4

Classical electromagnetism

Maxwell's theory of electromagnetism is, along with Einstein's theory of gravitation, one of the most beautiful of classical field theories. In this chapter we exhibit the Lorentz covariance of Maxwell's equations and show how they may be obtained from Hamilton's principle. The important idea of a gauge transformation is introduced, and related to the conservation of electric charge. We analyse some properties of solutions of the field equations. Finally, we generalise the Lagrangian to describe massive vector fields, which will figure in later chapters.

4.1 Maxwell's equations

In common with much of the literature, we shall use units in which the force between charges q_1 and q_2 is $q_1 q_2/4\pi r^2$, and the velocity of light $c = 1$. (Thus in these units $\mu_0 = 1,\ \varepsilon_0 = 1$.) Maxwell's equations then take the form

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\mathbf{E} = \rho \quad \text{(a)}, \quad \boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial \mathrm{E}}{\partial t} = \mathbf{J} \quad \text{(b)}, \\ &\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \quad \text{(c)}, \quad \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial \mathrm{B}}{\partial t} = 0 \quad \text{(d)}. \end{aligned} \tag{4.1}$$

E and **B** are the electric and magnetic fields, ρ and **J** are the electric charge and current densities. In this chapter we do not consider the dynamics of ρ and **J**, but take them to be 'external' fields that we are free to manipulate. The inhomogeneous equations (a) and (b) are consistent with the observed fact of charge conservation, which is expressed by the continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathrm{J} = 0.$$

This equation takes the Lorentz invariant form

$$\partial_\mu J^\mu = 0 \tag{4.2}$$

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if we postulate that the charge-current densities

$$J^{\mu} = (\rho, \mathbf{J}) \tag{4.3}$$

make up a contravariant four-vector field.

Introducing a scalar potential ϕ and a vector potential $\mathbf{A}$, the homogeneous equations (c) and (d) of the set (4.1) are satisfied identically by

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{4.4}$$

We postulate that the potentials

$$A^{\mu} = (\phi, \mathbf{A}) \tag{4.5}$$

make up a contravariant four-vector field also.

Maxwell's equations may be written in terms of the antisymmetric tensor $F^{\mu\nu}$, defined by

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{4.6}$$

It is apparent that the electromagnetic field is a tensor field. For example,

$$F^{01} = \partial A^1/\partial x_0 - \partial A^0/\partial x_1 = \partial A_x/\partial t + \partial\phi/\partial x = -E_x.$$

Thus the components of the electromagnetic field transform under a Lorentz transformation like the elements of a tensor.

The homogeneous Maxwell equations correspond to the identitities

$$\partial^{\lambda} F^{\mu\nu} + \partial^{\nu} F^{\lambda\mu} + \partial^{\mu} F^{\nu\lambda} \equiv 0, \tag{4.7}$$

where λ, μ, ν are any three of 0, 1, 2, 3, as the reader may easily verify. The inhomogeneous equations take the manifestly covariant form

$$\partial_{\mu} F^{\mu\nu} = J^{\nu}. \tag{4.8}$$

For example, with $\nu = 0$, looking at the first column of $F^{\mu\nu}$, and noting $\partial_{\mu} = (\partial/\partial t, \ \nabla)$, gives

$$\nabla \cdot \mathbf{E} = \rho.$$

4.2 A Lagrangian density for electromagnetism

We now seek a Lagrangian density $\mathcal{L}$ that will yield Maxwell's equations from Hamilton's principle. If $\mathcal{L}$ is Lorentz invariant, the action

$$S = \int \mathcal{L}\, \mathrm{d}^4 x = \int \mathcal{L}\, \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \tag{4.9}$$

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is also Lorentz invariant, since d^4x is invariant (Section 2.4 and Section 3.4), and the field equations which follow from the condition $\delta S = 0$ will take the same form in every inertial frame of reference.

Although Maxwell's equations do not refer explicitly to the potentials A^μ, to derive the equations from Hamilton's principle requires the potentials to be taken as the basic fields which are to be varied. The 'stretched string' example of Section 3.4 suggests that $\mathcal{L}$ should be quadratic in the first derivatives of the field. A suitable Lorentz invariant choice is found to be

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu. \tag{4.10}$$

Varying the fields A^μ, while keeping the charge and current densities J_μ fixed, yields Maxwell's equations, as we shall show in some detail. (Subsequent arguments will be more terse!)

We may write

$$S = \int \left[-\frac{1}{4} g_{\mu\lambda} g_{\nu\rho} F^{\lambda\rho} F^{\mu\nu} - J^\mu A_\mu \right] d^4x. \tag{4.11}$$

Then

$$\begin{aligned} \delta S &= \int \left[-\frac{1}{2} g_{\mu\lambda} g_{\nu\rho} F^{\lambda\rho} \delta F^{\mu\nu} - J^\mu \delta A_\mu \right] d^4x \\ &= \int \left[-\frac{1}{2} F^{\lambda\rho} (\partial_\lambda \delta A_\rho - \partial_\rho \delta A_\lambda) - J^\mu \delta A_\mu \right] d^4x \\ &= \int [-F^{\lambda\rho} \partial_\lambda \delta A_\rho - J^\mu \delta A_\mu] d^4x, \quad \text{since } F^{\lambda\rho} = -F^{\rho\lambda}. \end{aligned}$$

The first term we integrate by parts. The boundary terms vanish for suitable conditions on the fields, so that we are left with

$$\delta S = \int [\partial_\lambda F^{\lambda\rho} - J^\rho] \delta A_\rho \, d^4x.$$

Setting $\delta S = 0$ for arbitrary δA_ρ gives the inhomogeneous Maxwell equations (4.8). (The homogeneous equations (4.7) are no more than identities.)

4.3 Gauge transformations

The four-potential $A^\mu = (\phi, \mathbf{A})$ is *not unique*: the same electromagnetic field tensor $F^{\mu\nu}$ is obtained from the potential

$$A^\mu + \partial^\mu \chi = (\phi + \partial\chi/\partial t, \mathbf{A} - \nabla\chi), \tag{4.12}$$

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where $\chi(x)$ is an arbitrary scalar field, since the additional terms which appear in $F^{\mu\nu}$ are identically zero:

$$\partial^\mu \partial^\nu \chi - \partial^\nu \partial^\mu \chi = 0.$$

The transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi$ is called a *gauge transformation*.

Under a gauge transformation, the action (4.11) acquires an additional term ΔS, where

$$\begin{aligned} \Delta S &= -\int J_\mu \partial^\mu \chi \, \mathrm{d}^4 x \\ &= \int (\partial^\mu J_\mu) \chi \, \mathrm{d}^4 x. \end{aligned}$$

We have integrated by parts to obtain the second line and again assumed that the boundary terms vanish. ΔS is zero for arbitrary χ if, and only if,

$$\partial^\mu J_\mu = \partial_\mu J^\mu = 0,$$

which is just equation (4.2). Thus the gauge invariance of the action requires, and follows from, the conservation of electric charge.

4.4 Solutions of Maxwell's equations

In terms of the potentials, the field equations (4.8) are

$$(\partial_\mu \partial^\mu) A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu. \tag{4.13}$$

We stress again that there is much arbitrariness in the solutions to these equations. Equivalent solutions differ by gauge transformations. It is usual to impose a gauge-fixing condition. For example in the 'radiation gauge' we set $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, everywhere and at all times (Problem 4.2). This has the disadvantage of not being a Lorentz invariant condition – it will not be true in another, moving, frame – but it does display important features of the theory. In the radiation gauge the field equation for A^0 becomes

$$(\partial_i \partial^i) A^0 = -\boldsymbol{\nabla}^2 A^0 = J^0$$

(setting $\nu = 0$ in (4.13), and noting $\partial_\mu A^\mu = \partial_0 A^0$ since in the radiation gauge $\partial_i A^i = 0$). This equation has the solution

$$A^0(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 \mathbf{r}'.$$

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Hence, in the radiation gauge, A^0 is determined entirely by the charge density to which it is rigidly attached! There are no wave-like solutions. The vector components A^i ($i = 1, 2, 3$) satisfy the inhomogeneous wave equation

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{A} = \mathbf{J} - \frac{\partial}{\partial t} \nabla A^0. \tag{4.14}$$

Charges and currents act as a source (and sink) of the field $\mathbf{A}$.

In free space $\mathbf{J} = 0$, $\rho = 0$, $A^0 = 0$, and there are plane wave solutions with wave vector $\mathbf{k}$, frequency $\omega_{\mathbf{k}} = |\mathbf{k}|$, of the form

$$\mathbf{A}(\mathbf{r}, t) = a\boldsymbol{\varepsilon} \cos(\mathbf{k} \cdot \mathbf{r} - \omega_{\mathbf{k}} t).$$

Here $\boldsymbol{\varepsilon}$ is a unit vector and a is the wave amplitude. The gauge condition requires $\mathbf{k} \cdot \boldsymbol{\varepsilon} = 0$. Thus for a given $\mathbf{k}$ there are only two independent states of polarisation, $\boldsymbol{\varepsilon}_1(\mathbf{k})$ and $\boldsymbol{\varepsilon}_2(\mathbf{k})$ say, perpendicular to $\mathbf{k}$. The general solution in free space is

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{\boldsymbol{\varepsilon}_\alpha(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} [a_{\mathbf{k}\alpha} \mathrm{e}^{\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)} + a^*_{\mathbf{k}\alpha} \mathrm{e}^{-\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)}]. \tag{4.15}$$

The complex number $a_{\mathbf{k}\alpha}$ represents an amplitude and a phase, and the plane waves are normalised in a volume V, with periodic boundary conditions. The factor $\sqrt{2\omega_{\mathbf{k}}}$ is put in for convenience later.

An important point apparent in the radiation gauge is that although the vector potential has four components A^μ, one of these, A^0, has no independent dynamics and another is a gauge artifact, which is eliminated by fixing the gauge. There are only *two* physically significant dynamical fields.

The fields in any other gauge are related to the fields in the radiation gauge by a gauge transformation; the physics is the same but the mathematics is different. For some purposes it is better to work in the relativistically invariant 'Lorentz gauge'. In the Lorentz gauge

$$\partial_\mu A^\mu = 0 \tag{4.16}$$

and the field equations become

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \right) A^\mu = J^\mu. \tag{4.17}$$

4.5 Space inversion

We now consider the operation of space inversion of the coordinate axes in the origin: $\mathbf{r} \to \mathbf{r}' = -\mathbf{r}$, $\boldsymbol{\nabla} \to \boldsymbol{\nabla}' = -\boldsymbol{\nabla}$ (Fig. 4.1), which was excluded from the group of proper Lorentz transformations. We shall also refer to this as the *parity* operation. The transformed coordinate axes are left-handed. By convention the

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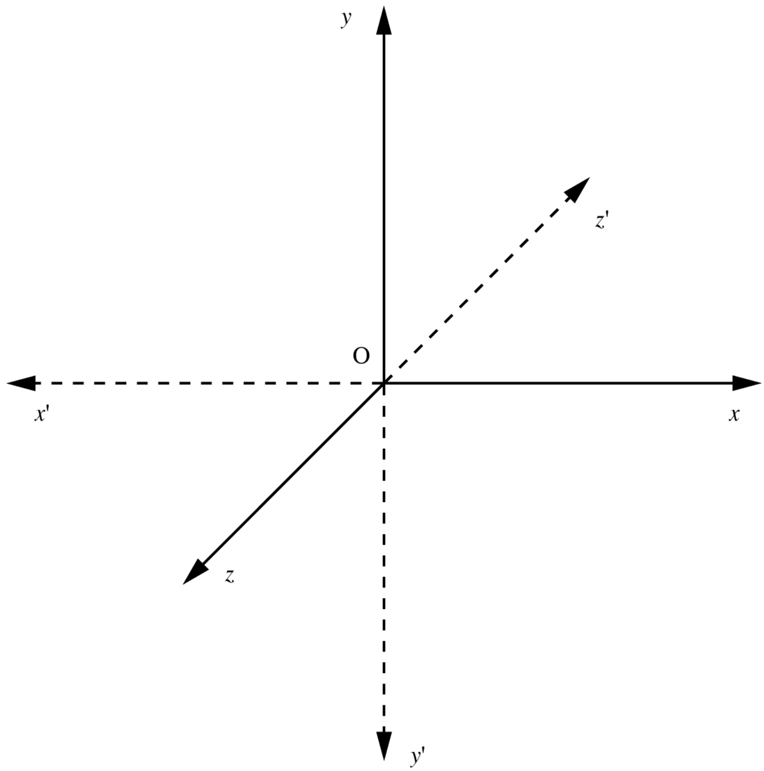

Figure 4.1 A normal right-handed set of axes (solid lines) and a space-inverted set (dashed lines). The space-inverted set is said to be left-handed. (Oz is out of the plane of the page.)

charge density is taken to be invariant under this transformation: if at some instant of time $\rho^{\boldsymbol{P}}(\mathbf{r}')$ is the charge density referred to the inverted coordinate axes, then $\rho^{\boldsymbol{P}}(\mathbf{r}') = \rho(\mathbf{r})$ when $\mathbf{r}' = -\mathbf{r}$. The current density $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\,\mathbf{u}(\mathbf{r})$, where $\mathbf{u}(\mathbf{r})$ is a velocity, and therefore transforms like $\mathrm{d}\mathbf{r}/\mathrm{d}t$, an ordinary vector: $\mathbf{J}^{\boldsymbol{P}}(\mathbf{r}') = -\mathbf{J}(\mathbf{r})$. Maxwell's equations (4.1) retain the same form in the primed coordinate system if $\mathbf{E}(\mathbf{r}')$ also transforms like a vector, $\mathbf{E}^{\boldsymbol{P}}(\mathbf{r}') = -\mathbf{E}(\mathbf{r})$, and $\mathbf{B}(\mathbf{r})$ transforms like an axial vector, $\mathbf{B}^{\boldsymbol{P}}(\mathbf{r}') = \mathbf{B}(\mathbf{r})$.

In terms of the potentials, equation (4.4) shows that we must take

$$\phi^P(\mathbf{r}') = \phi(\mathbf{r}), \qquad \mathbf{A}^P(\mathbf{r}') = -\mathbf{A}(\mathbf{r}). \tag{4.18}$$

The field equations in a left-handed frame then have the same form as in a right-handed frame. The Lagrangian density (4.10) is invariant under space inversion. Electromagnetism is indifferent to handedness.

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4.6 Charge conjugation

It will also be of interest to note that Maxwell's equations can be made to take the same form if matter is replaced by antimatter. As a consequence of this replacement both the charge and current densities change sign so that

$$\rho(\mathbf{r}) \to \rho^C(\mathbf{r}) = -\rho(\mathbf{r}) \qquad \text{and} \qquad \mathbf{J}(\mathbf{r}) \to \mathbf{J}^C(\mathbf{r}) = -\mathbf{J}(\mathbf{r}).$$

Maxwell's equations take the same form if we define

$$\phi^C(\mathbf{r}) = -\phi(\mathbf{r}), \quad \mathbf{A}^C(\mathbf{r}) = -\mathbf{A}(\mathbf{r}). \tag{4.19}$$

This operation is called *charge conjugation*. As with Lorentz transformations and the parity transformation, the Lagrangian is invariant under the charge conjugation transformation.

4.7 Intrinsic angular momentum of the photon

Without embarking here on the full quantisation of the electromagnetic field, we can discuss the quantised intrinsic angular momentum, or spin, of the photons associated with plane waves of the form (4.15).

The spin $\mathbf{S}$ of a particle with mass is defined as its angular momentum in a frame of reference in which it is at rest. In such a frame its orbital angular momentum $\mathbf{L} = 0$, and its total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{S}$. This definition is inapplicable to a massless particle, which moves with the velocity of light in every frame of reference. However, for a massless particle moving in, say, the z-direction, it is possible to define the z-component S_z of its spin, since the z-component of the orbital angular momentum is $L_z = xp_y - yp_x$, and $p_x = p_y = 0$ for a particle moving in the z-direction, hence $L_z = 0$, and $J_z = S_z$.

In quantum mechanics, the component J_z of the total angular momentum operator of a system is given by

$$J_z = \mathrm{i}\hbar r_z = \mathrm{i}\hbar \lim_{\phi \to 0} [R_z(\phi) - 1]\, \phi, \tag{4.20}$$

where $R_z(\phi)$ is the operator that rotates the system through an angle ϕ about Oz in a positive sense.

Consider a term from (4.15) with $\mathbf{k} = (0, 0, k)$ along Oz:

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{2\omega V}}[(a_1 \boldsymbol{\varepsilon}_x + a_2 \boldsymbol{\varepsilon}_y)\mathrm{e}^{\mathrm{i}(kz-\omega t)} + \text{complex conjugate}]. \tag{4.21}$$

The wave amplitudes a_1 and a_2 are complex numbers, and we have taken the polarisation vectors $\boldsymbol{\varepsilon}_x$ and $\boldsymbol{\varepsilon}_y$ to be unit vectors aligned with the x- and y-axes. A

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rotation of **A** through an angle ϕ about Oz makes a change in the amplitudes that can be expressed by the rotation matrix equation

$$R_z(\phi)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

In the limit $\phi \to 0$, we have

$$\lim[R_z(\phi) - 1]/\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$J_z = \hbar\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

The eigenvectors of $J_z/\hbar$ are

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

with eigenvalue $+\,1$,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$$

with eigenvalue -1.

Thus we may say that a photon represented by the plane wave (4.21) has ‘spin one’, with just two spin states aligned and anti-aligned with its direction of motion. No meaning can be given to spin components perpendicular to the direction of motion. Classically these waves are right circularly polarised and left circularly polarised, respectively (Problem 4.4).

A plane wave of any polarisation can be constructed by a suitable superposition of right-handed and left-handed circularly polarised waves.

4.8 The energy density of the electromagnetic field

The analysis of the energy density of the electromagnetic field in free space is a generalisation of the analysis for a scalar field set out in Section 3.6. Equation (3.25) becomes

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\lambda)}\partial_\nu A^\lambda - \delta^\mu_\nu\mathcal{L}, \tag{4.22}$$

and using this formula gives

$$T^0_0 = -F_{0\mu}F^{0\mu} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.23}$$

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(Problem 4.5). In terms of the physical fields $\mathbf{E}$ and $\mathbf{B}$, (4.23) is the familiar expression

$$\text{energy density} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2). \tag{4.24}$$

We can also express the fields in terms of the field amplitudes $a_{\mathbf{k}\alpha}$ introduced in equation (4.15) and obtain for the total energy of the field

$$H = \int T^0_0 \mathrm{d}^3\mathbf{x} = \sum_{\mathbf{k},\alpha} a^*_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} \omega_{\mathbf{k}}. \tag{4.25}$$

Similarly the total momentum of the field is

$$\mathbf{P} = \sum_{\mathbf{k},\alpha} a^*_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} \mathbf{k}. \tag{4.26}$$

4.9 Massive vector fields

Let us modify the Lagrangian density (4.10) by adding an additional Lorentz invariant term, and consider

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - J^\mu A_\mu \tag{4.27}$$

where J^μ is an external current. The additional term in the action is easily seen to modify the field equations to

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu. \tag{4.28}$$

Since $\partial_\nu \partial_\mu F^{\mu\nu} \equiv 0$, it follows from (4.28) that

$$m^2 \partial_\nu A^\nu = \partial_\nu J^\nu. \tag{4.29}$$

This equation is a necessary consequence of the field equations: it is not a Lorentz gauge-fixing condition like equation (4.16), but it does imply that the A^ν are not independent. Using this equation, the field equations simplify to

$$\partial_\mu \partial^\mu A^\nu + m^2 A^\nu = J^\nu + \partial^\nu(\partial_\mu J^\mu)/m^2. \tag{4.30}$$

Hence in free space each component of A^ν of the field satisfies

$$\frac{\partial^2 A^\nu}{\partial t^2} - \boldsymbol{\nabla}^2 A^\nu + m^2 A^\nu = 0. \tag{4.31}$$

This wave equation is related by the quantisation rules $E \to \mathrm{i}\partial/\partial t$, $\mathbf{p} \to -\mathrm{i}\boldsymbol{\nabla}$, to the Einstein equation for a free particle,

$$E^2 = p^2 + m^2.$$

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We may conclude that our modified Lagrangian, when quantised, describes particles of mass m associated with a four-component field, of which three components are independent.

Plane wave solutions of (4.31) are of the form

$$A^\nu = a\varepsilon^\nu \cos(\mathbf{k}\cdot\mathbf{r} - \omega_k t) = a\varepsilon^\nu \cos(k_\mu x^\mu),$$

where $\omega_k = k^0 = \sqrt{m^2 + \mathbf{k}^2}$. To satisfy the condition $\partial_\nu A^\nu = 0$ we need

$$k_\nu \varepsilon^\nu = 0. \tag{4.32}$$

For example, if we consider a plane wave in the z-direction with $k^\nu = (k^0, 0, 0, k)$ there are three independent polarisations, labelled 1, 2, 3, which we may take as the contravariant four-vectors

$$\begin{aligned}\varepsilon_1^\nu &= (0, 1, 0, 0),\\ \varepsilon_2^\nu &= (0, 0, 1, 0),\\ \varepsilon_3^\nu &= (k, 0, 0, k^0)/m.\end{aligned}$$

The intrinsic spin of a particle is its angular momentum in a frame of reference in which it is at rest (Section 4.7). In such a frame $\mathbf{k} = 0$, and $\varepsilon_1 = (0, \varepsilon_x)$, $\varepsilon_2 = (0, \varepsilon_y)$, $\varepsilon_3 = (0, \varepsilon_z)$. As in Section 4.7, the states with polarisation $\varepsilon_x \pm \mathrm{i}\varepsilon_y$ correspond to $J_z = \pm 1$, but we now have also the state with polarisation ε_z, which corresponds to $J_z = 0$, since the operator r_z acting on ε_z gives $r_z \varepsilon_z = 0$.

Thus our modified Lagrangian describes massive particles having intrinsic spin $\mathbf{S}$ with $S = 1$ and $S_z = 1, 0, -1$. That such particles are important in the Standard Model will become evident in later chapters.

Problems

4.1 Show that the Lagrangian density of equation (4.10) can also be written

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - J^\mu A_\mu.$$

4.2 Suppose that in a certain gauge $\nabla \cdot \mathbf{A} = f(\mathbf{r}, t) \neq 0$. Find an expression for a gauge transforming function $\chi(\mathbf{r}, t)$ such that the new potentials given by equation (4.12) satisfy the radiation gauge condition.

4.3 Show that the tensor field $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ has the same form as $F^{\mu\nu}$ but with the electric and magnetic fields interchanged. Show that

$$\frac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B}$$

and that it is a scalar field under Lorentz transformations but a pseudoscalar under the parity operation.

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4.4 Show that the electric field of the wave of equation (4.21) with $a_1 = 1$, $a_2 = \mathrm{i}$, is

$$(E_x, E_y, E_z) = -\sqrt{\frac{2\omega}{V}}[\sin(kz - \omega t), \cos(kz - \omega t), 0].$$

Show that as a function of time, at a fixed z, $\mathbf{E}$ rotates in a positive sense about the z-axis. This is the definition of right circular polarisation.

4.5 Show that equation (4.22) gives immediately

$$T^0_0 = -F^{0\mu}\partial_0 A_\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Show that the term $\partial_\mu(A_0 F^{0\mu}) = \partial_i(A_0 F^{0i})$ can be added to this without changing the total energy. Hence arrive at the form for T^0_0 given in equation (4.23).

4.6 A particle of mass m, charge q, is moving in a fixed external electromagnetic field described by the four-potential $(\phi, \mathbf{A})$. Show that the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}} \cdot \mathbf{A}$$

gives the non-relativistic equation of motion

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}),$$

and the Hamiltonian is

$$H(\mathbf{p}, \mathbf{x}) = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi,$$

where $\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A}$.

4.7 Show that for a particle the action $S = \int L\, \mathrm{d}t$ is Lorentz invariant if γL is Lorentz invariant. Verify that this condition is satisfied by the Lagrangian

$$L = -m/\gamma - qA^\mu(\mathrm{d}x_\mu/\mathrm{d}t).$$

(This gives the relativistic version of Problem 4.6.)

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