THE LATTICE OF CONGRUENCES ON A BAND OF GROUPS

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It is implicit in a result of Kapp and Schneider [3] that, if S is a completely simple semigroup, then the lattice $\Lambda(S)$ of congruences on S can be embedded in the product of certain sublattices. In this paper we consider the problem of embedding $\Lambda(S)$ in a product of sublattices, when S is an arbitrary band of groups. The principal tool is the θ-relation of Reilly and Scheiblich [7]. The class of θ-modular bands of groups is defined by means of a type of modularity condition on $\Lambda(S)$. It is shown that the θ-modular bands of groups are precisely those for which a certain function is an embedding of $\Lambda(S)$ into a product of sublattices. The problem of embedding the inverse semigroup congruences into a certain product lattice is also considered.

1. Terminology and preliminary results. A semigroup that is a union of groups is called a *band of groups*, provided that Green's $\mathscr{H}$-relation is a congruence. It is rather well known [**1**, Theorem 4.6] that on any band of groups S (and in fact on any union of groups), the $\mathscr{D}$-relation is the minimum semilattice congruence, and the $\mathscr{D}$-classes of S are completely simple semigroups. The " fine structure " of such semigroups has recently been studied by Leech [**5**].

If S is any regular semigroup, then the θ-relation on $\Lambda(S)$, first studied by Reilly and Scheiblich in [**7**], is defined by $(\rho, \tau) \in \theta$ if and only if $\rho \cap (E_S \times E_S) = \tau \cap (E_S \times E_S)$. In [**7**] it is proved that, if S is an inverse semigroup, then θ is a complete lattice congruence on $\Lambda(S)$. Scheiblich, in [**8**], later extended this result to regular semigroups.

The notation in this paper will be that of Clifford and Preston [**1**], with the exception of the following list of symbols.

x^{-1}: the inverse of x in H_x, in a band of groups.
$B(S)$: the lattice of band congruences on S.
$M(S)$: the lattice of idempotent-separating congruences on S.
$D(S)$: the lattice of congruences on S that are contained in $\mathscr{D}$.
$I(S)$: the lattice of inverse semigroup congruences on S.
$Y(S)$: the lattice of semilattice congruences on S.
$\Delta(S)$: the θ-class of $\mathscr{D}$.
1_S: the universal congruence $S \times S$.
0_S: the diagonal congruence $\Delta S^2 = \{(x, x) \mid x \in S\}$.
β: the minimum band congruence on S.
μ: the maximum idempotent-separating congruence on S.
σ: the minimum group congruence on S.
δ: the minimum inverse semigroup congruence on S.
η: the minimum semilattice congruence on S.

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The congruences β, η, and σ are discussed in [2], as well as in other places. The congruence μ is discussed in [6]. In [7] it is pointed out that, on any regular semigroup, the idempotent-separating congruences are precisely those that are contained in $\mathcal{H}$. Combining this with the result of Munn [6] that the congruences contained in $\mathcal{H}$ form a sublattice of $\Lambda(S)$ with a greatest and a least element, yields the result that μ exists on any regular semigroup, and that $\mu \subseteq \mathcal{H}$.

We have the following characterization of bands of groups, in terms of μ and β.

LEMMA 1.1. *Let S be any regular semigroup. Then the following statements are equivalent.*

(i) *S is a band of groups.*
(ii) $\mu = \mathcal{H} = \beta$.
(iii) $\mu = \beta$.

Proof. In [2] it is shown that $\mathcal{H} \subseteq \beta$. Thus $\mu \subseteq \mathcal{H} \subseteq \beta$, in any regular semigroup. Now, if S is a band of groups, $\mathcal{H}$ is a band congruence; so we must have $\mathcal{H} = \beta$. Also, each $\mathcal{H}$-class contains exactly one idempotent; so $\mathcal{H}$ is also idempotent-separating. Thus $\mathcal{H} = \mu$, and we see that (i) implies (ii). Since $\mu \subseteq \mathcal{H} \subseteq \beta$, it is clear that (ii) is equivalent to (iii). Now, if $\mu = \mathcal{H} = \beta$, then $\mathcal{H}$ is a band congruence. It then follows from [4, Lemma 2.2] that each $\mathcal{H}$-class contains an idempotent. So, by [1, Theorem 2.16], S is a union of groups, and hence a band of groups. Thus (ii) implies (i).

2. The θ-relation and $\Lambda(S)$.

The following two lemmas are due to Scheiblich in [8].

LEMMA 2.1. *Let S be a regular semigroup, and* $\rho, \tau \in \Lambda(S)$, *such that* ρ *separates idempotents. Then* $(\rho \vee \tau, \tau) \in \theta$.

LEMMA 2.2. *If S is a regular semigroup, then* θ *is a complete lattice congruence on* $\Lambda(S)$.

Now suppose that S is a band of groups. It is then the case that $\mathcal{H}$ is idempotent-separating; so we have the following immediate corollary.

COROLLARY 2.3. *Let S be a band of groups. Then, for any* $\rho \in \Lambda(S)$, $(\rho \vee \mathcal{H}, \rho) \in \theta$.

We also note that a congruence τ on a regular semigroup is a band congruence if and only if τ contains β, the minimum band congruence. We therefore have

PROPOSITION 2.4. *Let S be a regular semigroup. Then each* θ*-class of* $\Lambda(S)$ *contains at most one band congruence. In addition, if S is a band of groups, then each* θ*-class contains exactly one band congruence.*

Proof. Suppose that α and γ are band congruences in the same θ-class. Since $\beta \subseteq \alpha$ and $\beta \subseteq \gamma$, the α- and γ-classes are unions of β-classes. Also, by [4, Lemma 2.2], each β-class contains an idempotent. Now suppose that $x \alpha y$. Let e and f be idempotents such that $e \beta x$, $f \beta y$. Then $e \beta x \alpha y \beta f$, so that $e \alpha f$. Hence, since $(\alpha, \gamma) \in \theta$, we have $e \gamma f$. But then $x \beta e \gamma f \beta y$, so that $x \gamma y$. Thus $\alpha \subseteq \gamma$. Similarly, $\gamma \subseteq \alpha$, proving the first part. Now, if S is a band of groups, we have $(\rho \vee \mathcal{H}, \rho) \in \theta$ for every $\rho \in \Lambda(S)$, by Corollary 2.3. Since $\beta = \mathcal{H} \subseteq \rho \vee \mathcal{H}$, $\rho \vee \mathcal{H}$ is a band congruence in the θ-class of ρ. This completes the proof.

The following proposition will prove to be useful.

PROPOSITION 2.5. *Let S be a band of groups and let $\rho, \tau \in \Lambda(S)$. Then $(\rho, \tau) \in \theta$ if and only if $\rho \vee \mathscr{H} = \tau \vee \mathscr{H}$.*

Proof. Suppose that $(\rho, \tau) \in \theta$. Combining this with $(\rho \vee \mathscr{H}, \rho) \in \theta$ and $(\tau \vee \mathscr{H}, \tau) \in \theta$, we obtain $(\rho \vee \mathscr{H}, \tau \vee \mathscr{H}) \in \theta$ by transitivity of θ. Hence, since $\rho \vee \mathscr{H}$ and $\tau \vee \mathscr{H}$ are both band congruences, we have $\rho \vee \mathscr{H} = \tau \vee \mathscr{H}$, by Proposition 2.4. Conversely, if $\rho \vee \mathscr{H} = \tau \vee \mathscr{H}$, then $(\rho \vee \mathscr{H}, \rho) \in \theta$ and $(\tau \vee \mathscr{H}, \tau) \in \theta$ imply that $(\rho, \tau) \in \theta$.

In [7] it is proved that the θ-classes of a regular semigroup S are very nice. We now record this for future reference.

LEMMA 2.6. [7, Theorem 3.4(ii)] *Let S be a regular semigroup. Then each θ-class is a complete modular sublattice of $\Lambda(S)$ (having a greatest and a least element).*

The following proposition gives a necessary and sufficient condition for these greatest elements to be band congruences.

PROPOSITION 2.7. *Let S be a regular semigroup. Then the greatest element of each θ-class is a band congruence if and only if S is a band of groups.*

Proof. If S is a band of groups, then $\mathscr{H} = \beta$. We have also seen that, if ρ is any congruence, then $(\rho \vee \mathscr{H}, \rho) \in \theta$. So, if τ is the greatest element of the θ-class of ρ, then $\mathscr{H} \subseteq \rho \vee \mathscr{H} \subseteq \tau$, which implies that τ is a band congruence. Conversely, if the greatest element of each θ-class is a band congruence, then in particular μ, which is the greatest element of the θ-class of 0_S, is a band congruence. But $\mu \subseteq \mathscr{H} \subseteq \beta$; so we obtain $\mu = \mathscr{H} = \beta$, whence S is a band of groups.

The θ-relation is a useful means of viewing $\Lambda(S)$, particularly in the case that S is a band of groups. For example, if S is a band of groups, the θ-class of 0_S consists of those congruences that partition the idempotents of S in the same manner as 0_S; that is, the θ-class of 0_S is the set of idempotent-separating congruences on S. Its greatest element is $\mu = \mathscr{H} = \beta$. Similarly, the θ-class of 1_S consists of all congruences that identify all idempotents of S; that is, it is the lattice of group congruences on S. The greatest element in this θ-class is, of course, 1_S, and the least element is σ, the minimum group congruence.

The θ-relation, being a congruence, partitions $\Lambda(S)$, and, in view of Propositions 2.7 and 2.4, $B(S)$ cross-sections the θ-classes. This naturally leads to the problem of describing $\Lambda(S)$ in terms of $B(S)$ and some other sublattice; for $B(S)$ is isomorphic to $\Lambda(S/\beta) = \Lambda(S/\mathscr{H})$, and hence is more accessible than $\Lambda(S)$ itself. This problem is considered in the following section.

3. Embedding $\Lambda(S)$ in a product lattice. In this section it is shown that the lattice $D(S)$ on a band of groups S can be embedded in the product lattice $B(S) \times M(S)$, but that the embedding does not always extend to an embedding of $\Lambda(S)$. A necessary and sufficient condition on $\Lambda(S)$ is then found, under which the natural extension of this map is an embedding of $\Lambda(S)$.

We begin with the following easy lemma, whose proof is omitted.

LEMMA 3.1. *Let S be a band of groups and let* $\rho \in \Lambda(S)$. *Then* $\rho \wedge \mathscr{H}$ *is an idempotent-separating congruence; that is,* $(\rho \wedge \mathscr{H}, \mathscr{H}) \in \theta$.

LEMMA 3.2. *Let S be a band of groups, let* $\rho \in \Lambda(S)$ *and suppose that* $(x, y) \in \rho$. *Let* $e \in E_S \cap H_x, f \in E_S \cap H_y$. *Then* $(e, f) \in \rho$.

Proof. We have $(e, f) \in \mathscr{H} \circ \rho \circ \mathscr{H} \subseteq \rho \vee \mathscr{H}$. Then, since $(\rho \vee \mathscr{H}, \rho) \in \theta$, by Corollary 2.3, we have $(e, f) \in \rho$.

PROPOSITION 3.3. *Let S be a band of groups and let* $\psi : \Lambda(S) \to B(S) \times M(S)$ *be defined by* $\psi(\rho) = (\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$. *Then* ψ *is one-to-one.*

Proof. Suppose that $\rho, \tau \in \Lambda(S)$ are such that $(\rho \vee \mathscr{H}, \rho \wedge \mathscr{H}) = (\tau \vee \mathscr{H}, \tau \wedge \mathscr{H})$. Then, from Proposition 2.5, we have $(\rho, \tau) \in \theta$, and also $\rho \wedge \mathscr{H} = \tau \wedge \mathscr{H}$. Suppose that $(x, y) \in \rho$, and let $e \in E_S \cap H_x, f \in E_S \cap H_y$. Then, by Lemma 3.2, we have $(e, f) \in \rho$; and, since $(\rho, \tau) \in \theta$, this implies that $(e, f) \in \tau$. Hence $x = xe \, \tau \, xf$, and $y = fy \, \tau \, ey$. But $ey \, \rho \, fy = y \, \rho \, x = xe \, \rho \, xf$, so that $ey \, \rho \, xf$. Also, $ey \, \mathscr{H} \, xf$, since $\mathscr{H}$ is a congruence, and thus $ey \, (\rho \wedge \mathscr{H}) \, xf$. Since $\rho \wedge \mathscr{H} = \tau \wedge \mathscr{H}$, we then have $ey (\tau \wedge \mathscr{H}) xf$. Thus $x \, \tau \, xf (\tau \wedge \mathscr{H}) ey \, \tau \, y$, so that $(x, y) \in \tau$. Thus $\rho \subseteq \tau$. Likewise $\tau \subseteq \rho$, and the result follows.

We remark that, by this proposition, every congruence ρ on a band of groups can be "factored" into a band congruence (namely $\rho \vee \mathscr{H}$), and an idempotent-separating congruence (namely $\rho \wedge \mathscr{H}$). The next proposition shows, to some extent, how the congruence ρ can be recovered from this factorization.

PROPOSITION 3.4. *Let S be a band of groups and let* $\rho \in \Lambda(S)$. *Then* $\rho = \bar{\rho} \vee (\rho \wedge \mathscr{H})$, *where* $\bar{\rho}$ *is the smallest element of the* θ*-class of* ρ.

Proof. It will suffice to show that $\psi(\rho) = \psi(\bar{\rho} \vee (\rho \wedge \mathscr{H}))$, where ψ is as in Proposition 3.3. By Lemma 3.1, Corollary 2.3, and Lemma 2.2, we have $[\bar{\rho} \vee (\rho \wedge \mathscr{H})] \, \theta \, [\bar{\rho} \vee \mathscr{H}] \, \theta \, \bar{\rho} \, \theta \rho$, and hence, by Proposition 2.5, $\rho \vee \mathscr{H} = [\bar{\rho} \vee (\rho \wedge \mathscr{H})] \vee \mathscr{H}$. Thus it remains to show that $\rho \wedge \mathscr{H} = [\bar{\rho} \vee (\rho \wedge \mathscr{H})] \wedge \mathscr{H}$. But $\bar{\rho}$, $\rho \wedge \mathscr{H} \subseteq \rho$; so we have $\bar{\rho} \vee (\rho \wedge \mathscr{H}) \subseteq \rho$. Thus $[\bar{\rho} \vee (\rho \wedge \mathscr{H})] \wedge \mathscr{H} \subseteq \rho \wedge \mathscr{H}$. Also, $\rho \wedge \mathscr{H} \subseteq \bar{\rho} \vee (\rho \wedge \mathscr{H})$, and $\rho \wedge \mathscr{H} \subseteq \mathscr{H}$. So we have $\rho \wedge \mathscr{H} \subseteq [\bar{\rho} \vee (\rho \wedge \mathscr{H})] \wedge \mathscr{H}$. Thus $[\bar{\rho} \vee (\rho \wedge \mathscr{H})] \wedge \mathscr{H} = \rho \wedge \mathscr{H}$, and the result follows.

A more interesting question concerns the problem of when the function ψ of Proposition 3.3 is an embedding. Needless to say, ψ is not always an embedding. It is always $\wedge$-preserving, however, as the next proposition shows.

PROPOSITION 3.5. *Let S be a band of groups and let* $\psi : \Lambda(S) \to B(S) \times M(S)$ be as in *Proposition* 3.3. *Then* ψ *is* $\wedge$*-preserving; that is,* $((\rho \wedge \tau) \vee \mathscr{H}, (\rho \wedge \tau) \wedge \mathscr{H}) = ((\rho \vee \mathscr{H}) \wedge (\tau \vee \mathscr{H}), (\rho \wedge \mathscr{H}) \wedge (\tau \wedge \mathscr{H}))$, *for each* $\rho, \tau \in \Lambda(S)$.

Proof. It is obvious that $(\rho \wedge \tau) \wedge \mathscr{H} = (\rho \wedge \mathscr{H}) \wedge (\tau \wedge \mathscr{H})$. For the other equality, since both $(\rho \wedge \tau) \vee \mathscr{H}$ and $(\rho \vee \mathscr{H}) \wedge (\tau \vee \mathscr{H})$ are band congruences, it suffices, by Proposition 2.4, to show that these congruences are θ-related. But $[(\rho \wedge \tau) \vee \mathscr{H}] \, \theta \, (\rho \wedge \tau)$, $(\rho \vee \mathscr{H}) \, \theta \, \rho$, and $(\tau \vee \mathscr{H}) \, \theta \, \tau$. And, since θ is a congruence, the last two relations imply that $[(\rho \vee \mathscr{H}) \wedge (\tau \vee \mathscr{H})] \, \theta \, (\rho \wedge \tau)$. The result then follows by the transitivity of θ.

COROLLARY 3.6. *Let S be a band of groups. Then $B(S)$ is lattice-isomorphic with $\Lambda(S)/\theta$.*

Proof. By Proposition 2.4, the map $\phi : \Lambda(S)/\theta \to B(S)$ defined by $\phi(\theta^\natural(\rho)) = \rho \vee \mathscr{H}$ is a bijection. (It is well-defined by Proposition 2.5.) Since θ is a lattice congruence, we have $\phi(\theta^\natural(\rho) \vee \theta^\natural(\tau)) = \phi(\theta^\natural(\rho \vee \tau)) = (\rho \vee \tau) \vee \mathscr{H} = (\rho \vee \mathscr{H}) \vee (\tau \vee \mathscr{H}) = \phi(\theta^\natural(\rho)) \vee \phi(\theta^\natural(\tau))$; and $\phi(\theta^\natural(\rho) \wedge \theta^\natural(\tau)) = \phi(\theta^\natural(\rho \wedge \tau)) = (\rho \wedge \tau) \vee \mathscr{H} = (\rho \vee \mathscr{H}) \wedge (\tau \vee \mathscr{H}) = \phi(\theta^\natural(\rho)) \wedge \phi(\theta^\natural(\tau))$, by Proposition 3.5.

We now give a simple example to show that the function ψ of Proposition 3.3 need not be $\vee$-preserving.

EXAMPLE 3.7. Let $S = \{e, a, f, b\}$ be the semigroup given by the following table:

	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f.

S is then in fact a semilattice of the groups $\{e, a\}$ and $\{f, b\}$. It is not hard to show that S has exactly five congruences; the classes of the three non-trivial congruences are listed below:

σ: $\{e, f\}$, $\{a, b\}$;
$\mathscr{H}$: $\{e, a\}$, $\{f, b\}$;
α: $\{e\}$, $\{a\}$, $\{f, b\}$.

The congruence σ is the minimum group congruence and α is the Rees congruence associated with the ideal $\{f, b\}$. We note that $\psi(\sigma) = (\sigma \vee \mathscr{H}, \sigma \wedge \mathscr{H}) = (1_S, 0_S)$, and $\psi(\alpha) = (\alpha \vee \mathscr{H}, \alpha \wedge \mathscr{H}) = (\mathscr{H}, \alpha)$, so that $\psi(\sigma) \vee \psi(\alpha) = (1_S, \alpha)$. But $\psi(\sigma \vee \alpha) = \psi(1_S) = (1_S, \mathscr{H})$; so we see that ψ is not $\vee$-preserving.

We now turn our attention to a portion of $\Lambda(S)$ on which ψ is $\vee$-preserving. Let S be a band of groups, and consider the function $\tilde{\psi} : D(S) \to B(S) \times M(S)$ defined by $\tilde{\psi}(\rho) = (\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$. That is, $\tilde{\psi}$ is the restriction to $D(S)$ of the function ψ of Proposition 3.3. It follows immediately from Propositions 3.3 and 3.5 that $\tilde{\psi}$ is one-to-one and $\wedge$-preserving. The restriction $\tilde{\psi}$ behaves better than ψ, however, in the following sense.

PROPOSITION 3.8. *Let S be a band of groups, and define $\tilde{\psi} : D(S) \to B(S) \times M(S)$ by $\tilde{\psi}(\rho) = (\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$. Then $\tilde{\psi}$ is $\vee$-preserving; that is, $((\rho \vee \tau) \vee \mathscr{H}, (\rho \vee \tau) \wedge \mathscr{H}) = ((\rho \vee \mathscr{H}) \vee (\tau \vee \mathscr{H}), (\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H}))$ for each $\rho, \tau \in D(S)$.*

Proof. It is clear that $(\rho \vee \tau) \vee \mathscr{H} = (\rho \vee \mathscr{H}) \vee (\tau \vee \mathscr{H})$. For the other equality, we note that $(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})$ is the smallest congruence containing $\rho \wedge \mathscr{H}$ and $\tau \wedge \mathscr{H}$.

But $(\rho \vee \tau) \wedge \mathscr{H}$ is certainly such a congruence. Hence we have $(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H}) \subseteq (\rho \vee \tau) \wedge \mathscr{H}$. On the other hand, suppose that $(x, y) \in (\rho \vee \tau) \wedge \mathscr{H}$. Then $x \mathscr{H} y$, and $(x, y) \in \rho \vee \tau = \bigcup_{n=1}^{\infty} (\rho \circ \tau)^n$. Thus there exist a positive integer n and elements x_i, x_i' $(i = 1, \dots, n)$ of S such that

$$x \rho x_1 \tau x_1' \rho x_2 \tau x_2' \rho \dots \rho x_n \tau x_n' = y.$$

Furthermore, since $\rho, \tau \subseteq \mathscr{D}$, all of the x_i and x_i' are in $D_x = D_y$. Now let e be the idempotent in $H_x = H_y$. Then

$$x = exe \rho ex_1 e \tau ex_1' e \rho \dots \rho ex_n e \tau ex_n' e = eye = y.$$

But $D_x = D_y$ is a completely simple semigroup, and so, for each i, $ex_i e$, $ex_i' e \in eD_x e = H_x$. Thus we in fact have $(x, y) \in \bigcup_{n=1}^{\infty} [(\rho \wedge \mathscr{H}) \circ (\tau \wedge \mathscr{H})]^n = (\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})$, completing the proof.

As a corollary, we now have

THEOREM 3.9. *Let S be a band of groups. Then $D(S)$ is lattice-isomorphic with a sublattice of the product lattice $B(S) \times M(S)$; specifically, $\tilde{\psi} : D(S) \to B(S) \times M(S)$ is an embedding.*

Since a completely simple semigroup has the property that $\mathscr{D} = 1_S$, and thus $D(S) = \Lambda(S)$, the following corollary is obvious.

COROLLARY 3.10. *Let S be a completely simple semigroup. Then $\Lambda(S)$ is lattice-isomorphic with a sublattice of the product lattice $B(S) \times M(S)$; specifically, $\psi : \Lambda(S) \to B(S) \times M(S)$ is an embedding.*

We shall now find a necessary and sufficient condition on $\Lambda(S)$, where S is an arbitrary band of groups, under which ψ is actually an embedding.

Recall that an arbitrary lattice L is called *modular* if, whenever $a, b, c \in L$ with $a \geqq b$, then $a \wedge (c \vee b) = (a \wedge c) \vee b$. It is well known that a lattice L is modular if and only if the conditions $a \geqq b$, $a \wedge c = b \wedge c$, and $a \vee c = b \vee c$, for elements $a, b, c \in L$, imply that $a = b$. This motivates the following definition.

DEFINITION 3.11. Let L be a lattice, and ζ a lattice congruence on L. We say that L is *ζ-modular* if the conditions $a \geqq b$, $(a, b) \in \zeta$, $a \wedge c = b \wedge c$, and $a \vee c = b \vee c$, for elements $a, b, c \in L$, imply that $a = b$.

For convenience, if S is a semigroup, and ζ is a lattice congruence on $\Lambda(S)$, we agree to call S *ζ-modular*, provided that $\Lambda(S)$ is ζ-modular. Since θ is a lattice congruence on $\Lambda(S)$, we may speak of θ-modularity of S. It is in this specialization of the above definition that we are interested.

As examples, we note that all bands are θ-modular; for all their congruences are band congruences, and so the θ-classes are trivial. All groups are θ-modular, for the lattice of congruences on a group consists of a single θ-class, which is in fact modular, by Lemma 2.6. Of course, not all bands of groups are θ-modular, as Example 3.7 readily shows. We shall see shortly that the class of θ-modular bands of groups is particularly interesting.

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We begin with a technical lemma.

LEMMA 3.12. *Let S be a θ-modular band of groups. Then, for any $\rho, \tau \in \Lambda(S)$, $\rho \vee [(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})] = \rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]$.*

Proof. We first note that $(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H}) \subseteq (\rho \vee \tau) \wedge \mathscr{H}$, since $(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})$ is the smallest congruence containing $\rho \wedge \mathscr{H}$ and $\tau \wedge \mathscr{H}$, and since $(\rho \vee \tau) \wedge \mathscr{H}$ is such a congruence. Thus $\rho \vee [(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})] \subseteq \rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]$. Now note that $\rho \vee [(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})] = [\rho \vee (\rho \wedge \mathscr{H})] \vee (\tau \wedge \mathscr{H}) = \rho \vee (\tau \wedge \mathscr{H})$. Now $\rho \vee (\tau \wedge \mathscr{H})$ and $\rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]$ are θ-related, for, by Lemma 3.1, $(\tau \wedge \mathscr{H})\ \theta\ (\rho \vee \tau) \wedge \mathscr{H}$, and then, by Lemma 2.2, $[\rho \vee (\tau \wedge \mathscr{H})]\ \theta\ \rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]$. Thus, by θ-modularity, it will suffice to show that $\tau \vee [\rho \vee (\tau \wedge \mathscr{H})] = \tau \vee [\rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]]$ and $\tau \wedge [\rho \vee (\tau \wedge \mathscr{H})] = \tau \wedge [\rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]]$. Now we have already seen that $\rho \vee (\tau \wedge \mathscr{H}) \subseteq \rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]$. Thus $\tau \vee \rho \subseteq \tau \vee [\rho \vee (\tau \wedge \mathscr{H})] \subseteq \tau \vee [\rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]] \subseteq \tau \vee [\rho \vee (\rho \vee \tau)] = \tau \vee (\rho \vee \tau) = \tau \vee \rho$, implying the first equality above. For the other equality, we have $\tau \wedge [\rho \vee (\tau \wedge \mathscr{H})] \subseteq \tau \wedge [\rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]] \subseteq \tau \wedge [\rho \vee \mathscr{H}]$. Hence it suffices to show that $\tau \wedge [\rho \vee \mathscr{H}] \subseteq \tau \wedge [\rho \vee (\tau \wedge \mathscr{H})]$. For this, it is sufficient to show that $\tau \wedge (\rho \vee \mathscr{H}) \subseteq \rho \vee (\tau \wedge \mathscr{H})$; for then $\tau \wedge (\rho \vee \mathscr{H}) = \tau \wedge [\tau \wedge (\rho \vee \mathscr{H})] \subseteq \tau \wedge [\rho \vee (\tau \wedge \mathscr{H})]$. So suppose that $(x, y) \in \tau \wedge (\rho \vee \mathscr{H})$. Let $e \in E_S \cap H_x$, $f \in E_S \cap H_y$ and $g \in E_S \cap H_{xy}$. Since $(x, y) \in \rho \vee \mathscr{H}$, we have $(e, f) \in \rho \vee \mathscr{H}$, by Lemma 3.2. Hence, since $(\rho, \rho \vee \mathscr{H}) \in \theta$, we have $(e, f) \in \rho$. Thus $e = ee\, \rho\, ef\, \mathscr{H}\, xy\, \mathscr{H}\, g$, so that $(e, g) \in \rho \vee \mathscr{H}$. Again, since $(\rho, \rho \vee \mathscr{H}) \in \theta$, we have $(e, g) \in \rho$, and hence also $(f, g) \in \rho$, by the transitivity of ρ. Moreover, using the fact that the $\mathscr{D}$-class D_g is completely simple, we have $gxg\, \mathscr{H}\, gyg$. Thus $x = exe\, \rho\, gxg\, (\tau \wedge \mathscr{H})\, gyg\, \rho\, fyf = y$, so that $(x, y) \in \rho \vee (\tau \wedge \mathscr{H})$. This completes the proof.

PROPOSITION 3.13. *Let S be a θ-modular band of groups. Then the function $\psi : \Lambda(S) \to B(S) \times M(S)$ defined by $\psi(\rho) = (\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$ is $\vee$-preserving; that is,*

$$((\rho \vee \tau) \vee \mathscr{H}, (\rho \vee \tau) \wedge \mathscr{H}) = ((\rho \vee \mathscr{H}) \vee (\tau \vee \mathscr{H}), (\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H}))$$

for each $\rho, \tau \in \Lambda(S)$.

Proof. It is obvious that $(\rho \vee \tau) \vee \mathscr{H} = (\rho \vee \mathscr{H}) \vee (\tau \vee \mathscr{H})$. For the other equality, we have already noted in the proof of Lemma 3.12 that $(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H}) \subseteq (\rho \vee \tau) \wedge \mathscr{H}$. Also, both $(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})$ and $(\rho \vee \tau) \wedge \mathscr{H}$ are contained in $\mathscr{H}$, and are therefore θ-related. Thus, by θ-modularity, it will suffice to show that $\rho \vee [(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})] = \rho \vee [(\rho \vee \tau) \wedge \mathscr{H}]$, and $\rho \wedge [(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})] = \rho \wedge [(\rho \vee \tau) \wedge \mathscr{H}]$. The first of these equalities is the content of Lemma 3.12. Also, since $\rho \wedge \mathscr{H} \subseteq (\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})$, we have $\rho \wedge \mathscr{H} = \rho \wedge (\rho \wedge \mathscr{H}) \subseteq \rho \wedge [(\rho \wedge \mathscr{H}) \vee (\tau \wedge \mathscr{H})] \subseteq \rho \wedge [(\rho \vee \tau) \wedge \mathscr{H}] = [\rho \wedge (\rho \vee \tau)] \wedge \mathscr{H} = \rho \wedge \mathscr{H}$, from which the second equality follows.

Combining Propositions 3.3, 3.5, and 3.13, we obtain

THEOREM 3.14. *Let S be a θ-modular band of groups. Then the function $\psi : \Lambda(S) \to B(S) \times M(S)$ defined by $\psi(\rho) = (\rho \vee \mathscr{H}, \rho \wedge \mathscr{H})$ is an embedding.*

The converse of this theorem is also true.

THEOREM 3.15. *Let S be a band of groups, and suppose that $\psi : \Lambda(S) \to B(S) \times M(S)$ as defined above is an embedding. Then S is θ-modular.*

Proof. Let $\rho \subseteq \tau$ be θ-related congruences, and suppose that α is a congruence such that $\rho \vee \alpha = \tau \vee \alpha$ and $\rho \wedge \alpha = \tau \wedge \alpha$. Clearly $\rho \wedge \mathscr{H} \subseteq \tau \wedge \mathscr{H}$; and, since ψ is an embedding, we have $(\rho \wedge \mathscr{H}) \vee (\alpha \wedge \mathscr{H}) = (\rho \vee \alpha) \wedge \mathscr{H} = (\tau \vee \alpha) \wedge \mathscr{H} = (\tau \wedge \mathscr{H}) \vee (\alpha \wedge \mathscr{H})$. Also, $(\rho \wedge \mathscr{H}) \wedge (\alpha \wedge \mathscr{H}) = (\rho \wedge \alpha) \wedge \mathscr{H} = (\tau \wedge \alpha) \wedge \mathscr{H} = (\tau \wedge \mathscr{H}) \wedge (\alpha \wedge \mathscr{H})$. Hence, since, by Lemma 2.6, the θ-class of $\mathscr{H}$ is a modular sublattice of $\Lambda(S)$, we conclude that $\rho \wedge \mathscr{H} = \tau \wedge \mathscr{H}$. Also, since $\rho\,\theta\,\tau$, we have $\rho \vee \mathscr{H} = \tau \vee \mathscr{H}$, by Proposition 2.5. Since ψ is one-to-one, we conclude that $\rho = \tau$. Thus S is θ-modular.

The above two theorems characterize θ-modular bands of groups as being those whose lattice of congruences can be naturally embedded in a certain product lattice. The class of θ-modular bands of groups is studied further in [**9**].

4. The inverse semigroup congruences. In this final section, we study the connection between the θ-relation and the sublattice $I(S)$ of inverse semigroup congruences on a band of groups S.

PROPOSITION 4.1. *Let S be a band of groups, and let $\tau \in Y(S)$. Let ρ be a congruence θ-related to τ. Then $\rho \in I(S)$.*

Proof. It will suffice to show that S/ρ is a semilattice of groups. (See Exercise 2 on page 129 in [**1**].) Write $S = \bigcup_{\alpha \in S/\tau} S_\alpha$, where the S_α are the τ-classes of S. Since $\mathscr{D} = \eta \subseteq \tau$, each S_α is a union of $\mathscr{D}$-classes, and is hence a regular subsemigroup of S. Since τ is a semilattice congruence (and thus *a fortiori* a band congruence), it follows from Propositions 2.7 and 2.4 that τ is the greatest element of its θ-class. In particular then, $\rho \subseteq \tau$. Since $\rho \subseteq \tau$, it follows that the sets $\rho^\natural[S_\alpha]$ are disjoint subsemigroups of S/ρ. Now, since S is a semilattice of the S_α and $\rho^\natural$ is a homomorphism, it follows that S/ρ is a semilattice of the $\rho^\natural[S_\alpha]$. Moreover, since $(\rho, \tau) \in \theta$, ρ identifies all the idempotents in the τ-class S_α. Hence $\rho^\natural[S_\alpha]$ is a group, and it follows that S/ρ is an inverse semigroup.

The converse of this proposition is also true.

PROPOSITION 4.2. *Let S be a band of groups, and let $\rho \in I(S)$. Then there is some congruence $\tau \in Y(S)$ such that $(\rho, \tau) \in \theta$.*

Proof. S/ρ is an inverse semigroup which is a union of groups; that is, S/ρ is a semilattice of groups. Let $Y = (S/\rho)/\mathscr{D}_{S/\rho}$ be the structure semilattice of S/ρ, and let ϕ denote $\mathscr{D}^\natural_{S/\rho} : S/\rho \rightarrow Y$. Let τ be the congruence on S determined by $\phi \circ \rho^\natural$. Then clearly τ is a semilattice congruence. Moreover, we have $(\tau, \rho) \in \theta$. For, if $e, f \in E_S$, then $e\,\rho\,f$ clearly implies $e\,\tau\,f$. And conversely, if $e\,\tau\,f$, then $\phi \circ \rho^\natural(e) = \phi \circ \rho^\natural(f)$; but, since the $\mathscr{D}$-classes of S/ρ are groups, ϕ is an idempotent-separating homomorphism, and so we must have $\rho^\natural(e) = \rho^\natural(f)$; that is, $e\,\rho\,f$.

As an immediate corollary, we now deduce

THEOREM 4.3. *Let S be a band of groups. Then the θ-saturation of $Y(S)$ is $I(S)$; that is, the inverse semigroup congruences on S are precisely those that are θ-related to some semilattice congruence.*

We now give an alternative characterization of the inverse semigroup congruences on a band of groups.

PROPOSITION 4.4. *Let S be a band of groups. Then a congruence ρ is an inverse semigroup congruence if and only if $\rho \vee \mathscr{D} = \rho \vee \mathscr{H}$.*

Proof. Suppose that $\rho \vee \mathscr{D} = \rho \vee \mathscr{H}$. Since $(\rho, \rho \vee \mathscr{H}) \in \theta$, we have $(\rho, \rho \vee \mathscr{D}) \in \theta$. But $\eta = \mathscr{D} \subseteq \rho \vee \mathscr{D}$, so that $\rho \vee \mathscr{D}$ is a semilattice congruence. It then follows from Proposition 4.1 that ρ is an inverse semigroup congruence. To prove the converse, we first note that $\rho \circ \mathscr{D} = \rho \circ \mathscr{H}$. For certainly $\rho \circ \mathscr{H} \subseteq \rho \circ \mathscr{D}$. On the other hand, if $(x, y) \in \rho \circ \mathscr{D}$, say $x \rho z \mathscr{D} y$, let y' be the inverse of z^{-1} in H_y. Since S/ρ is an inverse semigroup, we have uniqueness of inverses in S/ρ, and thus $\rho^\natural(y') = \rho^\natural(z)$; that is, $z \rho y'$. Thus $x \rho y' \mathscr{H} y$, so that $(x, y) \in \rho \circ \mathscr{H}$. We thus have $\rho \circ \mathscr{H} = \rho \circ \mathscr{D}$. Hence $\rho \vee \mathscr{D} = \bigcup_{n=1}^{\infty} (\rho \circ \mathscr{D})^n = \bigcup_{n=1}^{\infty} (\rho \circ \mathscr{H})^n = \rho \vee \mathscr{H}$, completing the proof.

We now have the following corollary.

COROLLARY 4.5. *Let S be a band of groups. Then δ, the minimum inverse semigroup congruence on S, is the least element of the θ-class of $\mathscr{D}$.*

Proof. Since $\mathscr{D} = \eta$ is an inverse semigroup congruence, we must have $\delta \subseteq \mathscr{D}$. Hence, by Proposition 4.4, $\delta \vee \mathscr{H} = \delta \vee \mathscr{D} = \mathscr{D}$. But $(\delta, \delta \vee \mathscr{H}) \in \theta$; so $(\delta, \mathscr{D}) \in \theta$. But, by Proposition 4.1, every congruence in the θ-class of $\mathscr{D}$ is an inverse semigroup congruence. Hence δ must be the least element of this θ-class, since it is to be contained in all inverse semigroup congruences.

A natural question to ask at this point is whether one obtains an embedding theorem for $I(S)$ similar to Theorem 3.9. The answer is that one does not, as is illustrated by the semigroup of Example 3.7. We shall show that θ-modularity of the semigroup S/δ is a necessary and sufficient condition for such a result.

Now let S be an arbitrary semigroup. If $\rho, \gamma \in \Lambda(S)$ and $\gamma \subseteq \rho$, then the relation ρ/γ on S/γ defined by $\rho/\gamma = \{(\gamma^\natural(x), \gamma^\natural(y)) \mid (x, y) \in \rho\}$ is a congruence. Moreover, the lattice $\gamma \vee \Lambda(S)$ is isomorphic with $\Lambda(S/\gamma)$ under the map $\gamma \vee \tau \to (\gamma \vee \tau)/\gamma$. In particular, if $\gamma \subseteq \rho, \tau$, then $(\rho \wedge \tau)/\gamma = (\rho/\gamma) \wedge (\tau/\gamma)$ and $(\rho \vee \tau)/\gamma = (\rho/\gamma) \vee (\tau/\gamma)$. These facts are readily verified, as is pointed out in [7].

We now have

LEMMA 4.6. *Let S be a band of groups. Then $\mathscr{H}_{S/\delta} = \mathscr{D}/\delta$.*

Proof. Suppose that $\delta^\natural(x)\ \mathscr{D}/\delta\ \delta^\natural(y)$. Then $x \mathscr{D} y$, so that $\delta^\natural(x) \mathscr{D}_{S/\delta} \delta^\natural(y)$. But S/δ is an inverse semigroup; that is, S/δ is a semilattice of groups. Hence $\mathscr{D}_{S/\delta} = \mathscr{H}_{S/\delta}$, and we thus have $\delta^\natural(x) \mathscr{H}_{S/\delta} \delta^\natural(y)$. Conversely, suppose that $\delta^\natural(x) \mathscr{H}_{S/\delta} \delta^\natural(y)$. Then $\mathscr{D}^\natural(x) = (\mathscr{D}/\delta)^\natural$ $(\delta^\natural(x))\ \mathscr{H}_{S/\mathscr{D}}\ (\mathscr{D}/\delta)^\natural\ (\delta^\natural(y)) = \mathscr{D}^\natural(y)$. But $S/\mathscr{D}$ is a semilattice; so its $\mathscr{H}$-relation is trivial. Hence we get $\mathscr{D}^\natural(x) = \mathscr{D}^\natural(y)$; that is, $x \mathscr{D} y$. Thus $\delta^\natural(x)\ \mathscr{D}/\delta\ \delta^\natural(y)$, and the result follows.

PROPOSITION 4.7. *Let S be a band of groups such that S/δ is θ-modular. Then the function $\hat{\psi} : I(S) \rightarrow Y(S) \times \Delta(S)$ defined by $\hat{\psi}(\rho) = (\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding.*

Proof. We first note that the function $\hat{\psi}$ is indeed well-defined; for $\rho \vee \mathscr{D}$ contains the minimum semilattice congruence $\mathscr{D}$, and is thus itself a semilattice congruence. And, by Corollary 4.5, $\rho \wedge \mathscr{D} \; \theta \; \rho \wedge \delta = \delta \, \theta \, \mathscr{D}$ since ρ is an inverse semigroup congruence. Since S/δ is θ-modular, the function $\psi : \Lambda(S/\delta) \rightarrow B(S/\delta) \times M(S/\delta)$ defined by $\psi(\rho/\delta) = (\rho/\delta \vee \mathscr{H}_{S/\delta}, \rho/\delta \wedge \mathscr{H}_{S/\delta})$ is an embedding. Now, by Lemma 4.6, we have $\rho/\delta \vee \mathscr{H}_{S/\delta} = \rho/\delta \vee \mathscr{D}/\delta = (\rho \vee \mathscr{D})/\delta$, and likewise $\rho/\delta \wedge \mathscr{H}_{S/\delta} = (\rho \wedge \mathscr{D})/\delta$. But $I(S) = \delta \vee \Lambda(S)$ is isomorphic to $\Lambda(S/\delta)$, under the isomorphism $\rho \rightarrow \rho/\delta$. Thus the composition $\rho \rightarrow \rho/\delta \xrightarrow{\psi} ((\rho \vee \mathscr{D})/\delta, (\rho \wedge \mathscr{D})/\delta) \rightarrow (\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding. This completes the proof.

Before proving the converse of this proposition, we need the following lemma.

LEMMA 4.8. *Let S be any regular semigroup, and $\rho, \tau, \alpha \in \Lambda(S)$ such that $\alpha \subseteq \rho, \tau$. Then $\rho \, \theta \, \tau$ if and only if $\rho/\alpha \, \theta \, \tau/\alpha$.*

Proof. We note first that, by [4, Lemma 2.2], $E_{S/\alpha} = \{\alpha^\natural(e) \mid e \in E_S\}$. Hence, if $\rho \, \theta \, \tau$, we have $\alpha^\natural(e) \; \rho/\alpha \; \alpha^\natural(f) \Leftrightarrow e \, \rho \, f \Leftrightarrow e \, \tau \, f \Leftrightarrow \alpha^\natural(e) \; \tau/\alpha \; \alpha^\natural(f)$, so that $\rho/\alpha \, \theta \, \tau/\alpha$. Conversely, if $\rho/\alpha \, \theta \, \tau/\alpha$, then $e \, \rho \, f \Leftrightarrow \alpha^\natural(e) \, \rho/\alpha \, \alpha^\natural(f) \Leftrightarrow \alpha^\natural(e) \, \tau/\alpha \, \alpha^\natural(f) \Leftrightarrow e \, \tau \, f$, and so $\rho \, \theta \, \tau$.

PROPOSITION 4.9. *Let S be a band of groups, and suppose that the function $\hat{\psi} : I(S) \rightarrow Y(S) \times \Delta(S)$ defined by $\hat{\psi}(\rho) = (\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding. Then S/δ is θ-modular.*

Proof. Suppose that $\rho/\delta \subseteq \tau/\delta$, $\rho/\delta \, \theta \, \tau/\delta$, and that, for some $\alpha/\delta \in \Lambda(S/\delta)$, $\rho/\delta \vee \alpha/\delta = \tau/\delta \vee \alpha/\delta$ and $\rho/\delta \wedge \alpha/\delta = \tau/\delta \wedge \alpha/\delta$. We then have $(\rho \vee \alpha)/\delta = (\tau \vee \alpha)/\delta$, so that $\rho \vee \alpha = \tau \vee \alpha$; and likewise, $\rho \wedge \alpha = \tau \wedge \alpha$. Then, since $\hat{\psi}$ is $\vee$-preserving, we have $(\rho \wedge \mathscr{D}) \vee (\alpha \wedge \mathscr{D}) = (\rho \vee \alpha) \wedge \mathscr{D} = (\tau \vee \alpha) \wedge \mathscr{D} = (\tau \wedge \mathscr{D}) \vee (\alpha \wedge \mathscr{D})$. Moreover, $(\rho \wedge \mathscr{D}) \wedge (\alpha \wedge \mathscr{D}) = (\rho \wedge \alpha) \wedge \mathscr{D} = (\tau \wedge \alpha) \wedge \mathscr{D} = (\tau \wedge \mathscr{D}) \wedge (\alpha \wedge \mathscr{D})$. Also, $\rho/\delta \subseteq \tau/\delta$ implies $\rho \subseteq \tau$, so that $\rho \wedge \mathscr{D} \subseteq \tau \wedge \mathscr{D}$. Now $\rho \wedge \mathscr{D}$, $\tau \wedge \mathscr{D}$, and $\alpha \wedge \mathscr{D}$ are inverse semigroup congruences contained in $\mathscr{D}$, and are hence in $\Delta(S)$. But $\Delta(S)$ is modular by Lemma 2.6; so we have $\rho \wedge \mathscr{D} = \tau \wedge \mathscr{D}$. Also, since $\rho/\delta \, \theta \, \tau/\delta$, we have $\rho \, \theta \, \tau$, by Lemma 4.8, and hence, by Propositions 2.5 and 4.4, $\rho \vee \mathscr{D} = \rho \vee \mathscr{H} = \tau \vee \mathscr{H} = \tau \vee \mathscr{D}$. Since $\hat{\psi}$ is one-to-one, we conclude that $\rho = \tau$, and hence $\rho/\delta = \tau/\delta$, completing the proof.

Combining Propositions 4.7 and 4.9, we immediately deduce

THEOREM 4.10. *Let S be a band of groups. Then $\hat{\psi} : I(S) \rightarrow Y(S) \times \Delta(S)$ defined by $\hat{\psi}(\rho) = (\rho \vee \mathscr{D}, \rho \wedge \mathscr{D})$ is an embedding if and only if S/δ is θ-modular.*

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