J. Austral. Math. Soc. (*Series A*) **54** (1993), 204–212

CROSSED PRODUCTS OF C^*-ALGEBRAS BY $*$-ENDOMORPHISMS

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(Received 1 November 1990)

Communicated by C. Sutherland

Abstract

Crossed products of C^*-algebras by $*$-endomorphisms are defined in terms of a universal property for covariant representations implemented by families of isometries and some elementary properties of covariant representations and crossed products are obtained.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 46 L 40, 46 L 05.

1. Introduction

In the fundamental paper [3] on the algebras O_n generated by families of isometries, Cuntz showed how O_n could be regarded as a crossed product $\mathrm{UHF}(n^\infty) \times_\alpha \mathbb{N}$ of the UHF algebra $\mathrm{UHF}(n^\infty)$ by the (non-unital) shift endomorphism $\alpha: x \mapsto e_{11} \otimes x$. As he remarked in the subsequent paper [4] this procedure can be extended to a $*$-endomorphism α of an arbitrary C^*-algebra A by firstly considering the inductive limit A_∞ of the system $A \underset{\alpha}{\to} A \underset{\alpha}{\to} A \underset{\alpha}{\to} \cdots$. (The notation used in this paper for the associated maps from A to A_∞ will be $\alpha_0, \alpha_1, \alpha_2, \dots$.)

The system

$$\begin{tikzcd}
A \arrow[r, "\alpha"'] \arrow[d, "\alpha"'] & A \arrow[r, "\alpha"'] \arrow[d, "\alpha"'] & A \arrow[r, "\alpha"'] \arrow[d, "\alpha"'] & \cdots \\
A \arrow[r, "\alpha"'] & A \arrow[r, "\alpha"'] & A \arrow[r, "\alpha"'] & \cdots
\end{tikzcd}$$

gives rise to an automorphism α_∞ of A_∞: let $A_\infty \times_{\alpha_\infty} \mathbb{Z}$ be the associated crossed product, with canonical unitary $u_\infty \in M(A_\infty \times_{\alpha_\infty} \mathbb{Z})$, and canonical injection $i_\infty : A_\infty \to A_\infty \times_{\alpha_\infty} \mathbb{Z}$, and, when $p \in M(A_\infty \times_{\alpha_\infty} \mathbb{Z})$, define the crossed product $A \times_\alpha \mathbb{N}$ to be $p(A_\infty \times \alpha_\infty \mathbb{Z})p$, where $p = (i_\infty \circ \alpha_0)(1)$. (Throughout this paper $M(B)$ denotes the multiplier algebra of a C^*-algebra B and the same symbol is used for a $*$-homomorphism from B onto C and its natural extension from $M(B)$ into $M(C)$.) It is easy to see that $v = pu_\infty p$ is an isometry in $A \times_\alpha \mathbb{N}$ implementing α and that $A \times_\alpha \mathbb{N}$ has a universal property for non-degenerate representations in which α is implemented by a single isometry; for completeness this is included as Proposition 3.3 of the present paper.

Another approach to crossed products of certain C^*-algebras by $*$-endomorphisms occurs in work by Doplicher and Roberts. In [6] they focus attention on endomorphisms of unital C^*-algebra which are covariantly implemented not by a single isometry but by a family $v_1, \dots, v_n$ of isometries with $v_i^* v_j = \delta_{ij} 1$; this situation is also described by them in terms of an appropriate Hilbert space of isometries. The purpose of the present note is to indicate a common framework for looking at both types of crossed products: given a $*$-endomorphism α of a C^*-algebra A an infinite family of crossed products will be defined. For each cardinal n the crossed product $A \times_\alpha^n \mathbb{N}$ will be the appropriate universal object for covariant representations of (A, α) implemented by a family of n isometries; in the case $n = 1$ the crossed product is that defined by Cuntz and for other finite n it is an extension of that defined by Doplicher and Roberts.

I am grateful to Iain Raeburn for pointing out an error in an earlier version of this paper.

2. Covariant representations

Let π be a non-degenerate representation of a C^*-algebra A on a Hilbert space H (that is, a representation with $\pi(1) = 1_H$, where $1 \in M(A)$) and let α be a $*$-endomorphism of A. There may (but need not) be a $*$-endomorphism β of $B(H)$ with $\beta(\pi(a)) = \pi(\alpha(a))$ for all $a \in A$: if there is then, by [1, Proposition 2.3] (or, more precisely, the minor generalisation in [12, Proposition 2.3], which also extends to non-separable spaces), there exists a family $\{T_i\}_{i \in I}$ of isometries on H such that $\pi(\alpha(a)) = \sum_i T_i \pi(a) T_i^*$ for each $a \in A$ (where weak convergence is used for infinite I). We will describe this situation by saying that the pair $(\pi, \{T_i\}_{i \in I})$ is a *covariant* *representation* of (A, α) of multiplicity $|I|$. The following examples illustrate some of the range of possibilities.

EXAMPLES 2.1. (a) Let α be the unilateral shift, defined by $\alpha((x_1, x_2, \ldots)) = (x_2, x_3, \cdots)$ on c_0. Then, for each $x \in c_{00}$, the dense subalgebra of sequences of finite support, $\alpha^n(x) = 0$ for some n. Hence, if $(\pi, \{T_i\})$ is a covariant representation of (c_0, α) on H and β is the corresponding $*$-endomorphism of $B(H)$, then $\beta^n(\pi(x)) = 0$ and hence $\pi(x) = 0$. It follows that π is zero on c_{00} and therefore, by continuity, $\pi = 0$ so that (c_0, α) possesses no (non-degenerate) covariant representations.

(b) Let α be a $*$-automorphism of a C^*-algebra A, let (ψ, U) be a covariant representation (in the usual sense) of (A, α) on H and let $\{S_i\}_{i\in I}$ be a family of isometries on some Hilbert space K with $S_i^* S_j = \delta_{ij} 1$ and $\sum_i S_i S_i^* = 1$. Then $(\psi \otimes 1, \{U \otimes S_i\}_{i\in I})$ is a covariant representation with multiplicity $|I|$ of (A, α) on $H \otimes K$: hence (A, α) possesses covariant representations of all multiplicities.

(c) Let α be the canonical shift endomorphism of O_n, where $n < \infty$, defined by $\alpha(x) = \sum_{i=1}^n S_i x S_i^*$. Then, for every non-degenerate representation π of O_n, $(\pi, \{\pi(S_i)\})$ is a covariant representation of multiplicity n. Some of these representations also have other multiplicities: for example let π be a type III representation on a separable Hilbert space and note that, as described in [11, 2.9.26], the endomorphism $x \mapsto \sum \pi(S_i) x \pi(S_i^*)$ of $\pi(O_n)''$ also extends to an automorphism $Ad(U)$ of $B(H)$.

Example 2.1(b) indicates how covariant representations of arbitrary multiplicity can be constructed from representations of multiplicity one; combining this method with Cuntz's approach to multiplicity one crossed products gives the following result. In the proof and throughout this paper $T_\mu T_\nu^*$ will be used to denote $T_{\mu_1} \cdots T_{\mu_r} T_{\nu_s}^* \cdots T_{\nu_1}^*$, where $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_s)$ have lengths $r = |\mu|$ and $s = |\nu|$.

PROPOSITION 2.2. *Let α be a $*$-endomorphism of a C^*-algebra A and let n be a cardinal number. (A, α) possesses (non-degenerate) covariant representations $(\pi, \{T_i\})$ of multiplicity n if and only if $A_\infty \neq \{0\}$.*

PROOF. If $A_\infty = \{0\}$ and $a \in A$ then, by the definition of A_∞ as described, for example, in [8, Chapter 2], there exists $k \in \mathbb{N}$ with $\alpha^k(a) = 0$. If $(\pi, \{T_i\})$ is a covariant representation of multiplicity n of (A, α) on H then $0 = \pi(\alpha^k(a)) = \sum_{|\mu|=k} T_\mu \pi(a) T_\mu^*$ and hence $\pi(a) = 0$, giving a contradiction. Conversely, let $A_\infty \neq \{0\}$, let ψ be a faithful representation of $A_\infty \times_{\alpha_\infty} \mathbb{Z}$ on some Hilbert space H and let $\{S_i\}$ be a family of n isometries on some Hilbert space K with $\sum S_i S_i^* = 1_K$. Then $((\psi \circ i_\infty \circ \alpha_0) \otimes 1,$

https://doi.org/10.1017/S1446788700037113 Published online by Cambridge University Press

$\{\psi(u_\infty)\otimes S_i\})$ is a covariant representation of multiplicity n of (A, α) on $\psi(p)H\otimes K$ (which is non-zero since $\alpha_0(1)\neq 0$).

The connection between A_∞ and covariant representations of (A, α) is further elaborated in the next result.

Proposition 2.3. *Let α be a $*$-endomorphism of a C^*-algebra A, let $(\pi, \{T_j\})$ be a covariant representation of (A, α) on H and let $\widetilde{H}$ be the Hilbert space inductive limit of the sequence $H \xrightarrow{T_1} H \xrightarrow{T_1} H \xrightarrow{T_1} \cdots$ (with corresponding embeddings $W_0: H\to\widetilde{H}$, $W_1: H\to\widetilde{H}, \cdots$). Then there exists a unitary U on $\widetilde{H}$ and a covariant representation $(\tilde{\pi}, U)$ of $(A_\infty, \alpha_\infty)$ on $\widetilde{H}$ such that $\tilde{\pi}\circ\alpha_0$ restricts to $Ad(W_0)\circ\pi$ on $W_0(H)$ and $UW_0 = W_0T_1$.*

Proof. Define U to be the unitary on $\widetilde{H}$ arising from the sequence

$$\begin{tikzcd} H \arrow[r, "T_1"] \arrow[d, "T_1"'] & H \arrow[r, "T_1"] \arrow[d, "T_1"'] & H \arrow[r, "T_1"] \arrow[d, "T_1"'] & \cdots \\ H \arrow[r, "T_1"'] & H \arrow[r, "T_1"'] & H \arrow[r, "T_1"'] & \cdots \end{tikzcd}$$

and observe that, for each $h\in H$, $UW_m(h) = W_{m-1}(h)$ for each $m\geq 1$ and $UW_0(h) = W_0(T_1h)$. Let π_0 be the representation of A on $\widetilde{H}$ defined by $\pi_0(a)W_m(h) = W_m(\pi(\alpha^m(a))h)$ for each $a\in A$ and each $h\in H$ (which is well-defined because $\pi(\alpha^m(a))T_1h = \sum_i T_i\pi(\alpha^{m-1}(a))T_i^*T_1h = T_1\pi(\alpha^{m-1}(a))h)$ and let $\pi_n = (Ad(U^*)^n)\pi_0$. A routine calculation shows that $\pi_{n+1}(\alpha(a))W_m(h) = \pi_n(a)W_m(h)$ for each $m\geq n+1$, each $h\in H$ and each $a\in A$, from which it follows that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\pi_0"'] & A \arrow[r, "\alpha"] \arrow[d, "\pi_1"] & A \arrow[r, "\alpha"] \arrow[dl, "\pi_2"] & \cdots \\ & B(\widetilde{H}) & & \end{tikzcd}$$

Hence there exists a representation $\tilde{\pi}$ of A_∞ on $\widetilde{H}$ with $\tilde{\pi}\circ\alpha_n = \pi_n$ for each $n\geq 0$, from which it follows that $\tilde{\pi}\circ\alpha_0$ restricts to $Ad(W_0)\circ\pi$ on $W_0(H)$. If $a\in A$ then, for each $n\geq 1$, $(\tilde{\pi}\circ\alpha_\infty\circ\alpha_n)(a) = (\tilde{\pi}\circ\alpha_n\circ\alpha)(a) = \pi_n(\alpha(a)) = \pi_{n-1}(a) = Ad(U)\pi_n(a) = Ad(U)(\tilde{\pi}\circ\alpha_n)(a)$ and hence $\tilde{\pi}\circ\alpha_\infty = (Ad(U))\circ\tilde{\pi}$, as required.

3. Crossed products

As in [10], crossed products of C^*-algebras by $*$-endomorphisms can be defined by their universal property.

DEFINITION 3.1. A crossed product of multiplicity n for a $*$-endomorphism α of a C^*-algebra A with $A_\infty \neq \{0\}$ is a C^*-algebra B together with a $*$-homomorphism $i_A\colon A \to B$ with $i_A(1_{M(A)}) = 1_{M(B)}$, and a family $\{t_i\}$ of n isometries in $M(B)$ with $t_i^* t_j = \delta_{ij} 1$ for each i, j such that

(a) $i_A(\alpha(a)) = \sum_j t_j i_A(a) t_j^*$ for each $a \in A$,
(b) for every covariant representation $(\pi, \{T_i\})$ of (A, α) with multiplicity n there exists a non-degenerate representation $\pi \times T$ of B with $(\pi \times T) \circ i_A = \pi$ and $(\pi \times T)(t_i) = T_i$ for each i,
(c) B is generated by elements of the form $i_A(a) t_\mu t_\nu^*$.

When n is infinite the expression $\sum_j t_j i_A(a) t_j^*$ is to be interpreted using weak convergence in B^{**}: condition (a) requires that the limit exists and belongs to B.

If $A_\infty = \{0\}$ then, by Proposition 2.2, the pair (A, α) possesses no covariant representations so that condition (b) in Definition 3.1 becomes vacuous: we therefore choose not to define a crossed product when $A_\infty = \{0\}$.

PROPOSITION 3.2. *Let α be a $*$-endomorphism of a C^*-algebra A for which $A_\infty \neq \{0\}$. Then, for any cardinal n, there exists a unique crossed product $A \times_\alpha^n \mathbb{N}$ of multiplicity n.*

PROOF. Following the proof of [10, Proposition 3] let S be a set of covariant representations of multiplicity n of (A, α) such that every cyclic covariant representation of multiplicity n is equivalent to a member $(\pi, \{T_i^\pi\})$ of S, let $H = \bigoplus\{H_\pi\colon (\pi, \{T_i^\pi\}) \in S\}$, let $i_A = \bigoplus \pi\colon A \to B(H)$, let $T_i = \bigoplus_\pi T_i^\pi$ and let $A \times_\alpha^n \mathbb{N}$ be the C^*-algebra generated by elements of the form $i_A(a) T_\mu T_\nu^*$. Since S is non-empty, $A \times_\alpha^n \mathbb{N}$ is the required unique crossed product (as in [10, Proposition 3]).

To complete the paper, the description of the crossed product in Proposition 3.2 will be supplemented by others, starting with special cases in which the crossed product takes a simple form.

PROPOSITION 3.3. *Let α be a $*$-endomorphism of a C^*-algebra A with $A_\infty \neq \{0\}$ and with $p = i_\infty(\alpha_0(1)) \in M(A_\infty \times_{\alpha_\infty} \mathbb{Z})$. Then the multiplicity one crossed product $A \times_\alpha^1 \mathbb{N}$ is isomorphic to $p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$.*

https://doi.org/10.1017/S1446788700037113 Published online by Cambridge University Press

PROOF. Let $t = pu_\infty p$. Then $t^*t = pu_\infty^* pu_\infty p = pi_\infty(\alpha_\infty^{-1}\alpha_0(1))p = pi_\infty(\alpha_1(1))p = p$, so that t is an isometry in $M(p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p)$. Let $i_A: A \to p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$ be defined by $i_A(a) = (i_\infty \circ \alpha_0)(a)$: it then follows that $ti_A(a)t^* = pu_\infty(i_\infty \circ \alpha_0)(a)u_\infty^* p = p(i_\infty \circ \alpha_\infty \circ \alpha_0(a))p = p(i_\infty \circ \alpha_0 \circ \alpha)(a)p = i_A(\alpha(a))$ for each $a \in A$.

To see that elements of the form $i_A(a)t_\mu t_\nu^*$ generate $p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$ note firstly that $p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$ is the closed linear span of the elements $p(i_\infty \circ \alpha_m)(a)u_\infty^n p$ where $a \in A$, $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m \geq -n$. However

$$\begin{aligned} t = pu_\infty p &= u_\infty u_\infty^*(i_\infty \circ \alpha_0)(1)u_\infty(i_\infty \circ \alpha_0)(1) \\ &= u_\infty(i_\infty \circ \alpha_1)(1)(i_\infty \circ \alpha_0)(1) = u_\infty p \end{aligned}$$

and so

$$\begin{aligned} p(i_\infty \circ \alpha_m)(a)u_\infty^n p &= pu_\infty^{*m}(i_\infty \circ \alpha_m)(\alpha^m(a))u_\infty^{m+n} p \\ &= pu_\infty^{*m}(i_\infty \circ \alpha_0)(a)u_\infty^{m+n} p = t^{*m} i_A(a)t^{m+n}. \end{aligned}$$

Using an approximate identity it is easily checked that each of these elements is in the C^*-algebra generated by elements of the form $i_A(a)t_\mu t_\nu^*$.

Finally let (π, T) be a multiplicity one covariant representation of (A, α) on H and let $(\tilde{\pi}, U)$ be the associated representation of A_∞ on $\widetilde{H}$ given by Proposition 2.3. If $\tilde{\pi} \times U$ is the corresponding non-degenerate representation of $A_\infty \times_{\alpha_\infty} \mathbb{Z}$, note that $P_0 = (\tilde{\pi} \times U)(p) = \tilde{\pi}(\alpha_0(1))$ is the projection from $\widetilde{H}$ onto $W_0(\widetilde{H})$, so that $(\tilde{\pi} \times U)(pA_\infty \times_{\alpha_\infty} \mathbb{Z}p)$ leaves $W_0(H)$ invariant: hence a representation $\pi \times T$ of $pA_\infty \times_{\alpha_\infty} \mathbb{Z}p$ on H is defined by $Ad(W_0^*) \circ (\tilde{\pi} \times U)$. By Proposition 2.3, $(\pi \times T)(t) = Ad(W_0^*) \circ (\tilde{\pi} \times U)(pu_\infty p) = W_0^* P_0 U P_0 W_0 = W_0^* P_0 U W_0 = W_0^* P_0 W_0 T = W_0^* W_0 T = T$ and $(\pi \times T)(i_A(a)) = W_0^*(\tilde{\pi} \times U)((i_\infty \circ \alpha_0)(a))W_0 = W_0^* \tilde{\pi}(\alpha_0(a))W_0 = \pi(a)$, as required.

The next special case describes the multiplicity n crossed product of a $*$-automorphism α of A as the twisted tensor product $A \times_U O_n$ defined by Cuntz in [5]. He remarks that the methods of [3] show that if $A \subseteq B(H)$ and U is a unitary in $B(H)$ implementing the $*$-automorphism α of A, then the C^*-subalgebra $A \times_U O_n$ of $B(H) \otimes O_n$ generated by elements of the form $aU \otimes S_i$ is (up to isomorphism) independent of the unitary U implementing α: hence we have chosen to use the notation $A \times_\alpha O_n$ for this twisted tensor product. Notice that when O_n is $\mathbb{Z}$-graded (as described following [7, Theorem 3.1]), with O_n^m the closed subspace of O_n spanned by elements of the form $S_\mu S_\nu^*$ with $|\mu| - |\nu| = m$, then the twisted tensor product $A \times_\alpha O_n$ is the closure of the linear span L of elements of the form $aU^m \otimes b$ where $a \in A$ and $b \in O_n^m$. As in [3, 1.9] we can assume that the

https://doi.org/10.1017/S1446788700037113 Published online by Cambridge University Press

norm on $A \times_\alpha O_n$ is given on the $*$-subalgebra L by $\|x\| = \sup\{\|\pi(x)\| : \pi$ is a representation of L on a separable Hilbert Space$\}$.

PROPOSITION 3.4. *Let α be a $*$-automorphism of a C^*-algebra A. Then, for each $n \in \mathbb{N}$, $A \times_\alpha^n \mathbb{N} = A \times_\alpha O_n$.*

PROOF. Define $i_A : A \to A \times_\alpha O_n$ by $i_A(a) = a \otimes 1$ and $t_i \in M(A \times_\alpha O_n)$ by $t_i = U \otimes S_i$. (It is easy to see, using an approximate identity, that $i_A(a)$ does indeed belong to $A \times_\alpha O_n$.) By construction, $A \times_\alpha O_n$ is generated by elements of the form $i_A(a)t_\mu t_\nu^*$ and $i_A(\alpha(a)) = \sum_j t_j i_A(a) t_j^*$ for each $a \in A$. So let $(\pi, \{T_i\})$ be a multiplicity n covariant representation of (A, α) on H, let θ be the $*$-isomorphism from O_n into $B(H)$ taking S_i to T_i for each i and let ψ be the linear map on the linear span of elements of the form $aU^m \otimes b$, where $a \in A$ and $b \in O_n^m$, defined by $\psi(aU^m \otimes b) = \pi(a)\theta(b)$. A simple calculation shows that ψ is a $*$-homomorphism and hence, by the choice of the norm on $A \times_\alpha O_n$, ψ extends to a $*$-homomorphism from $A \times_\alpha O_n$ into $B(H)$, taking $i_A(a)$ to $\pi(a)$ and t_i to T_i.

COROLLARY 3.5. *Let α be an inner $*$-automorphism of a C^*-algebra A. Then, for each $n \in \mathbb{N}$, $A \times_\alpha^n \mathbb{N} = A \otimes O_n$.*

The next special case extends that described in Corollary 3.5. It concerns inner $*$-endomorphisms, that is, those of the form $a \mapsto \sum_i S_i a S_i^*$ for some family $\{S_i\}$ of isometries in $M(A)$. If A and B are C^*-algebras, then $A * B$ will be used to denote the maximal free product of A and B (defined, for example, in [2, Section 0]; T_n will be used to denote the C^*-subalgebra of O_{n+1} generated by $S_1, \ldots, S_n$, which is discussed in [3, Proposition 3.1].

PROPOSITION 3.6. *Let A be a C^*-algebra, let $S_1, \ldots, S_m \in M(A)$ satisfy $S_i^* S_j = \delta_{ij} 1$ and let α be the $*$-endomorphism of A defined by $\alpha(a) = \sum_{i=1}^m S_i a S_i^*$ for all $a \in A$. Let E be the C^*-subalgebra of $O_m * O_n$ or $T_m * T_n$ (depending on whether or not α is unital) generated by the elements $r_i^* R_j$ for $1 \le i \le m$ and $1 \le j \le n$, where $\{r_i\}$, $\{R_j\}$ are appropriate generating sets of isometries, and let I be the $*$-ideal of E generated by the elements $\sum_j r_i^* R_j R_j^* r_k - \delta_{ik} 1$ and $\sum_j R_i^* r_j r_j^* R_k - \delta_{ik} 1$. Then $A \times_\alpha^n \mathbb{N} = A \bigotimes_{\max} (E/I)$.*

PROOF. Let $i_A(a) = a \otimes (1 + I)$ and let $t_i = \sum_j S_j \otimes (r_j^* R_i + I) \in M(A) \bigotimes_{\max} (E/I)$. Then $t_i^* t_k = \sum_{j,m} S_j^* S_m \otimes (R_i^* r_j r_m^* R_k + I) = \delta_{ik} 1$ and, for each $a \in A$, $\sum_j t_j i_A(a) t_j^* = \sum_{i,j,m} S_j a S_m^* \otimes (r_j^* R_i R_i^* r_m + I) = \sum_j S_j a S_j^* \otimes (1 + I) = i_A(\alpha(a))$. The elements of the form $i_A(a) t_\mu t_\nu^*$ generate $A \bigotimes_{\max} (E/I)$

because $a \otimes (r_i^* R_j + I) = i_A(aS_i^*)t_j$ for each $a \in A$. So let $(\pi, \{T_i\})$ be a covariant representation of multiplicity n on H. It is easily checked that $\pi(S_i^*)T_j \in \pi(A)'$ for each i, j and hence there exists a $*$-homomorphism θ from $A \bigotimes_{\max} E$ into $B(H)$ mapping $a \otimes r_j^* R_i$ to $\pi(a)\pi(S_j^*)T_i$. The generators for $A \bigotimes_{\max} I$ belong to the kernel of θ which therefore gives a $*$-homomorphism from $A \bigotimes_{\max}(E/I)$ into $B(H)$ mapping $i_A(a)$ to $\pi(a)$ and t_i to T_i.

Corollary 3.7. *If $\alpha(a) = \sum_{i=1}^m S_i a S_i^*$ is an inner $*$-endomorphism of A then $A \times_\alpha^1 \mathbb{N} = A \otimes O_m$.*

Proof. Let $\{r_i\}$ be a set of generators for T_m and let R be an isometry generating the Toeplitz algebra T_1. Then, using the notation of Proposition 3.6, $\{r_i^* R\}$ generate E and $\{r_i^* RR^* r_k - \delta_{ik} 1\}$, $\{\sum_i R^* r_i r_i^* R - 1\}$ generate I, so that $R^* r_i + I$ are isometries of sum one generating E/I. (In the case $m = 1$ the universal property of the crossed product ensures that E/I is isomorphic to $C(S^1)$.)

The general crossed product $A \times_\alpha^n \mathbb{N}$ can also be described in terms of free products. For convenience the following proposition is stated for unital α although, as in Proposition 3.6, there is an analogous version for α non-unital, with T_n replacing O_n.

Proposition 3.8. *Let α be a $*$-endomorphism of a C^*-algebra A with a unital extension to $M(A)$, let $n \in \mathbb{N}$, let B be the C^*-subalgebra of $(A_\infty \times_{\alpha_\infty} \mathbb{Z}) * O_n$ generated by elements of the form $i_\infty(\alpha_0(a))t_\mu t_\nu^*$ where $a \in A$ and $t_i = u_\infty S_i$ and let I be the ideal of B generated by elements of the form $(i_\infty \circ \alpha_0 \circ \alpha)(a) - \sum_i t_i(i_\infty \circ \alpha_0)(a))t_i^*$. Then $A \times_\alpha^n \mathbb{Z} = B/I$.*

Proof. It is routine to check each of the required properties.

Finally it is appropriate to point out that, although the work by Doplicher and Roberts provided motivation for this investigation, their crossed products do not, in general, agree with those defined here. If A is a C^*-algebra with centre $\mathbb{C}1$ and α is an endomorphism with permutation symmetry of dimension $d > 1$ satisfying the special conjugate property (using terminology defined in [6, Section 4]) then the Doplicher and Roberts crossed product of A by α provides a covariant implementation of α and hence is a quotient of $A \times_\alpha^d \mathbb{N}$ as defined here. However their crossed product does not necessarily possess the universal property for covariant representations. On the other hand the crossed product defined by Paschke in [9] does coincide with that given here (by the remarks after the proof of his Theorem 1.).

Note added in proof

Another use of a universal property to specify a crossed product extending the usual one occurs in the paper 'Ordered groups and crossed products of C^*-algebras' Pacific J. Math. **148** (1991), 319–343 by G. J. Murphy. However Murphy considers semigroups of $*$-automorphisms and a different version of covariance, so there is only a slight relation to the crossed products studied here.

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