\mathbb{N}$ with the division relation. We see that Y_∞ is faithfully flat and pro-finite étale over Y.

Proposition 8.21 (Abhyankar's lemma). *Under the assumptions in 8.20 and with the same notation, for any finite étale Y_∞-scheme V_∞, the integral closure $X^{V_\infty}_\eta$ is finite étale over $X^{Y_\infty}_\eta$.*

Proof. We set $Z=X^Y_\eta$. Passing to a strict Henselization of Z where D is nonempty, we may assume that Z is local and regular and that $f_1,\dots,f_r$ $(r\geq 1)$ are all contained in the maximal ideal. We set $Y_n=\mathrm{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r))$ and $Z_n=\mathrm{Spec}_{\mathcal{O}_Z}(\mathcal{O}_Z[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r))$ for any integer $n>0$. Notice that Z_n is still local and regular (thus isomorphic to $X^{Y_n}_\eta$) and that $g_0=f_0^{1/n},\dots,g_r=f_r^{1/n}$ form a subset of a regular system of parameters for Z_n (see the proof of [SGA 1, XIII.5.1]). Using [EGA IV$_3$, 8.8.2, 8.10.5] and [EGA IV$_4$, 17.7.8], there exists an integer $n_0>0$ and a finite étale Y_{n_0}-scheme V_{n_0} such that $V_\infty=Y_\infty\times_{Y_{n_0}}V_{n_0}$. We set $V_n=Y_n\times_{Y_{n_0}}V_{n_0}$ for any $n\geq n_0$. According to [SGA 4$_{\mathrm{III}}$, XIII.5.2], there exists a multiple n_1 of n_0 such that $Z^{V_{n_1}}_{n_1}$ is finite étale over Z_{n_1}. As $Z_\infty=\lim Z_n$ is normal, $Z_\infty\times_{Z_{n_1}}Z^{V_{n_1}}_{n_1}$ is also normal and thus isomorphic to $Z^{V_\infty}_\infty=X^{V_\infty}_\eta$, which shows that the latter is also finite étale over $Z_\infty=X^{Y_\infty}_\eta$. □

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Corollary 8.22. *Under the assumptions in 8.20 and with the same notation, the natural functor* $\mathbf{E}^{\text{proét}}_{X_\eta^{Y_\infty}\to X}\to\mathbf{E}^{\text{proét}}_{Y_\infty\to X}$ *sending* $V\to U$ *to* $Y_\infty\times_{X_\eta^{Y_\infty}}V\to U$ *induces an equivalence of ringed sites* $(\mathbf{E}^{\text{proét}}_{Y_\infty\to X},\overline{\mathscr{B}})\to(\mathbf{E}^{\text{proét}}_{X_\eta^{Y_\infty}\to X},\overline{\mathscr{B}})$.

Proof. For the equivalence of categories, it suffices to show that the induced functor $u^+:\mathbf{E}^{\text{ét}}_{X_\eta^{Y_\infty}\to X}\to\mathbf{E}^{\text{ét}}_{Y_\infty\to X}$ is an equivalence by 7.14.(6). Since u^+ is a morphism of fibred categories over $X_{\text{ét}}$, it suffices to show that for each object U of $X_{\text{ét}}$, the fibre functor $u^+_U:U^{Y_\infty\times_X U}_{\eta,\text{fét}}\to(Y_\infty\times_X U)_{\text{fét}}$ induced by u^+ is an equivalence of categories. Notice that if we replace $Y\to X$ in 8.20 by $Y\times_X U\to U$, then $(Y\times_X U)_\infty=Y_\infty\times_X U$. Therefore, the equivalence of categories follows from applying 8.21 to $Y\times_X U\to U$.

To show the equivalence of categories identifies their topologies, it suffices to show that it identifies the vertical coverings and Cartesian coverings in 7.23. For a finite family $\{(V_m\to U)\to(V\to U)\}_{m\in M}$ in $\mathbf{E}^{\text{proét}}_{X_\eta^{Y_\infty}\to X}$, its image in $\mathbf{E}^{\text{proét}}_{Y_\infty\to X}$ is $\{(Y_\infty\times_{X_\eta^{Y_\infty}}V_m\to U)\to((Y_\infty\times_{X_\eta^{Y_\infty}}V\to U)\}_{m\in M}$. Notice that $Y_\infty\times_{X_\eta^{Y_\infty}}V$ is a dense open subset of V as V is flat over $X_\eta^{Y_\infty}$ ([EGA IV$_2$, 2.3.7]), and the same holds for V_m. Thus, the integral morphism $\coprod_{m\in M}V_m\to V$ is surjective if and only if $\coprod_{m\in M}Y_\infty\times_{X_\eta^{Y_\infty}}V_m\to Y_\infty\times_{X_\eta^{Y_\infty}}V$ is surjective. On the other hand, it is tautological that the equivalence identifies the Cartesian coverings. Hence, the two sites are naturally equivalent.

The identification of the structural sheaves by the equivalence of sites follows from the fact that V is integrally closed in $Y_\infty\times_{X_\eta^{Y_\infty}}V$ for any object $V\to U$ of $\mathbf{E}^{\text{proét}}_{X_\eta^{Y_\infty}\to X}$ as V is pro-étale over $X_\eta^{Y_\infty}$ (3.19). □

Corollary 8.23. *Under the assumptions in 8.20 and with the same notation, let* $V\to U$ *be an object of* $\mathbf{E}^{\text{proét}}_{Y_\infty\to X}$ *such that* U^V *is the spectrum of an* $\mathcal{O}_K$*-algebra which is almost pre-perfectoid, and let* $V'\to U'$ *be an object of* $\mathbf{E}^{\text{proét}}_{V\to U}$ *with* U' *affine. Then,* $U'^{V'}$ *is the spectrum of an* $\mathcal{O}_K$*-algebra which is almost pre-perfectoid.*

Proof. It follows directly from 8.22 and 8.19. □

Theorem 8.24. *Under the assumptions in 8.20 and with the same notation, let* $V\to U$ *be an object of* $\mathbf{E}^{\text{proét}}_{Y_\infty\to X}$*. Then, the following statements are equivalent:*

(1) *The morphism* $V\to U$ *is Faltings acyclic.*
(2) *The scheme* U *is affine and* $U^V=\mathrm{Spec}(A)$ *is the spectrum of an* $\mathcal{O}_K$*-algebra* A *which is almost pre-perfectoid.*

Proof. (2)$\Rightarrow$(1): Let $V'\to U$ be an object of $\mathbf{E}^{\text{proét}}_{X_\eta^{Y_\infty}\to X}$ whose image under the equivalence in 8.22 is isomorphic to $V\to U$. Then, $U^{V'}=\mathrm{Spec}(A)$, $V'=U_\eta^{V'}$, and $\mathrm{R}\Gamma(\mathbf{E}^{\text{proét}}_{V\to U},\overline{\mathscr{B}}/p\overline{\mathscr{B}})=\mathrm{R}\Gamma(\mathbf{E}^{\text{proét}}_{V'\to U},\overline{\mathscr{B}}/p\overline{\mathscr{B}})$. The conclusion follows from 8.13.

(1) $\Rightarrow$ (2): Firstly, notice that the objects $V'\to U'$ of $\mathbf{E}^{\text{proét}}_{Y_\infty\to X}$ satisfying the condition (8.24) form a topological generating family by 8.22 and 8.10. Let $p_1\in\mathcal{O}_K$ be a p-th root of p up to a unit (5.4). Then, we see that the Frobenius induces an almost isomorphism $\overline{\mathscr{B}}/p_1\overline{\mathscr{B}}\to\overline{\mathscr{B}}/p\overline{\mathscr{B}}$ by evaluating these sheaves at the objects $V'\to U'$. The Frobenius also induces an almost isomorphism $A/p_1A\to A/pA$ by 8.3, which shows that A is almost pre-perfectoid. □

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