J. Aust. Math. Soc. **92** (2012), 127–136
doi:10.1017/S1446788712000110

EMBEDDING PERMUTATION GROUPS INTO WREATH PRODUCTS IN PRODUCT ACTION

CHERYL E. PRAEGER ✉ and CSABA SCHNEIDER

(Received 28 April 2011; accepted 16 January 2012)

Communicated by I. E. Shparlinski

This paper is dedicated to the memory of Alf van der Poorten

Abstract

We consider the wreath product of two permutation groups $G \leq \mathrm{Sym}\,\Gamma$ and $H \leq \mathrm{Sym}\,\Delta$ as a permutation group acting on the set Π of functions from Δ to Γ. Such groups play an important role in the O'Nan–Scott theory of permutation groups and they also arise as automorphism groups of graph products and codes. Let X be a subgroup of $\mathrm{Sym}\,\Gamma \wr \mathrm{Sym}\,\Delta$. Our main result is that, in a suitable conjugate of X, the subgroup of $\mathrm{Sym}\Gamma$ induced by a stabiliser of a coordinate $\delta \in \Delta$ only depends on the orbit of δ under the induced action of X on Δ. Hence, if X is transitive on Δ, then X can be embedded into the wreath product of the permutation group induced by the stabiliser X_δ on Γ and the permutation group induced by X on Δ. We use this result to describe the case where X is intransitive on Δ and offer an application to error-correcting codes in Hamming graphs.

2010 *Mathematics subject classification*: primary 05C25; secondary 20B05, 20B25, 20B35, 20D99.
Keywords and phrases: wreath products, product action, permutation groups, embedding theorems.

1. Introduction

Subgroups of wreath products in product action arise in a number of different contexts. Their importance for group actions is due to the fact that such subgroups give rise to several of the 'O'Nan–Scott types' of finite primitive permutation groups (see [9, Ch. 2] or [8, Sections 1.10 and 4.3]) and finite quasiprimitive groups [13]. They have received special attention recently in the work of Aschbacher [1, 2] aimed at studying intervals in subgroup lattices [3] (with Shareshian), and of the authors [4–6, 14] investigating invariant Cartesian decompositions (with Baddeley). The product

Praeger is supported by Australian Research Council Federation Fellowship FF0776186. Schneider acknowledges the support of the grants PEst-OE/MAT/UI0143/2011 and PTDC/MAT/101993/2008 of the Fundação para a Ciência e a Tecnologia (Portugal) and of the Hungarian Scientific Research Fund (OTKA) grant 72845.

action of the wreath product $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ is its natural action on the set $\Pi = \mathrm{Func}(\Delta, \Gamma)$ of functions from Δ to Γ, described in Section 1.1. If $\Delta = \{1, \ldots, m\}$ then Π can be identified with the set Γ^m of ordered m-tuples of elements of Γ, and in this case subgroups of W arise as automorphism groups of various kinds of graph products, as automorphism groups of codes of length m over the alphabet Γ (regarded as subsets of Γ^m), and as automorphism groups of a special class of chamber systems in the sense of Tits. To study subgroups X of W, and the structures on which they act, one considers the subgroup H of $\mathrm{Sym}\ \Delta$ induced by X along with the 'components' X^{Γ_δ}, which are permutation groups on Γ, defined in Section 1.2, for each $\delta \in \Delta$.

We are interested in X up to permutation isomorphism, and wish to replace X by some conjugate in W which gives a simple form with respect to the product action, both for X and the structures on which it acts. This has been done in detail by Kovács [12] in the case where X is primitive on Π. Kovács also provides a simple form for subgroups inducing a transitive group H on Δ; indeed, his statement [12, (2.2)] is the first assertion of Theorem 1.1(b). One way to handle general subgroups X is to proceed indirectly by appeal to the embedding theorem for subgroups of W using a different action, namely its imprimitive action on $\Gamma \times \Delta$ (see, for example, [7, Theorem 8.5]). However this indirect method does not allow us to keep track of important properties of the underlying product structure. For example, if X is an automorphism group of a code $C \subset \Gamma^m$ then we may wish to maintain the property that C contains a specified codeword, say $(\gamma, \ldots, \gamma)$ for a fixed $\gamma \in \Gamma$, as well as to obtain a simple form for the group X. Thus a direct approach is highly desirable, and the results of this paper provide such an approach. Our main result shows how to choose a form for X so that the δ-component depends only on the X-orbit in Δ containing δ.

Theorem 1.1. *Suppose that* $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ *acts in product action on* $\mathrm{Func}(\Delta, \Gamma)$ *with base group* $B = \mathrm{Func}(\Delta, \mathrm{Sym}\ \Gamma)$. *Let* $X \leq W$, $\varphi \in \mathrm{Func}(\Delta, \Gamma)$ *and* $\delta_1 \in \Delta$. *Then the following hold.*

(a) *There is an element* $x \in B$ *such that the components of* $x^{-1}Xx$, *as defined in (1.4), are constant on each* X*-orbit in* Δ. *Moreover, if the* δ*-component of* X *is transitive on* Γ *for each* $\delta \in \Delta$, *then the element* x *can be chosen to fix* φ.

(b) *If the group* H *induced by* X *on* Δ *is transitive, and if* G *is the* δ_1*-component of* X, *then the element* x *may be chosen in* $\mathrm{Func}(\Delta, \mathrm{Sym}\ \Gamma)$ *such that* $X^x \leq G \wr H$, *(and also such that* $\varphi x = \varphi$ *if* G *is transitive on* Γ*).*

Note that, in part (b), $G \wr H$ denotes a particular subgroup of W (defined in Section 1.1) and not just an isomorphism class of groups. If the subgroup X is transitive on Π then all of its components are transitive (Theorem 1.2), so the additional condition on the element x in Theorem 1.1 to fix a given point is possible.

Theorem 1.2. *Let* $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ *act in product action on* $\mathrm{Func}(\Delta, \Gamma)$ *with base group* $B = \mathrm{Func}(\Delta, \mathrm{Sym}\ \Gamma)$*, where* Δ, Γ *are finite sets. If* X *is a transitive subgroup of* W, *then each component of* X *is transitive on* Γ. *Moreover, if* X *acts transitively on* Δ *then each component of the intersection* $X \cap B$ *is transitive on* Γ.

In many instances the group X will be far from transitive on Π, but may still satisfy some transitivity conditions. We give a simple application of Theorem 1.1 in the context of codes. It is most conveniently stated using coordinate notation. So $\Delta = \{1, \ldots, m\}$ and the code C is a subset of Γ^m with automorphism group being the setwise stabiliser X of C in $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$. The image of C under some element of W is a code *equivalent* to C. Equivalence preserves most important properties, such as the *minimum distance* d of C, which is the minimum number of entries in which distinct elements of C differ. For codes in several interesting families, such as completely transitive codes [11] and neighbour transitive codes [10] with minimum distance at least 3, X is transitive on Δ and all components of X are 2-transitive on Γ. For such codes with minimum distance d, we show that, for our two 'favourite' elements γ, ν of Γ, there is a code equivalent to C containing both the m-tuple (γ^m), and the m-tuple (ν^d, γ^{m-d}) with the first d entries ν and the remaining entries γ, while maintaining a simple form for X.

THEOREM 1.3. *Let $\Delta = \{1, \ldots, m\}$, and let γ, ν be distinct elements of Γ. Suppose that $C \subset \Gamma^m$ has minimum distance d, cardinality $|C| > 1$, and automorphism group $X \leq W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ such that X induces a transitive group H on Δ and some component G of X is 2-transitive on Γ. Then there exists $x \in W$ such that the equivalent code C^x has automorphism group $X^x \leq G \wr K$ (with K conjugate to H in $\mathrm{Sym}\ \Delta$) and C^x contains the m-tuples (γ^m) and (ν^d, γ^{m-d}).*

1.1. Wreath products and the product action. For our proofs, it is most convenient to use 'function notation' for defining the wreath product and its product action.

Let Γ, Δ be sets and let G, H be subgroups of $\mathrm{Sym}\ \Gamma$, $\mathrm{Sym}\ \Delta$ respectively. Set $B = \mathrm{Func}(\Delta, \mathrm{G})$, the set of functions from Δ to G. Then B is a group with respect to pointwise multiplication of its elements: the product of the functions f and g is the function fg that maps $\delta \mapsto (\delta f)(\delta g)$. Moreover, B is isomorphic to the direct product of $|\Delta|$ copies of G (or the Cartesian product if Δ is infinite): for $\delta \in \Delta$, set

$$G_\delta = \{f \in \mathrm{Func}(\Delta, \mathrm{G}) \mid \delta' f = 1 \forall \delta' \in \Delta \setminus \{\delta\}\}$$

and define the map $\sigma_\delta : \mathrm{Func}(\Delta, \mathrm{G}) \to G_\delta$ by

$$\sigma_\delta : f \mapsto f_\delta \quad \text{where } \delta' f_\delta = \begin{cases} \delta f & \text{if } \delta' = \delta, \\ 1 & \text{if } \delta' \neq \delta. \end{cases}$$

Then G_δ is a subgroup isomorphic to G, B is the direct product of the subgroups G_δ (the Cartesian product if Δ is infinite), and σ_δ is the natural projection map $G \to G_\delta$.

We define a homomorphism τ from H to $\mathrm{Aut}\, B$: for $f \in B$ and $h \in H$ let $f(h\tau)$ be the function that maps $\delta \mapsto \delta h^{-1} f$. Now the *wreath product* $G \wr H$ is defined as the semidirect product $B \rtimes H$ with respect to the homomorphism τ. The normal subgroup B is called the *base group* of the wreath product, and H is the *top group*. A useful and easy computation shows that

$$(\delta h^{-1}) f = \delta f^h \quad \forall h \in H, f \in \mathrm{Func}(\Delta, \mathrm{G}), \delta \in \Delta. \tag{1.1}$$

The *product action* of $G \wr H$ on $\Pi = \mathrm{Func}(\Delta, \Gamma)$ is defined as follows. Let $f \in \mathrm{Func}(\Delta, \mathrm{G})$, $h \in H$ and set $g = fh$. For $\varphi \in \Pi$ we define φg as the function that maps $\delta \in \Delta$ to

$$\delta(\varphi g) = (\delta h^{-1}\varphi)(\delta h^{-1} f). \tag{1.2}$$

Note that $\delta h^{-1}\varphi \in \Gamma$, and $\delta h^{-1} f \in \mathrm{Sym}\ \Gamma$. Thus $(\delta h^{-1}\varphi)(\delta h^{-1} f) \in \Gamma$, and so $\varphi g \in \mathrm{Func}(\Delta, \Gamma) = \Pi$, as required. It is straightforward to verify that this action of $G \wr H$ on Π is well defined and faithful (see also [9, Section 2.7]).

Let γ be a fixed element of Γ and let φ be the element of $\mathrm{Func}(\Delta, \Gamma)$ that maps $\delta \mapsto \gamma$ for all $\delta \in \Delta$. Let us compute the stabiliser $(G \wr H)_\varphi$ of φ in $G \wr H$. The subgroup $H \leq (G \wr H)_\varphi$ since, if $h \in H$, then $\delta(\varphi h) = (\delta h^{-1}\varphi) = \gamma$. Therefore $(G \wr H)_\varphi = B_\varphi H$. Suppose that $f \in B$. Then the image of $\delta \in \Delta$ under φf is

$$(\delta\varphi)(\delta f) = \gamma(\delta f).$$

Hence $f \in B_\varphi$ if and only if $\delta f \in G_\gamma$ for all $\delta \in \Delta$. Thus

$$(G \wr H)_\varphi = \{fh \mid \delta f \in G_\gamma, \forall \delta \in \Delta, h \in H\} = \mathrm{Func}(\Delta, \mathrm{G}_\gamma)H.$$

In order to facilitate our discussion of subgroups of wreath products we invoke the language of Cartesian decompositions which was introduced by Baddeley and the authors [4] and was subsequently used to describe innately transitive subgroups of wreath products in product action [5, 6, 14]. Consider the set $\Pi = \mathrm{Func}(\Delta, \Gamma)$, and define, for each $\delta \in \Delta$, a partition Γ_δ of Π as follows. Set

$$\Gamma_\delta = \{\gamma_\delta \mid \gamma \in \Gamma\} \quad \text{where } \gamma_\delta := \{\varphi \in \Pi \mid \delta\varphi = \gamma\}. \tag{1.3}$$

It is routine to check that Γ_δ is indeed a partition of Π. Our notation reflects two important facts. Firstly, the map $\delta \mapsto \Gamma_\delta$ is a bijection between Δ and $\{\Gamma_\delta \mid \delta \in \Delta\}$. Secondly, for a fixed $\delta \in \Delta$, the map $\gamma \mapsto \gamma_\delta$ is a bijection between Γ and Γ_δ. For $\gamma \in \Gamma$ and $\delta \in \Delta$, the element $\gamma_\delta \in \Gamma_\delta$ can be considered as the 'copy' of γ in Γ_δ, and is usually called the *γ-part of* Γ_δ.

The Cartesian product $\prod_{\delta\in\Delta} \Gamma_\delta$ can be bijectively identified with the original set Π. Namely, choosing $\gamma_\delta \in \Gamma_\delta$, one for each $\delta \in \Delta$, the intersection $\bigcap_{\delta\in\Delta} \gamma_\delta$ consists of a single point of Π, and this gives rise to a bijection from the Cartesian product $\prod_{\delta\in\Delta} \Gamma_\delta$ to Π. Therefore, in the terminology of [4], the set $\{\Gamma_\delta \mid \delta \in \Delta\}$ is called a *Cartesian decomposition* of Π. In fact, this set of partitions is viewed as the natural Cartesian decomposition of Π. As $\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ is a permutation group acting on Π, the action of $\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ can be extended to subsets of Π, subsets of subsets, and so on. Hence one can consider the action of $\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ on the set of partitions of Π. It is easy to see that $\{\Gamma_\delta \mid \delta \in \Delta\}$ is invariant under this action, and we will see that the $(\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta)$-action on this set is permutationally isomorphic to the induced action of $\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ on Δ (defined in Section 1.2) under the bijection $\delta \mapsto \Gamma_\delta$. The natural product action of $\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ on $\prod_{\delta\in\Delta} \Gamma_\delta$ is permutationally isomorphic to its action on Π, and indeed the stabiliser in $\mathrm{Sym}\ \Pi$ of this Cartesian decomposition is the wreath product $\mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$. See [4] for a more detailed discussion.

https://doi.org/10.1017/S1446788712000110 Published online by Cambridge University Press

In the case where $\Delta = \{1, \ldots, m\}$, it is worth expressing the product action of the wreath product in coordinate notation. View $\mathrm{Func}(\Delta, \mathrm{G})$ and $\Pi = \mathrm{Func}(\Delta, \Gamma)$ as G^m and Γ^m, respectively. Then, for $(\gamma_1, \ldots, \gamma_m) \in \Gamma^m$ and $(g_1, \ldots, g_m)h \in G \wr H$,

$$(\gamma_1, \ldots, \gamma_m)((g_1, \ldots, g_m)h) = (\gamma_{1h^{-1}} g_{1h^{-1}}, \ldots, \gamma_{mh^{-1}} g_{mh^{-1}}).$$

1.2. Subgroups of wreath products and their components. Suppose that $X \leq \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$. We define, for $\delta \in \Delta$, the *δ-component* X^{Γ_δ} of X as a subgroup of $\mathrm{Sym}\ \Gamma$ as follows. Recall that each element of X is of the form fh, where $f \in \mathrm{Func}(\Delta, \Gamma)$ and $h \in \mathrm{Sym}\ \Delta$. Recall also the definition of Γ_δ in (1.3). Now X permutes the partitions Γ_δ and we denote the stabiliser $\{x \in X \mid \Gamma_\delta x = \Gamma_\delta\}$ in X of Γ_δ by X_{Γ_δ}. Then $X_{\Gamma_\delta} = \{fh \in X \mid \delta h = \delta\}$, and the δ-component X^{Γ_δ} of X is the image of X_{Γ_δ} in $\mathrm{Sym}\ \Gamma$ under the map $fh \mapsto \delta f$, namely

$$X^{\Gamma_\delta} := \{\delta f \mid \exists fh \in X_{\Gamma_\delta} \text{ for some } h\}. \tag{1.4}$$

The bijection $\gamma_\delta \mapsto \gamma$ is equivariant with respect to the actions of X_{Γ_δ} on Γ_δ and X^{Γ_δ} on Γ. Later (when we define the induced action of X on Δ) we will see that $\Gamma_\delta x = \Gamma_{\delta x}$, for $x \in X$.

In order to prove Proposition 1.2, we need more information about subgroups of $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ which do not act transitively on Δ. It turns out that such subgroups X may be viewed as subgroups of a direct product in product action: for sets Ω_1 and Ω_2, and permutation groups $G \leq \mathrm{Sym}\ \Omega_1$ and $H \leq \mathrm{Sym}\ \Omega_2$, the *product action* of the direct product $G \times H$ is the natural action of $G \times H$ on $\Omega_1 \times \Omega_2$ given by $(g, h) : (\omega_1, \omega_2) \mapsto (\omega_1 g, \omega_2 h)$ for $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and $(g, h) \in G \times H$. We construct a *permutational embedding* (ϑ, χ) of X acting on $\Pi = \mathrm{Func}(\Delta, \Gamma)$ into $\mathrm{Sym}\ \Omega_1 \times \mathrm{Sym}\ \Omega_2$ acting on $\Omega_1 \times \Omega_2$, by which we mean a bijection $\chi : \Pi \to \Omega_1 \times \Omega_2$ and a monomorphism $\vartheta : X \to \mathrm{Sym}\ \Omega_1 \times \mathrm{Sym}\ \Omega_2$ such that, for all $\varphi \in \Pi$ and all $x \in X$, $(\varphi x)\chi = (\varphi\chi)x\vartheta$.

For a proper nonempty subset Δ' of Δ, and an element $\varphi \in \mathrm{Func}(\Delta, \Gamma)$, define $\varphi|_{\Delta'} \in \mathrm{Func}(\Delta', \Gamma)$ as the restriction of φ to Δ', so $\delta\varphi|_{\Delta'} = \delta\varphi$ for all $\delta \in \Delta'$. For $X \leq \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$, define the induced action of X on Δ by $fh : \delta \mapsto \delta h$; equivalently. this is the action $x : \delta \mapsto \delta x$ defined by $\Gamma_\delta x = \Gamma_{\delta x}$.

Proposition 1.4. *Let $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$, in product action on $\Pi = \mathrm{Func}(\Delta, \Gamma)$, and suppose that $X \leq W$, such that X leaves invariant a proper nonempty subset Δ_0 of Δ in the induced X-action on Δ. Let $\Delta_1 = \Delta \setminus \Delta_0$, and set $\Omega_0 = \mathrm{Func}(\Delta_0, \Gamma)$ and $\Omega_1 = \mathrm{Func}(\Delta_1, \Gamma)$. Then the following hold.*

(a) *The map $\vartheta : \Pi \to \Omega_0 \times \Omega_1$ defined by $\varphi\vartheta = (\varphi|_{\Delta_0}, \varphi|_{\Delta_1})$, for $\varphi \in \Pi$, is a bijection.*
(b) *The map $\chi : X \to \mathrm{Sym}\ \Omega_0 \times \mathrm{Sym}\ \Omega_1$ defined by $x\chi = (x_0, x_1)$, where $\varphi|_{\Delta_i}\, x_i = (\varphi x)|_{\Delta_i}$ for $\varphi|_{\Delta_i} \in \Omega_i$, is a monomorphism.*
(c) *For $i = 0, 1$, if $\sigma_i : \mathrm{Sym}\ \Omega_0 \times \mathrm{Sym}\ \Omega_1 \to \mathrm{Sym}\ \Omega_i$ is the projection map $(x_0, x_1)\sigma_i = x_i$, then $X\chi\sigma_i$ is contained in $W_i := \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta_i$, and for each $\delta \in \Delta_i$, the δ-components of X and $X\chi\sigma_i$ are the same subgroup of* $\mathrm{Sym}\ \Gamma$.

(d) (ϑ, χ) *is a permutational embedding of* X *on* Π *into the group* $\mathrm{Sym}\ \Omega_0 \times \mathrm{Sym}\ \Omega_1$ *in its product action on* $\Omega_0 \times \Omega_1$, *and* $X\chi \leq W_0 \times W_1$.

Proof. (a) This follows from the definition of the maps $\varphi|_{\Delta_i}$ as restrictions of φ.

(b) Let $x \in X$ and $x\chi = (x_0, x_1)$. Note that $\varphi x \in \Pi$, for $\varphi \in \Pi$, and hence $(\varphi x)|_{\Delta_i} \in \Omega_i$. It is straightforward to check that $\varphi|_{\Delta_i} \mapsto (\varphi x)|_{\Delta_i}$ is a bijection $\Omega_i \to \Omega_i$. Thus $x_i \in \mathrm{Sym}\ \Omega_i$, for each i, and the map χ is well defined. Let also $y \in X$ and $y\chi = (y_0, y_1)$. It follows immediately from the definition of the x_i and y_i that $(xy)_i = x_i y_i$ for each i, and hence that $x\chi y\chi = (xy)\chi$. Thus χ is a homomorphism. If $x \in \ker \chi$ then, for each $\varphi \in \Pi$ and each i, $\varphi|_{\Delta_i} = \varphi|_{\Delta_i} x_i = (\varphi x)|_{\Delta_i}$. Thus $\varphi = \varphi x$. Since this holds for all $\varphi \in \Pi$, $x = 1$.

(c) As in (1.3), for each $\delta \in \Delta_i$ we define a partition Γ^i_δ of Ω_i as follows. For $\gamma \in \Gamma$, we define $\gamma^i_\delta = \{\psi \in \Omega_i \mid \delta\psi = \gamma\}$ and $\Gamma^i_\delta = \{\gamma^i_\delta \mid \gamma \in \Gamma\}$. Since $(\varphi x)|_{\Delta_i} = \varphi_{\Delta_i} x_i$ we have $\gamma^i_\delta x_i = \gamma^i_{\delta x}$ so that $\Gamma^i_\delta x_i = \Gamma^i_{\delta x}$. Thus $X\chi\sigma_i$ leaves invariant the set of partitions $\{\Gamma^i_\delta \mid \delta \in \Delta_i\}$ which forms a Cartesian decomposition of Ω_i. Hence $X\chi\sigma_i$ is contained in W_i. The stabiliser of Γ^i_δ in $X\chi\sigma_i$ is $(X_{\Gamma_\delta})\chi\sigma_i$ and the δ-component of $X\chi\sigma_i$, defined as in (1.4), is equal to the δ-component X^{Γ_δ} of X.

(d) This follows since, for all $\varphi \in \Pi$ and all $x \in X$,

$$\varphi\vartheta x\chi = (\varphi|_{\Delta_0}, \varphi|_{\Delta_1})x\chi = ((\varphi x)|_{\Delta_0}, (\varphi x)|_{\Delta_1}) = (\varphi x)\vartheta$$

and since, by part (c), $X\chi\sigma_i \leq W_i$. □

2. Proof of Theorem 1.1

Suppose that $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ acts in product action on $\Pi = \mathrm{Func}(\Delta, \Gamma)$ with base group $B = \mathrm{Func}(\Delta, \mathrm{Sym}\ \Gamma)$. Let $X \leq W$, $\varphi \in \mathrm{Func}(\Delta, \Gamma)$ and $\delta_1 \in \Delta$. Note that B is the kernel of the induced action of W on Δ, so if $x \in B$, then the X-orbits in Δ coincide with the $x^{-1}Xx$-orbits in Δ. For the computations in the proof we often use the properties given in (1.1) and (1.2), and the equality $\delta(ff') = (\delta f)(\delta f')$, for $f, f' \in B$, $h \in \mathrm{Sym}\ \Delta$, $\delta \in \Delta$.

Let $\Delta_1, \ldots, \Delta_r$ be the X-orbits in Δ under the action induced by X on Δ. For $1 \leq i \leq r$ choose $\delta_i \in \Delta_i$, with δ_1 as in the previous paragraph if $i = 1$. For each $\delta \in \Delta_i$, choose $t_\delta \in X$ such that $\Gamma_{\delta_i} t_\delta = \Gamma_\delta$, and in particular take $t_{\delta_i} = 1$. Then $t_\delta = f_\delta h_\delta$ with $f_\delta \in B$ and $h_\delta \in \mathrm{Sym}\ \Delta$ such that $\delta_i h_\delta = \delta$. Also $X_{\Gamma_\delta} = (X_{\Gamma_{\delta_i}})^{t_\delta}$.

Claim 1. If the δ_i-component is transitive on Γ, then we may assume in addition that, for each $\delta \in \Delta_i$, $\delta_i f_\delta$ fixes the point $\delta\varphi$ of Γ.

Since we have $t_{\delta_i} = 1$, the element $\delta_i f_{\delta_i}$ is the identity of $\mathrm{Sym}\ \Gamma$ and hence fixes $\delta\varphi$. Let $\delta \in \Delta_i \setminus \{\delta_i\}$ and consider $s_\delta = fh \in X_{\Gamma_{\delta_i}}$ with $f \in B$ and $h \in \mathrm{Sym}\ \Delta$. Then $\delta_i h = \delta_i$, and the element $s_\delta t_\delta$ is equal to $f'_\delta h'_\delta$, with $f'_\delta = f f_\delta^{h^{-1}}$ and $h'_\delta = h h_\delta$, and satisfies $\Gamma_{\delta_i} s_\delta t_\delta = \Gamma_\delta$. Moreover, $\delta_i f'_\delta = (\delta_i f)((\delta_i h) f_\delta) = (\delta_i f)(\delta_i f_\delta)$, and we note that $\delta_i f \in \mathrm{Sym}\ \Gamma$ lies in the δ_i-component of X; see (1.4). If the δ_i-component is transitive on Γ, then we may choose s_δ in $X_{\Gamma_{\delta_i}}$ such that the element $(\delta_i f)((\delta_i) f_\delta)$ fixes $\delta\varphi$. Replacing t_δ by $s_\delta t_\delta$ gives an element with the required properties.

https://doi.org/10.1017/S1446788712000110 Published online by Cambridge University Press

Claim 2. For $\delta \in \Delta_i$, the δ-component X^{Γ_δ} equals $(X^{\Gamma_{\delta_i}})^{\delta_i f_\delta}$.

Let $\delta_i f \in X^{\Gamma_{\delta_i}}$. By (1.4), there exists $h \in \mathrm{Sym}\ \Delta$ such that $\delta_i h = \delta_i$ and $fh \in X_{\Gamma_{\delta_i}}$. Therefore X_{Γ_δ} contains

$$(fh)^{t_\delta} = f^{t_\delta} h^{f_\delta h_\delta} = f^{t_\delta}(f_\delta^{-1} f_\delta^{h^{-1}} h)^{h_\delta} = f^{t_\delta}(f_\delta^{-1})^{h_\delta} f_\delta^{h^{-1}h_\delta} h^{h_\delta}.$$

This implies that the δ-component X^{Γ_δ} contains

$$\delta(f^{t_\delta}(f_\delta^{-1})^{h_\delta} f_\delta^{h^{-1}h_\delta}) = ((\delta h_\delta^{-1}) f^{f_\delta})((\delta h_\delta^{-1}) f_\delta^{-1})((\delta h_\delta^{-1} h) f_\delta)$$

and using the facts that $\delta_i h_\delta = \delta$ and $\delta_i h = \delta_i$, this is equal to

$$((\delta_i) f^{f_\delta})((\delta_i) f_\delta^{-1})((\delta_i) f_\delta) = \delta_i(f^{f_\delta} f_\delta^{-1} f_\delta) = \delta_i f^{f_\delta} = (\delta_i f)^{\delta_i f_\delta}.$$

Thus X^{Γ_δ} contains $(X^{\Gamma_i})^{\delta_i f_\delta}$, and a similar argument proves the reverse inclusion. Hence equality holds and the claim is proved.

Definition of x: Define $x \in B = \mathrm{Func}(\Delta, \mathrm{Sym}\ \Gamma)$ as the function satisfying, for each i and each $\delta \in \Delta_i$, $\delta x = \delta_i f_\delta^{-1}$. If all components of X are transitive on Γ then we assume (as we may by Claim 1) in addition that, for each i and $\delta \in \Delta_i$, $\delta_i f_\delta$ fixes the point $\delta\varphi$, and hence $\delta x = \delta_i f_\delta^{-1} = (\delta_i f_\delta)^{-1}$ fixes $\delta\varphi$. Thus in this case x fixes φ.

Claim 3. The components of $x^{-1}Xx$ are constant on each of the Δ_i.

Since x acts trivially on Δ, the stabiliser $(X^x)_{\Gamma_\delta} = (X_{\Gamma_\delta})^x$ for each $\delta \in \Delta$. Thus δf lies in the δ-component X^{Γ_δ} if and only if there exists $h \in \mathrm{Sym}\ \Delta$ such that $fh \in X_{\Gamma_\delta}$ or, equivalently, $(fh)^x = f^x x^{-1} x^{h^{-1}} h \in (X^x)_{\Gamma_\delta}$. This implies that the δ-component of X^x contains

$$\delta(f^x x^{-1} x^{h^{-1}}) = \delta(x^{-1} f x^{h^{-1}}) = (\delta x^{-1})(\delta f)((\delta h)x) = (\delta f)^{\delta x}$$

since $\delta h = \delta$. Thus the δ-component of X^x contains $(X^{\Gamma_\delta})^{\delta x}$ and a similar argument proves the reverse inclusion, so equality holds. Now $\delta x = \delta_i f_\delta^{-1} = (\delta_i f_\delta)^{-1}$, which by Claim 2 conjugates X^{Γ_δ} to $X^{\Gamma_{\delta_i}}$. Thus

$$(X^x)^{\Gamma_\delta} = (X^{\Gamma_\delta})^{\delta x} = (X^{\Gamma_\delta})^{\delta_i f_\delta^{-1}} = X^{\Gamma_{\delta_i}}$$

for all $\delta \in \Delta_i$. This completes the proof of Claim 3, and part (a) follows.

To prove part (b) we assume that the group H induced by X on Δ is transitive, and let G be the δ_1-component of X. From what we have just proved, each component of X^x is equal to G. Let g' be an arbitrary element of X^x. Then $g' = x^{-1}gx$ for some $g \in X$, and we have $g = fh$ with $f \in B$ and $h \in \mathrm{Sym}\ \Delta$. By the definition of H, we have $h \in H$. Also

$$g' = x^{-1} f h x = (x^{-1} f x^{h^{-1}}) h = f' h, \quad \text{say.}$$

Thus, in order to prove that $g' \in G \wr H$, it is sufficient to prove that, for each $\delta \in \Delta$, $\delta f' \in G$.

https://doi.org/10.1017/S1446788712000110 Published online by Cambridge University Press

Let $\delta' := \delta h$. Then $hh_{\delta'}^{-1}h_\delta$ fixes δ, and so X_{Γ_δ} contains

$$gt_{\delta'}^{-1}t_\delta = fhh_{\delta'}^{-1}f_{\delta'}^{-1}f_\delta h_\delta = f(f_{\delta'}^{-1}f_\delta)^{h_{\delta'}h^{-1}}hh_{\delta'}^{-1}h_\delta.$$

Hence $(X^x)_{\Gamma_\delta} = (X_{\Gamma_\delta})^x$ contains

$$x^{-1}gt_{\delta'}^{-1}t_\delta x = (x^{-1}f(f_{\delta'}^{-1}f_\delta)^{h_{\delta'}h^{-1}}x^{h_\delta^{-1}h_{\delta'}h^{-1}})hh_{\delta'}^{-1}h_\delta$$

which equals $f''hh_{\delta'}^{-1}h_\delta$, say. This means that the δ-component G of X^x contains

$$\begin{aligned}\delta f'' &= (\delta x^{-1}f)((\delta hh_{\delta'}^{-1})(f_{\delta'}^{-1}f_\delta))((\delta hh_{\delta'}^{-1}h_\delta)x)\\ &= (\delta x^{-1}f)(\delta_1(f_{\delta'}^{-1}f_\delta))(\delta x).\end{aligned}$$

By the definition of x, $\delta_1(f_{\delta'}^{-1}f_\delta) = (\delta_1 f_{\delta'}^{-1})(\delta_1 f_\delta) = (\delta' x)(\delta x)^{-1}$. It follows that

$$\delta f'' = (\delta x^{-1}f)(\delta' x) = (\delta x^{-1}f)(\delta x^{h^{-1}}) = \delta(x^{-1}fx^{h^{-1}}) = \delta f'.$$

Therefore $\delta f' \in G$, as required. Thus part (b) is proved, completing the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Let $W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ act in product action on $\Pi = \mathrm{Func}(\Delta, \Gamma)$ with base group $B = \mathrm{Func}(\Delta, \mathrm{Sym}\ \Gamma)$, where Δ, Γ are finite sets. Suppose that X is a transitive subgroup of W, and let $K := X \cap B$.

Let $\delta \in \Delta$, let Δ_0 be the orbit of X in Δ containing δ, and let $\Delta_1 = \Delta \setminus \Delta_0$. By Proposition 1.4(d), the permutation actions of X on Π and on $\Omega_0 \times \Omega_1$ are equivalent. In particular, as X is transitive on Π, its projection $X\sigma_0$ is transitive on Ω_0. Further (defining Γ_δ^0 as in the proof of Proposition 1.4(c)), if $X^{\Gamma_\delta^0}$ is transitive then, by Proposition 1.4(c), X^{Γ_δ} is transitive. Thus it suffices to prove that all components of X are transitive in the case where X acts transitively on Δ. So assume that X is transitive on Δ. Let $r := |\Delta|$, and suppose that, for some $\delta \in \Delta$, the δ-component X^{Γ_δ} is intransitive. Now K is a normal subgroup of X_{Γ_δ} and hence, by (1.4), the δ-component K^{Γ_δ} of K is a normal subgroup of X^{Γ_δ}. Hence K^{Γ_δ} has s orbits in its action on Γ for some $s > 1$. Since X is transitive on Δ and normalises K, it follows that K^{Γ_δ} has s orbits for each $\delta \in \Delta$. Define $L := \{f \in B \mid \delta f \in K^\delta \text{ for each } \delta \in \Delta\}$. Then $L \cong \prod_{\delta\in\Delta} K^\delta$, L has s^r orbits in Π, and $K \le L \cap X$. Moreover, X normalises L and, since X is transitive on Π, it permutes the s^r orbits of L transitively and K lies in the kernel of this action. Thus $|X/K|$ is divisible by s^r. However X/K is isomorphic to the transitive group induced by X on Δ and hence $|X/K|$ divides $r!$. Thus s^r divides $r!$. However, this is impossible since for any prime p dividing s, the order of a Sylow p-subgroup of $\mathrm{Sym}\ \Delta$ is at most p^{r-1}. Thus $s = 1$. This proves both assertions of Theorem 1.2. □

https://doi.org/10.1017/S1446788712000110 Published online by Cambridge University Press

4. Proof of Theorem 1.3

Let $\Delta = \{1, \ldots, m\}$, and let γ, ν be distinct elements of Γ. Suppose that $C \subset \Gamma^m$ has minimum distance d, cardinality $|C| > 1$, and automorphism group $X \leq W = \mathrm{Sym}\ \Gamma \wr \mathrm{Sym}\ \Delta$ such that X induces a transitive group H on Δ and the 1-component X^{Γ_1} of X is a 2-transitive subgroup G of Sym Γ. In this context it is convenient to identify $\Pi = \mathrm{Func}(\Delta, \Gamma)$ with Γ^m, and the base group B of W with $(\mathrm{Sym}\ \Gamma)^m$. Under this identification, for example, the subgroup $L = \{f \in B \mid \delta f \in X^{\Gamma_\delta} \text{ for all } \delta \in \Delta\}$ of B is identified with the direct product $\prod_{\delta\in\Delta} X^{\Gamma_\delta}$ of the components of X. Moreover, since X acts transitively on Δ, each of the X^{Γ_δ} is 2-transitive on Γ.

Let $a := (\gamma_1, \ldots, \gamma_m), b := (\beta_1, \ldots, \beta_m) \in C$ be codewords at distance d. Since G is transitive on Γ, the subgroup L of the base group B is transitive on Γ^m, so there is an element $x_1 \in L$ such that $ax_1 = (\gamma^m)$. Then, since x_1 normalises each of the direct factors X^{Γ_δ} of L, it follows that X^{x_1} has the same components as X. Now we apply Theorem 1.1(b) and obtain an element $x_2 \in B$ such that $X^{x_1x_2} \leq G \wr H$ and $ax_1x_2 = (\gamma^m)x_2 = (\gamma^m)$. Now the image bx_1x_2 differs from (γ^m) in exactly d entries. Let I denote this d-subset of Δ. Choose x_3 in the top group Sym Δ of W such that $Ix_3 = \{1, \ldots, d\}$. Then $Cx_1x_2x_3$ contains $ax_1x_2x_3 = (\gamma^m)x_3 = (\gamma^m)$ and $bx_1x_2x_3$, and the latter m-tuple differs from (γ^m) precisely in the d-subset $Ix_3 = \{1, \ldots, d\}$. Thus entries $d + 1, \ldots, m$ of $bx_1x_2x_3$ are all equal to γ. The automorphism group $X^{x_1x_2x_3}$ of $Cx_1x_2x_3$ has the same components as $X^{x_1x_2}$ (which are all equal to G) and induces the transitive group $K := H^{x_2}$ on Δ. Thus $X^{x_1x_2x_3} \leq G \wr K$. Finally, since G is 2-transitive on Γ, for each $i \leq d$ there is an element $y_i \in G_\gamma$ which maps the ith entry of $bx_1x_2x_3$ to ν. Let $x_4 \in \mathrm{Func}(\Delta, G_\gamma) \leq B$ be any element such that $ix_4 = y_i$ for $i = 1, \ldots, d$, and set $x = x_1x_2x_3x_4$. Then $X^x \leq G \wr K$ and Cx contains (γ^m) and (ν^d, γ^{m-d}). □

Acknowledgement

The authors are grateful to an anonymous referee for suggestions leading to clarification and improvement of the exposition.

References

[1] M. Aschbacher, 'Overgroups of primitive groups', *J. Aust. Math. Soc.* **87** (2009), 37–82.

[2] M. Aschbacher, 'Overgroups of primitive groups. II', *J. Algebra* **322** (2009), 1586–1626.

[3] M. Aschbacher and J. Shareshian, 'Restrictions on the structure of subgroup lattices of finite alternating and symmetric groups', *J. Algebra* **322** (2009), 2449–2463.

[4] R. W. Baddeley, C. E. Praeger and C. Schneider, 'Transitive simple subgroups of wreath products in product action', *J. Aust. Math. Soc.* **77** (2004), 55–72.

[5] R. W. Baddeley, C. E. Praeger and C. Schneider, 'Innately transitive subgroups of wreath products in product action', *Trans. Amer. Math. Soc.* **358** (2006), 1619–1641.

[6] R. W. Baddeley, C. E. Praeger and C. Schneider, 'Intransitive Cartesian decompositions preserved by innately transitive permutation groups', *Trans. Amer. Math. Soc.* **360** (2008), 743–764.

[7] M. Bhattacharjee, D. Macpherson, R. G. Möller and P. M. Neumann, *Notes on Infinite Permutation Groups*, Texts and Readings in Mathematics, 12 (Hindustan Book Agency, New Delhi, 1997; co-published by Springer, Berlin, 1997).

[8] P. J. Cameron, *Permutation Groups*, London Mathematical Society Student Texts, 45 (Cambridge University Press, Cambridge, 1999).
[9] J. D. Dixon and B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics, 163 (Springer, New York, 1996).
[10] N. Gillespie, 'Neighbour transitive codes', PhD Thesis, The University of Western Australia, 2011.
[11] M. Giudici and C. E. Praeger, 'Completely transitive codes in Hamming graphs', *European J. Combin.* **20** (1999), 647–661.
[12] L. G. Kovács, 'Wreath decompositions of finite permutation groups', *Bull. Aust. Math. Soc.* **40** (1989), 255–279.
[13] C. E. Praeger, 'An O'Nan–Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs', *J. Lond. Math. Soc.* (2) **47** (1993), 227–239.
[14] C. E. Praeger and C. Schneider, 'Three types of inclusions of innately transitive permutation groups into wreath products in product action', *Israel J. Math.* **158** (2007), 65–104.

CHERYL E. PRAEGER, Centre for Mathematics of Symmetry and Computation, School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009, Australia
e-mail: cheryl.praeger@uwa.edu.au

CSABA SCHNEIDER, Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal
e-mail: csaba.schneider@gmail.com

https://doi.org/10.1017/S1446788712000110 Published online by Cambridge University Press