GEOMETRICAL AND MEASURE-THEORETIC STRUCTURES OF MAPS WITH A MOSTLY EXPANDING CENTER

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Abstract In this article we study physical measures for $C^{1+\alpha}$ partially hyperbolic diffeomorphisms with a mostly expanding center. We show that every diffeomorphism with a mostly expanding center direction exhibits a geometrical-combinatorial structure, which we call skeleton, that determines the number, basins and supports of the physical measures. Furthermore, the skeleton allows us to describe how physical measures bifurcate as the diffeomorphism changes under C^1 topology.

Moreover, for each diffeomorphism with a mostly expanding center, there exists a C^1 neighbourhood, such that diffeomorphism among a C^1 residual subset of this neighbourhood admits finitely many physical measures, whose basins have full volume.

We also show that the physical measures for diffeomorphisms with a mostly expanding center satisfy exponential decay of correlation for any Hölder observes. In particular, we prove that every C^2, partially hyperbolic, accessible diffeomorphism with 1-dimensional center and nonvanishing center exponent has exponential decay of correlations for Hölder functions.

Contents

1. Introduction and statement of results

Physical measures were introduced in the 1970s by Bowen, Ruelle and Sinai to study the large time behavior of Lebesgue typical points for Axiom A attractors. Such systems do not preserve volume (or any measure that is equivalent to the volume) due to the contraction near the attractor. For this reason, those measures are often supported on a zero-volume subset of the manifold but capture the behavior of points in a large set with positive Lebesgue measure. More precisely, an invariant measure μ is called a physical measure if the set

$$\mathcal{B}(\mu) := \left\{ x \in M : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \overset{weak*}{\longrightarrow} \mu \right\}$$

has positive volume. This set is known as the *basin of* μ. For Axiom A attractors, many properties of physical measures were studied by many different authors. We refer the readers to the review paper [55] and the book [10] for more details.

It is also known that the physical measures of Axiom A attractors have strong statistical properties, one of the most important of which is the *decay of correlations*. It can be seen as the speed at which the system loses dependence and starts to behave like a random system. To be more precise, we have the following definition.

Definition 1.1. Given observables $\phi, \psi : M \to \mathbb{R}$, we define the correlation function with respect to a measure μ as

$$C_\mu(\phi, \psi \circ f^n) = \left| \int \phi(\psi \circ f^n) d\mu - \int \phi d\mu \int \psi d\mu \right| \quad \text{for } n \geq 1.$$

We say that the system has decay of correlations if for all ϕ and ψ in some families of functions, $C_\mu(\phi, \psi \circ f^n)$ converges to zero as n goes to infinity.

With that we are ready to introduce the main application of the results in this article.

Theorem A. *Let f be a C^2 partially hyperbolic, volume-preserving diffeomorphism with 1-dimensional center. Assume that f is accessible and that the center Lyapunov exponent of the volume is nonvanishing. Then f has exponential decay of correlations: there is $d > 0$ such that*

$$C_{\text{vol}}(\phi, \psi \circ f^n) = \mathcal{O}(e^{-dn})$$

for all Hölder continuous $\phi : M \to \mathbb{R}$, and $\psi \in L^\infty(\text{vol})$.

This theorem generalises [18, Corollary 0.2], where it is shown that every C^2 accessible, partially hyperbolic diffeomorphism with 1-dimensional center is ergodic and has K-property. We remark that such systems are abundant; see the discussion in Subsection 3.2. Also note that the hyperbolicity assumption in the previous theorem (nonvanishing center exponent) is rather weak, yet we obtain a strong statistical property in the form of fast decay of correlations, central limit theorem and exponential large deviation control (the latter two results are the natural consequences of the decay of correlations; see [2] and [28]).

By [52, Section 8], if f is a C^2 partially hyperbolic, volume-preserving diffeomorphism with 1-dimensional center and $\lambda^c(\text{vol}) \neq 0$, then either f or f^{-1} has a *mostly expanding center*. The rest of this article is devoted to a general theory on such diffeomorphisms. In particular, Theorem A is a direct consequence of Theorem G.

1.1. Diffeomorphisms with a mostly expanding center

Shortly after the physical measures were introduced for Axiom A attractors, a program for investigating the physical measures of diffeomorphisms beyond uniform hyperbolicity was initiated by Alves, Bonatti and Viana in a sequence of papers, such as [1, 11], to name but a few. They introduced several classes of systems for which physical measures exist, and the number of physical measures is finite. Among them are *diffeomorphisms with a mostly contracting center* and *diffeomorphisms with a mostly expanding center*. In this article, we are particularly interested in the latter class.

Diffeomorphisms with a mostly expanding center are, roughly speaking, partially hyperbolic diffeomorphisms whose center Lyapunov exponents are positive. This class of systems was introduced by Alves, Bonatti and Viana ([1]) using a different, more technical definition. Later, another definition was given by Dolgopyat [25] and more recently by Andersson and Vásquez [3]. In [3], they also proposed the latter, somewhat stronger, definition as the official definition of having a mostly expanding center, which we will follow in this article.

We call a diffeomorphism f *partially hyperbolic* if there exists a decomposition $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle TM into three continuous invariant subbundles E^s_x and E^c_x and E^u_x such that $Df \mid E^s$ is a uniform contraction, $Df \mid E^u$ is a uniform expansion and $Df \mid E^c$ lies in between them:

$$\frac{\|Df(x)v^s\|}{\|Df(x)v^c\|} \leq \frac{1}{2} \quad \text{and} \quad \frac{\|Df(x)v^c\|}{\|Df(x)v^u\|} \leq \frac{1}{2}$$

for any unit vectors $v^s \in E^s_x$, $v^c \in E^c_x$, $v^u \in E^u_x$ and any $x \in M$. This notation was proposed by Brin and Pesin [14] and Pugh and Shub [44] independently as early as the 1970s. In this article, we will assume that both E^s and E^u bundles are nontrivial. In this case, it is well known (see, for example, [31]) that E^*, $* = s, u$ can be integrated into foliations $\mathcal{F}^*$, $* = s, u$, whose leaves are as smooth as the diffeomorphism f. A partially hyperbolic diffeomorphism f is called *accessible* if any point $x \in M$ can be reached from any other point $y \in M$ by an *su*-path, a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{F}^s$ or a single leaf of $\mathcal{F}^u$.

As shown by Bonatti, Díaz and Viana [10] and Dolgopyat [24], physical measures of any $C^{1+\alpha}$ partially hyperbolic diffeomorphism should be a *Gibbs u-state*, meaning that the conditional measures of μ with respect to the partition into local strongly unstable manifolds are absolutely continuous with respect to the Lebesgue measure along the unstable leaves.

Definition 1.2. A partially hyperbolic diffeomorphism $f : M \to M$ is *mostly expanding along the central direction* if f has positive central Lyapunov exponents almost everywhere with respect to every Gibbs u-state for f.

This definition is comparable to diffeomorphisms with a *mostly contracting center* (see, for example, [26]) and share similar properties with the latter. In particular, C^1 openness of the partially hyperbolic diffeomorphisms with a mostly expanding center was recently proved in [52]. Note, however, that the inverse of a diffeomorphism with a mostly expanding center may not be mostly contracting. This is because the space of Gibbs u-states of f could be very different from that of f^{-1}.

A list of examples for partially hyperbolic diffeomorphisms with a mostly expanding center will be provided in Section 3.

1.2. Index-$\dim(\mathbf{E^{cu}})$ skeleton

In this article, we will introduce a topological structure of f, known as the *skeleton*, and use it to study the structure of physical measures of f. To this end, for a C^1 partially hyperbolic diffeomorphism f with partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$, we denote by $i_{cu} = \dim(E^{cu})$ and $i_s = \dim(E^s)$, where $E^{cu} = E^c \oplus E^u$.

Definition 1.3. [1] We say that $\mathcal{S}$ is an *index i_s skeleton of f* if $\mathcal{S} = \{p_1, \cdots, p_k\}$ consists of finitely many hyperbolic saddles with stable index i_s such that

(a) $\bigcup_{i=1, \cdots k} \mathcal{F}^s(\mathrm{Orb}(p_i))$ is dense in M;
(b) $\mathcal{S}$ does not have a proper subset that satisfies property (a).

A set $\mathcal{S}$ consisting of finitely many hyperbolic saddles with stable index i_s and satisfying (a) above is called a *pre-skeleton* .

Let us observe that, in general, a partially hyperbolic diffeomorphism may not have any skeleton, because it may not have any hyperbolic periodic orbit at all. Even if it admits a set of periodic points such that the union of their stable manifolds is dense, such a set may have infinite cardinality. However, we will see in Section 4 that if f does have a skeleton, then all skeletons of f (with the same index) must have the same cardinality (Lemma 4.4). Furthermore, every pre-skeleton of f contains a skeleton (Lemma 4.5).

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[1]In [26], a different type of skeleton was defined for diffeomorphisms with a mostly contracting center, where the index of saddles in $\mathcal{S}$ equals $i_{cs} = \dim(E^s \oplus E^c)$. Instead of condition (a), there the union of a stable manifold of periodic orbits of the skeleton is a u-section. The existence of an index i_{cs} skeleton is a C^1 open property, but it is not necessarily true any more for an index i_s skeleton. For more discussion, see Section 4.

Finally, in Proposition 6.8 we will show that if f is $C^{1+\alpha}$ with a mostly expanding center (or if f is C^1 and close to a $C^{1+\alpha}$ diffeomorphism with a mostly contracting center), then f has an index i_s skeleton. Furthermore, in Section 7 we will see that the skeletons are robust under C^1 topology, in the sense that the continuation of a skeleton of f is a pre-skeleton for nearby C^1 maps. Note, however, that this property requires f to have a mostly expanding center, unlike the skeletons in [26].

The main result of this article shows that for such diffeomorphisms, skeletons provide rich geometrical information on the physical measures of f.

For simplicity, we will frequently suppress the dependence on the Hölder index α and write C^{1+}, because the Hölder index α does not play any particular role.

Theorem B. *Let f be a C^{1+} diffeomorphism with a mostly expanding center. Then f admits an index i_s skeleton. Moreover, let $\mathcal{S} = \{p_1, \cdots, p_k\}$ be any index i_s skeleton of f; then for each $p_i \in \mathcal{S}$ there exists a distinct physical measure μ_i such that*

(1) *both the closure of $W^u(Orb(p_i))$ and the homoclinic class of the orbit $Orb(p_i)$ coincide with $\mathrm{supp}(\mu_i)$;*

(2) *the closure of $\mathcal{F}^s(Orb(p_i))$ coincides with the closure of the basin of the measure μ_i.*

In particular, the number of physical measures of f is precisely $k = \#\mathcal{S}$. Moreover,

$$\mathrm{Int}(Cl(\mathcal{B}(\mu_i))) \cap \mathrm{Int}(Cl(\mathcal{B}(\mu_j))) = \emptyset$$

for $1 \leq i \neq j \leq k$, where $\mathcal{B}(\mu_i)$ is the basin of μ_i.

It is worth noting that the finiteness of the physical measures was known since the work of Alves, Bonatti and Viana [1]. However, Theorem B above provides a more detailed description on the geometric structure of the physical measures, which allows us to keep track of those measures as we perturb the system.

Remark 1.4. From the proof of Theorem B, we have more detailed description on the basins of μ_i: for every $p_i \in \mathcal{S}$, we write

$$\mathcal{O}_i = \bigcup_{x \in W^u(\mathrm{Orb}(p_i))} \mathcal{F}^s(x);$$

then $\mathcal{O}_i$ contains an open neighbourhood of $\mathrm{Orb}(p_i)$. We are going to show that $\mathcal{O}_i$ is open and dense in $Cl(\mathcal{F}^s(\mathrm{Orb}(p_i))) = Cl(\mathcal{B}(\mu_i))$. Moreover, $\mathcal{B}(\mu_i)$ is a full-volume subset of $\mathcal{O}_i$, and $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ for $1 \leq i \neq j \leq k$. This shows that the basin of different physical measures is *topologically* separated.

We would like to mention that the idea of using homoclinic classes to study measures was initiated by [30]; see also [26] and [19] for recent similar results.

As a corollary of the previous theorem, we are going to show that any iteration of f still has a mostly expanding center; furthermore, the number of physical measures of f^k is determined by the skeleton of f.

Corollary C. *Let f be a C^{1+} partially diffeomorphism with a mostly expanding center and $\mathcal{S} = \{p_1, \cdots, p_k\}$ be any index i_s skeleton of f. Then for any $n > 0$, f^n has a mostly*

expanding center and has finitely many physical measures with number bounded by

$$P = \prod_{i=1}^{k} \pi(p_i), \ where \ \pi(p_i) \ denotes \ the \ period \ of \ p_i. \tag{1}$$

Moreover, every physical measure of f^P is Bernoulli.

Recall that for C^2 diffeomorphisms, every hyperbolic measure that is mixing has the Bernoulli property [42, 22]. Here we will provide a direct proof using the general work of Ornstein and Weiss [41] and later show that the physical measures of f^P are indeed exponentially mixing.

1.3. Perturbation of physical measures

It was shown in [52] that partially hyperbolic diffeomorphisms with a mostly expanding center are C^1 open; that is, if a C^{1+} diffeomorphism f has a mostly expanding center, then any C^{1+} diffeomorphism g that is sufficiently C^1 close to f also has a mostly expanding center. In the following we will analyse how the physical measures vary with respect to the C^{1+} diffeomorphisms in C^1 topology, which generalises a similar result of Andersson and Vásquez ([4]) under $C^{1+\alpha}$ topology. The key observation here is that physical measures of f are associated with skeletons, which behaves well under C^1 topology.

Theorem D. *Let $f : M \to M$ be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center. Then there exists a C^1 neighbourhood $\mathcal{U}$ of f such that the number of physical measures depends upper semi-continuously in C^1 topology among diffeomorphisms in $Diff^{1+}(M) \cap \mathcal{U}$. Moreover, the number of physical measures is locally constant and the physical measures vary continuously in the weak* topology on a C^1 open and dense subset $\mathcal{U}^\circ \subset \mathcal{U}$.*

Indeed, the skeletons of f provide even more information on the physical measures for C^1 perturbed C^{1+} diffeomorphisms. In particular, the skeletons allow us to describe how the physical measures bifurcate as the diffeomorphism changes. To this end, we write $p_i(g)$ the continuation of the hyperbolic saddle p_i for g in a C^1 neighbourhood of f. Theorem D is a direct consequence of the following, more technical, result.

Theorem E. *Let f be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center and $\mathcal{S} = \{p_1, \cdots, p_k\}$ be a skeleton of f. There exists a C^1 neighbourhood $\mathcal{U}$ of f such that, for any C^1 diffeomorphism $g \in \mathcal{U}$, there is a subset of $\mathcal{S}(g) = \{p_1(g), \cdots, p_k(g)\}$ that is a skeleton. Consequently, for $g \in Diff^{1+}(M) \cap \mathcal{U}$, the number of physical measures of g is no larger than the number of physical measures of f. Moreover, these two numbers coincide if and only if there is no heteroclinic intersection within $\{p_i(g)\}$. In this case, each physical measure of g is close to some physical measure of f in the weak-* topology.*

In addition, restricted to any subset of $\mathcal{V} \subset \mathcal{U}$ where the number of physical measures is constant, the supports of the physical measures and the closures of their basins vary in a lower semi-continuous fashion, in the sense of the Hausdorff topology.

1.4. Existence of physical measures for C^1 generic diffeomorphisms

Previously, the study of physical measures mainly focused on maps that are sufficiently smooth; that is, with C^{1+} regularity. Recently, the new technique developed in [32, 23] enables us to shows the existence of physical measure for a large family of C^1 diffeomorphisms, such as those with a mostly contacting center.

In this article, we will further show the existence of physical measures for C^1 generic diffeomorphisms close to a partially hyperbolic diffeomorphism f that has a mostly expanding center.

Before stating the main theorem of this section, we need the following definition.

Definition 1.5. A set Λ of a homeomorphism f is *Lyapunov stable* if there is a sequence of open neighbourhoods $U_1 \supset U_2 \supset \cdots$ such that

(a) $\bigcap U_i = \Lambda$;
(b) $f^n(U_{i+1}) \subset U_i$ for any $n, i \geq 1$.

A set being Lyapunov stable means that points starting near Λ will not travel too far away from this set under forward iterations of *f*. However, this does not mean that Λ is an attractor.

We have the following C^1 locally generic result, which generalises Theorem E. We state it as a standalone result because the techniques involved are quite different from Theorem E.

Recall that a set $\mathcal{R}$ is called *residual* if it is a countable intersection of open and dense sets.

Theorem F. *Let $f : M \to M$ be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center and $\mathcal{S} = \{p_1, \cdots, p_k\}$ be a skeleton of f. Then there exists a C^1 neighbourhood $\mathcal{U}$ of f and a C^1 residual subset $\mathcal{R} \subset \mathcal{U}$ such that every C^1 diffeomorphism $g \in \mathcal{R}$ admits finitely many physical measures whose basins have full volume. The number of physical measures of g coincides with the cardinality of its skeleton, which is no more than the number of physical measures of f. Moreover, the physical measures of g are supported on disjoint Lyapunov stable chain recurrent classes, each of which is the homoclinic class of some saddle in its skeleton.*

1.5. Statistical properties

To study the speed of decay of correlations for systems beyond uniformly hyperbolic, in [53] Young used a type of Markov partition with infinitely many symbols to build towers for systems with nonuniform hyperbolic behavior. These structures are commonly referred to as Gibbs-Markov-Young (GMY) structures (see, for instance, [2].) It is well known that such maps have exponential speed of decay of correlations whenever the GMY structure has exponentially small tails. By Alves and Li in [2], which is built on the work of Gouëzel [28], the latter case happens if the center bundle has certain expansion and, moreover, the tail of *hyperbolic times* is exponentially small.

We are going to show that Alves and Li's criterion can be applied to partially hyperbolic diffeomorphisms with a mostly expanding center and, in particular, we prove exponential

decay of correlations and exponential large deviations for the physical measures of f, provided that f has a mostly expanding center.

Theorem G. *Let $f : M \to M$ be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center, $\mathcal{S} = \{p_1, \cdots, p_k\}$ be a skeleton of f and $P = \prod_{i=1}^{k} \pi(p_i)$. Then for every physical measure μ of f^P, there is $d > 0$ such that*

$$C_\mu(\phi, \psi \circ f^{Pn}) = \mathcal{O}(e^{-dn})$$

for Hölder continuous $\phi : M \to \mathbb{R}$, and $\psi \in L^\infty(\mu)$.

Corollary H. *Under the assumptions of Theorem G, for every physical measure μ of f^P and any Hölder continuous function ϕ, the following limit exists:*

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \int \left(\sum_{j=0}^{n-1} \phi \circ f^{jP} - n \int \phi d\mu \right)^2 d\mu.$$

Moreover, if $\sigma^2 > 0$, then there is a rate function $c(\varepsilon) > 0$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \mu \left(\left| \sum_{j=0}^{n-1} \phi \circ f^{jP} - n \int \phi d\mu \right| \geq \varepsilon \right) = -c(\varepsilon).$$

1.6. Robustly transitive partially hyperbolic diffeomorphisms

The diffeomorphisms with a mostly expanding center also provide a new mechanism to describe the topological transitivity property. To make this article more complete, we collect two results from two other papers without giving their proof. For more details, see the related papers and the references therein.

Theorem I ([52]). *Let f be a C^{1+} volume-preserving, partially hyperbolic diffeomorphism with a 1-dimensional center. Suppose that f is accessible and the center exponent is not vanishing; then f is C^1 robustly transitive; that is, every diffeomorphism g is transitive for g in a C^1 neighbourhood f, which is not necessarily volume preserving.*

Theorem J ([49]). *Let f be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center such that the stable foliation $\mathcal{F}^s$ is minimal. Then there is a C^1 neighbourhood $\mathcal{U}$ of f such that the stable foliation of any $g \in \mathcal{U}$ is minimal.*

1.7. Structure of the article

This article is organised as follows: In Section 2 we introduce the main tool of this article: a special space of probability measures, denoted by $\mathrm{G}(f)$, which is defined using the partial entropy along unstable leaves. This space will serve as the candidate space of physical measures.

Section 3 contains all the existing examples of diffeomorphisms with a mostly expanding center, as far as the author is aware. In particular, we collect some very recent examples from [52].

In Section 4, we provide some geometrical properties of skeletons, assuming that such structure exists (which will not be proven until Section 6). In particular, we will show that every skeleton of f must have the same cardinality and provide a useful criterion for the existence of a skeleton to be used in later sections.

Section 5 consists of a direct proof on the existence of physical measures for C^{1+} diffeomorphisms with a mostly expanding center. More important, we show that the space $G(f)$ is a finite-dimensional simplex that varies upper semi-continuously with respect to the diffeomorphism in C^1 topology; moreover, every extreme point of $G(f)$ is an ergodic physical measure of f.

The proof of Theorems B and E occupies the next two sections. We will carefully analyse the nonuniform expanding of f along E^c using hyperbolic times and use the shadowing lemma of Liao to show the existence of skeletons. We then build a one-to-one correspondence between elements of a skeleton and the physical measures of f, and show that physical measures bifurcate as heteroclinic intersections are created between different elements of a skeleton. Then in Section 7, we generalise the result of Theorem E to generic C^1 diffeomorphisms near f.

1.8. On the regularity assumption

Throughout this article, the regularity assumption on f is changed several times between C^1 and C^{1+}. For the convenience of the readers, we summarise those changes below:

(1) Having a mostly contracting center requires the diffeomorphism to be C^{1+}; as a result, the initial diffeomorphism f is always assumed to be C^{1+}.

(2) The topology is always C^1. Throughout this article, $\mathcal{U}$ is a neighbourhood of f under C^1 topology;

(3) The geometrical properties of skeletons only require the diffeomorphism to be C^1; this involves Section 4, Subsection 6.2 and certain parts of Section 7.

(4) The physical measure having absolutely continuous conditional measure on the unstable leaves and the stable holonomy being absolutely continuous requires C^{1+} regularity, as shown in the classical theory of physical measures. This affects Section 5, Subsection 6.3, certain parts of Section 7 and Section 9.

(5) Section 8 deals with C^1 generic diffeomorphisms in $\mathcal{U}$ and thus only requires C^1 smoothness.

2. Preliminary

In this section, we introduce some necessary notations and results that will be used later. Throughout this section, we assume f to be a partially hyperbolic diffeomorphism on the manifold M and μ an invariant probability measure of f. In Subsection 2.1 we will assume f to be C^{1+} for the discussion on the Gibbs u-states. In Subsections 2.2 and 2.3, f is assumed to be C^1 only.

2.1. Gibbs u-states

Following Pesin and Sinai [43] and Bonatti and Viana [11] (see also [10, Chapter 11]), we call *Gibbs u-state* any invariant probability measure whose conditional probabilities (Rokhlin [45]) along strongly unstable leaves are absolutely continuous with respect to the Lebesgue measure on the leaves. In fact, assuming that the derivative Df is Hölder continuous, the Gibbs-u state always exists, and the densities with respect to Lebesgue measures along unstable plaques are continuous. Moreover, the densities vary continuously with respect to the strongly unstable leaves. As a consequence, the space of Gibbs u-states of f, denoted by $Gibbs^u(\cdot)$, is compact relative to the weak-* topology in the probability space.

The set of Gibbs u-states plays important roles in the study of physical measures for partially hyperbolic diffeomorphisms. The proofs for the following basic properties of Gibbs u-states can be found in the book of Bonatti, Díaz and Viana [10, Subsection 11.2] (see also Dolgopyat [24]):

Proposition 2.1. *Suppose that f is a C^{1+} partially hyperbolic diffeomorphism; then*

(1) *$Gibbs^u(f)$ is nonempty, weak* compact and convex. Ergodic components of Gibbs u-states are Gibbs u-states.*

(2) *The support of every Gibbs u-state is $\mathcal{F}^u$-saturated; that is, it consists of entire strongly unstable leaves.*

(3) *For Lebesgue almost every point x in any disk inside some strongly unstable leaf, every accumulation point of $\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}$ is a Gibbs u-state.*

(4) *Every physical measure of f is a Gibbs u-state; conversely, every ergodic Gibbs u-state whose center Lyapunov exponents are negative is a physical measure.*

The semi-continuity of Gibbs u-states with respect to C^{1+} diffeomorphisms under C^1 topology was recently proved by the author of this article in [52].

Proposition 2.2. *Suppose that f_n $(n = 1, \cdots, \infty)$ and f are C^{1+} partially hyperbolic diffeomorphisms such that $f_n \xrightarrow{C^1} f$. Then*

$$\limsup Gibbs^u(f_n) \subset Gibbs^u(f),$$

where the convergence is in the Hausdorff topology of the probability space.

The following lemma shows the relation between the Gibbs u-states of a diffeomorphism and its iterations.

Lemma 2.3. *For any $n > 0$, $Gibbs^u(f) \subset Gibbs^u(f^n)$. Conversely, let ν be any Gibbs u-state of f^n; then $\frac{1}{n}\sum_{i=0}^{n-1} f^i(\nu)$ is a Gibbs u-state of f.*

Proof. Let μ be a Gibbs u-state of f; then it is also an invariant probability of f^n. Because f and f^n share the same unstable foliation, μ must have the same disintegration along the unstable plaques. Then it follows from the definition that μ is also a Gibbs u-state of f^n.

On the other hand, it is clear that $\frac{1}{n}\sum_{i=0}^{n-1}f^i(\nu)$ is an invariant probability of f. By a similar argument as above, $\frac{1}{n}\sum_{i=0}^{n-1}f^i(\nu)$ is a Gibbs u-state of f. $\qquad\square$

2.2. Partial entropy along unstable foliation

In this section, we give the precise definition of the partial metric entropy of μ along the unstable foliation $\mathcal{F}^u$ of f, which depends on a special class of measurable partitions. The partial entropy has been proven to be a powerful tool in the study of partially hyperbolic diffeomorphisms, thanks to its semi-continuity in the C^1 topology.

Definition 2.4. We say that a measurable partition ξ of M is *μ-subordinate* to the $\mathcal{F}$-foliation if for μ-a.e. x, we have

(1) $\xi(x)\subset\mathcal{F}(x)$ and $\xi(x)$ has uniformly small diameter inside $\mathcal{F}(x)$;

(2) $\xi(x)$ contains an open neighbourhood of x inside the leaf $\mathcal{F}(x)$;

(3) ξ is an increasing partition, meaning that $\xi\prec f\xi$.

Ledrappier and Strelcyn [34] proved that the Pesin unstable lamination admits some μ-subordinate measurable partition. The following result is contained in Lemma 3.1.2 of Ledrappier and Young [35].

Lemma 2.5. *For any measurable partitions ξ_1 and ξ_2 that are μ-subordinate to $\mathcal{F}$, we have $h_\mu(f,\xi_1)=h_\mu(f,\xi_2)$.*

This allows us to define the partial entropy of μ using any μ-subordinate partition.

Definition 2.6. For a C^1 partially hyperbolic diffeomorphism f and an invariant measure μ, the *partial μ-entropy along unstable foliation $\mathcal{F}^u$*, which we denote by $h_\mu(f,\mathcal{F}^u)$, is defined to be $h_\mu(f,\xi)$ for any μ-subordinate partition ξ.

Proposition 2.7 ([52]). *The partial entropy $h_\mu(f,\mathcal{F}^u)$ varies upper semi-continuously with respect to the measures and maps in C^1 topology.*

Although partially entropies are well defined for C^1 diffeomorphisms and behave well under C^1 topology, one still need higher regularity such as C^2 or at least C^{1+} in order to relate it with other quantities such as Lyapunov exponents or Gibbs u-states. The following upper bound for the partial entropy along the unstable foliation $\mathcal{F}^u$ follows [35, 36].

Proposition 2.8. *Let f be C^{1+} and μ be an invariant probability measure of f; then*

$$h_\mu(f,\mathcal{F}^u)\leq\int\log\mathrm{Jac}^u(x)d\mu(x).$$

Moreover,

$$h_\mu(f,\mathcal{F}^u)=\int\log\mathrm{Jac}^u(x)d\mu(x) \tag{2}$$

if and only if μ is a Gibbs u-state of f.

Proof. The inequality follows by [36, Theorem C'] when f is C^2. It was pointed out by [15] that the same inequality goes well for C^{1+} diffeomorphism.

The second part was stated in [33, Theorem 3.4]. $\qquad\square$

The following equality was built in [36, Proposition 5.1] when f is C^2. As explained above, it also holds under the general situation assuming only C^{1+}.

Proposition 2.9. *Let μ be a probability measure of f such that all of the center exponents of μ are nonpositive. Then*

$$h_\mu(f,\mathcal{F}^u) = h_\mu(f).$$

2.3. Other invariant measure subspaces

Proposition 2.1 (4) states that when f is C^{1+}, Gibbs u-states are the natural candidates of the physical measures of f. However, this statement falls apart when f is only C^1. Here let us recall that the main result of [32] (see also [23]) shows that for C^1 diffeomorphisms, every limit point of the empirical measures at Lebesgue almost every point must satisfy ((2), Proposition 2.8). However, such measures may not be Gibbs u-states due to the lack of Pesin's formula for C^1 diffeomorphisms. To this end, we introduce two candidate spaces of physical measures for such f. See [32], [23] and [21] for their properties.

Definition 2.10. We define

(A1)

$$\mathrm{G}^u(f) = \{\mu \in \mathcal{M}_{\mathrm{inv}}(f) : h_\mu(f,\mathcal{F}^u) \geq \int \log(\det(Df\mid_{E^u(x)}))d\mu(x)\};$$

(A2)

$$\mathrm{G}^{cu}(f) = \{\mu \in \mathcal{M}_{\mathrm{inv}}(f) : h_\mu(f) \geq \int \log(\det(Df\mid_{E^{cu}(x)}))d\mu(x)\}$$

where $E^{cu} = E^c \oplus E^u$.

We write

$$\mathrm{G}(f) = \mathrm{G}^u(f) \cap \mathrm{G}^{cu}(f).$$

Remark 2.11.

(a) When f is C^{1+}, by Ledrappier [33], $\mathrm{G}^u(f) = \mathrm{Gibbs}^u(f)$.

(b) By the Ruelle's inequality for partial entropy (see, for instance, [51]), one can replace the inequality in the definition of G^u by the equality

$$\mathrm{G}^u(f) = \{\mu \in \mathcal{M}_{\mathrm{inv}}(f) : h_\mu(f,\mathcal{F}^u) = \int \log(\det(Df\mid_{E^u(x)}))d\mu(x)\}.$$

However, the definition of G^{cu} remains unchanged due to the possibility of having negative Lyapunov exponents in E^c.

We first observe that the spaces above are nonempty; moreover, the space $\mathrm{G}(f)$ contains all of the candidates of physical measures.

Proposition 2.12. *For every C^1 partially hyperbolic diffeomorphism f, there is a full-volume subset Γ such that for any $x \in \Gamma$, any limit point of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ belongs to $\mathrm{G}(f)$.*

Proof. By [21], for x belonging to a full-volume subset, any limit of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ belongs to G^{cu}. Moreover, by [23, 32], for x belonging to a full-volume subset, any limit of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ belongs to G^u. We conclude the proof by taking the intersection of the two full-volume subsets. $\square$

The following property shows that $\mathrm{G}^u(\cdot)$ shares similar properties with $\mathrm{Gibbs}^u(\cdot)$ (Proposition 2.1).

Proposition 2.13 ([32] [Propositions 3.1, 3.5]). *The space $\mathrm{G}^u(f)$ is convex, compact and varies in a upper semi-continuous way with respect to the partially hyperbolic diffeomorphisms under C^1 topology. Moreover, for any invariant measure $\mu \in \mathrm{G}^u(f)$, every ergodic component of its ergodic decomposition still belongs to $\mathrm{G}^u(f)$.*

We need to observe that, in general, the space $\mathrm{G}(f)$ may not have such properties (especially when it comes to the ergodic components). Indeed, in Proposition 5.17, we will show that the above properties hold for $\mathrm{G}(g)$ when g is C^1 close to f which is C^{1+} with a mostly expanding center.

3. Examples of partially hyperbolic diffeomorphisms with a mostly expanding center

For a long time (before [52]), there were only two known examples of diffeomorphisms with a mostly expanding center (under the definition that is used in this article, which is stronger than that in [1]). These examples are due to Mañé [39] (see [1] and [3, Section 6]) and Dolgopyat [25]. We list these examples below, as well as some new examples provided in [52]. Let us recall that the set of partially hyperbolic diffeomorphisms with a mostly expanding center is C^1 open among $\mathrm{Diff}^{1+}(M)$.

3.1. Derived from Anosov diffeomorphisms

We assume A to be a linear Anosov diffeomorphism over $\mathbb{T}^3$ with three positive simple real eigenvalues $0 < k_1 < 1 < k_2 < k_3$.

3.1.1. Local derived from Anosov diffeomorphisms. Let us begin by recalling the construction of Mañé's example, which is a local C^0 perturbation of A. The statement below is a little different from the original construction in history.

Example 3.1. Let p be a fixed point of A and U a small neighbourhood of p. There is a partially hyperbolic diffeomorphism f_0 that coincides with A on $\mathbb{T}^3 \setminus U$. f_0 is topological Anosov, and

$$\mid Df_0\mid_{E^c(x)}\mid \geq 1, \tag{3}$$

where the equality holds if and only if $x = p$.

Because $Df_0 \mid_{E^c(\cdot)}$ is expanding everywhere except at the point p, it is clear that f_0 has a mostly expanding center. Thus, by [52], f_0 admits a C^1 neighbourhood $\mathcal{U}$ such that every C^{1+} diffeomorphism belonging to $\mathcal{U}$ has a mostly expanding center.

3.1.2. Generalised derived from Anosov diffeomorphisms. By the topological classification of partially hyperbolic diffeomorphisms that are isotopic to A ([13, 29, 48]), we call such diffeomorphisms *derived from Anosov A* and denote this set of diffeomorphisms by $\mathcal{DA}(A)$. The following example by Shi, Viana and the author of this article [47] revises the fact that C^{1+} volume preserving derived from Anosov diffeomorphisms have a mostly expanding center whenever the volume has large metric entropy.

Example 3.2. Let $f \in \mathcal{DA}(A)$ be a C^{1+} volume-preserving partially hyperbolic diffeomorphism and $h_{\mathrm{vol}}(f) > \log k_3$; then f has a mostly expanding center.

3.2. Perturbation of volume-preserving partially hyperbolic diffeomorphisms
In [25], Dolgopyat showed the following.

Example 3.3. Let X_1 be the time 1 map of a hyperbolic geodesic flow on a surface M; then for generic C^∞ perturbation f of X_1, either f or its inverse f^{-1} has a mostly expanding center.

The following result in [52] allows us to obtain more examples using C^1 perturbation.

Proposition 3.4. *Let f be a C^{1+} volume-preserving partially hyperbolic diffeomorphism with a 1-dimensional center. Suppose that the center exponent of the volume measure is positive and f is accessible. Then f admits an C^1 open neighbourhood, such that every C^{1+} diffeomorphism in this neighbourhood (not necessarily volume preserving) has a mostly expanding center.*

Proposition 3.4 contains an abundance of systems: by Avila [5], C^∞ volume-preserving diffeomorphisms are C^1 dense. And by Baraviera and Bonatti [6], the volume-preserving partially hyperbolic diffeomorphisms with a 1-dimensional center and nonvanishing center exponent are C^1 open and dense. Moreover, the subset of accessible systems is C^1 open and C^k dense for any $k \geq 1$ among all partially hyperbolic diffeomorphisms with a 1-dimensional center direction, due to the work of Burns et al. [17]; see also Theorem 1.5 in Niţică and Török [40].

Indeed, the accessibility assumption in the above proposition can be replaced by another hypothesis.

Example 3.5 (see [49]). Let f be a C^{1+} volume-preserving partially hyperbolic diffeomorphism with a 1-dimensional center. Suppose that the center exponent of the volume measure is positive and f^{-1} has a mostly contracting center. Then f admits an C^1 open neighbourhood such that every C^{1+} diffeomorphism in this neighbourhood has a mostly expanding center.

Remark 3.6. The hypothesis that f^{-1} has a mostly contracting center is equivalent to the assumption that $\mathcal{F}^s$ is minimal.

The diffeomorphisms with minimal strongly stable and unstable foliations are also quite common; they fill an open and dense subset of volume-preserving partially hyperbolic diffeomorphisms with a 1-dimensional center and have compact center leaves. This follows from a conservative version of the results in [9].

3.3. Product of diffeomorphisms with a mostly expanding center

The following was shown by Ures, Viana and the author of this article in [49].

Proposition 3.7. *Suppose that f_1 and f_2 are C^{1+} partially hyperbolic diffeomorphisms over manifolds M_1 and M_2. Assume that both f_1 and f_2 have a mostly expanding center. Then $f_1 \times f_2$ is a partially hyperbolic diffeomorphism over $M_1 \times M_2$ with a mostly expanding center. As a result, nearby C^{1+} diffeomorphisms (which may not be products anymore) also have a mostly expanding center.*

4. Properties of skeletons

In this section, we introduce several basic properties for skeletons, although the existence of skeletons will be postponed to Section 6. The main tool in this section is the inclination lemma, also known as the λ-lemma.

To state the properties of skeletons under general situations, throughout this section we assume f to be a C^1 partially hyperbolic diffeomorphism with dominated splitting $E^s \oplus E^c \oplus E^u$, and $\mathcal{S} = \{p_1, \cdots, p_k\}$ is an index i_s skeleton of f. In particular, we will not assume f to have a mostly expanding center. It is also worth noting that, unlike in [26], we will not discuss the robustness of skeletons under perturbation of f in this section. Such discussion requires f to have a mostly expanding center and is postponed to Section 7 (see Lemma 7.1).

The first three technical lemmas provide geometrical information on the structure of skeletons. The main result in this section is Lemma 4.4, which states that every skeleton of f must have the same cardinality. The last two lemmas provide useful criteria for skeletons, which will be used multiple times in later sections.

Lemma 4.1.

(1) *For any $1 \le i \le k$, $Cl(\mathcal{F}^s(Orb(p_i)))$ has a nonempty interior.*

(2) *For $1 \le i \ne j \le k$, there is no heteroclinic intersection between $Orb(p_i)$ and $Orb(p_j)$; that is, $\mathcal{F}^s(\ Orb(p_i)) \cap W^u(Orb(p_j)) = \emptyset$.*

(3) $Int(Cl(\mathcal{F}^s(Orb(p_i)))) \cap Int(Cl(\mathcal{F}^s(Orb(p_j)))) = \emptyset$.

Proof. Because $\mathcal{S}$ is a skeleton, from (a) of the definition of a skeleton is

$$\bigcup_{i=1}^{k} \mathrm{Cl}(\mathcal{F}^s(\ \mathrm{Orb}(p_i))) = M.$$

Suppose by contradiction that $\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i)))$ has empty interior for some $1 \le i \le k$; then $\bigcup_{j \ne i} \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_j))) = M$. Thus, $\mathcal{S} \setminus \{p_i\}$ also satisfies (a) of Definition 1.3, which contradicts with (b) of Definition 1.3 and the fact that $\mathcal{S}$ is a skeleton. This finishes the proof of (1).

We are ready to prove (2). First, by the unstable manifold theorem, $W^u(\mathrm{Orb}(p_j))$ is tangent to the bundle E^{cu}. Thus, if the intersection $\mathcal{F}^s(\mathrm{Orb}(p_i)) \cap W^u(\mathrm{Orb}(p_j))$ is not empty, it must be transversal. By the inclination lemma, $\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_j))) \subset \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i)))$, and thus $\mathcal{S} \setminus \{p_j\}$ is a pre-skeleton, a contradiction.

To prove (3), we assume by contradiction that there are $1 \le i \ne j \le k$ such that $U = \mathrm{Int}(\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i)))) \cap \mathrm{Int}(\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_j)))) \ne \emptyset$. Take $x \in \mathcal{F}_R^s(\mathrm{Orb}(p_i)) \cap U$ for some $R > 0$ where $\mathcal{F}_R^s(\cdot)$ is the disk in $\mathcal{F}^s(\cdot)$ with radius R under leaf metric; then there is $x_n \in \mathcal{F}^s(\mathrm{Orb}(p_j)) \cap U$ such that $x_n \to x$. By the continuity of stable foliation, we have $\mathcal{F}_{2R}^s(x_n) \to \mathcal{F}_{2R}^s(x)$ and thus for n sufficiently large, $\mathcal{F}_{2R}^s(x_n) \cap W^u(\mathrm{Orb}(p_i)) \ne \emptyset$. Because $x_n \in \mathcal{F}^s(\mathrm{Orb}(p_j))$, we have $\mathcal{F}^s(\mathrm{Orb}(p_j)) \cap W^u(\mathrm{Orb}(p_i)) \ne \emptyset$, which is a heteroclinic intersection between p_j and p_i, a contradiction with item (2).   □

In the following, instead of using the open set $\mathrm{Int}(\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i))))$, we are going to consider the set $\mathcal{O}_i = \bigcup_{x \in W^u(\mathrm{Orb}(p_i))} \mathcal{F}^s(x)$. By the transversality between E^{cu} and E^s and continuity of stable foliation, the set $\mathcal{O}_i$ is open. In the following we will reveal the relation between these two open sets.

For a hyperbolic saddle p, we denote by $H(p,f)$ the homoclinic class of p with respect to the map f; that is, the closure of homoclinic intersections between $W^s(\mathrm{Orb}(p))$ and $W^u(\mathrm{Orb}(p))$.

Proposition 4.2. *For every $p_i \in \mathcal{S}$,*

(i) *$Cl(W^u(Orb(p_i))) = H(p_i, f)$.*

(ii)

$$Cl(\mathcal{F}^s(Orb(p_i))) = Cl(\cup_{x \in W^u(Orb(p_i))} \mathcal{F}^s(x)); \tag{4}$$

thus, $\mathcal{O}_i$ is open and dense in $\mathrm{Int}(Cl(\mathcal{F}^s(Orb(p_i))))$.

Proof. We first prove (i). From the definition of homoclinic class, we have

$$\mathrm{Cl}(W^u(\mathrm{Orb}(p_i))) \supset H(p_i, g).$$

Now let us prove the other direction of the inclusion.

By the definition of skeleton, $\bigcup_{j=1,\cdots,k}(\mathcal{F}^s(\mathrm{Orb}(p_j)))$ is dense in the manifold M. Thus, for any $x \in W^u(\mathrm{Orb}(p_i))$, there is $p_j \in \mathcal{S}$ such that $x \in \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_j)))$. According to (2) of Lemma 4.1, there is no heteroclinic intersection between $\mathcal{F}^s(\mathrm{Orb}(p_j))$ and $W^u(\mathrm{Orb}(p_i))$ when $i \ne j$; thus, $i = j$. It then follows that $\mathcal{F}^s(\mathrm{Orb}(p_i))$ and $W^u(\mathrm{Orb}(p_i))$ have nontrivial intersections arbitrarily close x, meaning that $x \in H(p_i, f)$. This completes the proof of (i).

By the discussion above, we have shown that $\mathcal{F}^s(\mathrm{Orb}(p_i)) \cap W^u(\mathrm{Orb}(p_i))$ is dense inside $W^u(\mathrm{Orb}(p_i))$; thus,

$$\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i))) \supset \mathrm{Cl}(\cup_{x \in W^u(\mathrm{Orb}(p_i))} \mathcal{F}^s(x)).$$

Meanwhile, because $\mathrm{Orb}(p_i) \subset W^u(\mathrm{Orb}(p_i))$, the inclusion

$$\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i))) \subset \mathrm{Cl}(\cup_{x \in W^u(\mathrm{Orb}(p_i))} \mathcal{F}^s(x))$$

is trivially satisfied, and the equality (4) follows immediately.   □

The next two lemmas show that if one replaces $p_i \in \mathcal{S}$ by another hyperbolic periodic point $q \in \mathcal{O}_i$ with index i_s, the new set $\mathcal{S}' = \mathcal{S} \cup \{q\} \setminus \{p_i\}$ is still an index i_s skeleton; moreover, any skeleton of f can be obtained in this way.

Lemma 4.3. *Let q be an index i_s hyperbolic periodic point; then $q \in \mathcal{O}_i$ if and only if q and p_i are homoclinic related to each other. Moreover, $\mathcal{S}' = \{q\} \bigcup \mathcal{S} \setminus \{p_i\}$ remains an index i_s skeleton.*

Proof. If q and p_i are homoclinic related with each other, take $a \in \mathcal{F}^s(q) \pitchfork W^u(\mathrm{Orb}(p_i))$ and U a neighbourhood of a in $W^u(\mathrm{Orb}(p_i))$. By the continuity of stable foliation, $\bigcup_{x \in U} \mathcal{F}^s(x)$ contains a neighbourhood of q. Then by Proposition 4.2, $q \in \bigcup_{x \in W^u(\mathrm{Orb}(p_i))} \mathcal{F}^s(x) = \mathcal{O}_i$.

On the other hand, suppose that $q \in \bigcup_{x \in W^u(\mathrm{Orb}(p_i))} \mathcal{F}^s(x)$; then there exists an intersection point $a \in \mathcal{F}^s(q) \pitchfork W^u(\mathrm{Orb}(p_i))$. By Proposition 4.2[(i)], $a \in H(p_i, f)$ and thus can be approached by $\mathcal{F}^s(\mathrm{Orb}(p_i))$. By the continuity of stable foliation, $\mathcal{F}^s(\mathrm{Orb}(p_i)) \cap W^u(q) \neq \emptyset$. We conclude that q and p_i are homoclinic related.

Now suppose that q and p_i are homoclinic related. Then by the inclination lemma, we have $\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(q))) = \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i)))$, which means that

$$\bigcup_{p \in \mathcal{S}'} \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p))) = M.$$

It remains to show that $\mathcal{S}'$ does not have a proper subset $\mathcal{S}''$ that satisfies the above equality.

Assume by contradiction that $\mathcal{S}''$ is such a proper subset of $\mathcal{S}'$. Because $\mathcal{S}$ is a skeleton, $\mathcal{S}''$ has to contain q; otherwise, $\mathcal{S}''$ will be a proper subset of $\mathcal{S}$, which contradicts the fact that $\mathcal{S}$ is a skeleton. By the discussion above, $\tilde{\mathcal{S}} = \{p_i\} \bigcup \mathcal{S}'' \setminus \{q\}$ is a pre-skeleton. However, this is impossible because $\tilde{\mathcal{S}}$ is a proper subset of $\mathcal{S}$. $\qquad\square$

Lemma 4.4. *Suppose that $\mathcal{S}' = \{q_1, \cdots, q_l\}$ is a skeleton of f; then $l = k$, and after reordering, q_i and p_i are homoclinic related for $i = 1, \cdots, k$.*

Proof. By Definition 1.3 (a), for each $q_j \in \mathcal{S}'$ there is some $p_i \in \mathcal{S}$ such that $\mathcal{F}^s(\mathrm{Orb}(p_i))$ approaches p_i; thus, $W^u(q_j)$ intersects $\mathcal{F}^s(p_i)$ transversally.

Choose any such p_i (we will see in a moment that the choice is unique). The same argument applied on p_i shows that there exists some $q_k \in \mathcal{S}'$ such that $W^u(p_i)$ intersects $\mathcal{F}^s(\mathrm{Orb}(q_k))$ transversally. By the inclination lemma, there is transverse intersection between $W^u(\mathrm{Orb}(q_j))$ and $\mathcal{F}^s(\mathrm{Orb}(q_k))$. By Lemma 4.1[(2)], this can only happen if $j = k$. In particular, p_i and q_j are homoclinically related to one another.

Because being homoclinically related is an equivalent relation and different elements in a skeleton do not have heteroclinic intersections, it follows that the choice of p_i is unique, and the map $q_j \mapsto p_i$ is injective. Reversing the roles of $\mathcal{S}'$ and $\mathcal{S}$, we also get an injective map $p_i \mapsto q_j$, which, by construction, is the inverse of the previous one. Thus, both maps are bijective and, in particular, $\#\mathcal{S} = \#\mathcal{S}'$. Moreover, after reordering, q_i and p_i are homoclinic related for $i = 1, \cdots, k$. $\qquad\square$

The following lemma provides a useful criterion on the existence of skeletons, which will be used in Section 6.

Lemma 4.5. *Any pre-skeleton contains a subset that forms a skeleton.*

Proof. Let $\mathcal{S}' = \{p_1, \cdots, p_l\}$ be a pre-skeleton. We first define a relation between the elements of $\mathcal{S}'$: we say $p_i \prec p_j$ if $W^u(\mathrm{Orb}(p_i)) \pitchfork \mathcal{F}^s(p_j) \neq \emptyset$. By the inclination lemma, it is easy to see that $\prec$ is reflexive and transitive: if $p_i \prec p_j$, then

$$\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_j))) \supset \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i))). \tag{5}$$

Moreover, if we have $p_i \prec p_j$ and $p_j \prec p_i$, then we say that they belong to the same equivalent class. Two elements belong to the same equivalent class if and only if they are homoclinic related.

Now in the set of equivalent classes, $\prec$ induces a partial order. For every maximal equivalent class under this partial order, we pick up a representative element and then obtain a subset $\mathcal{S} \subset \mathcal{S}'$. By (5), $\mathcal{S}$ is clearly a pre-skeleton. Moreover, from the construction, the elements of $\mathcal{S}$ have no heteroclinic intersection. Then this lemma is a corollary of the following result. $\qquad\square$

Lemma 4.6. *Let $\mathcal{S} = \{p_1, \cdots, p_k\}$ be a pre-skeleton of f such that there is no heteroclinic intersection between $Orb(p_i)$ and $Orb(p_j)$ for $1 \leq i \neq j \leq k$; then $\mathcal{S}$ is a skeleton.*

Proof. We prove by contradiction. Suppose that $\mathcal{S}$ is not a skeleton; then by Definition 1.3 (b), it contains a proper subset $\mathcal{S}''$ that forms a pre-skeleton. After reordering, we may assume $\mathcal{S}'' = \{p_1, \cdots, p_l\}$, where $l < k$.

Then by the definition of skeleton, $\bigcup_{1 \leq i \leq l} \mathcal{F}^s(\mathrm{Orb}(p_i))$ is dense in the manifold M. As a result, there is $1 \leq i_0 \leq l$ such that $\mathcal{F}^s(\mathrm{Orb}(p_{i_0}))$ approaches p_k, and thus $\mathcal{F}^s(\mathrm{Orb}(p_{i_0})) \pitchfork W^u(p_k) \neq \emptyset$, which contradicts the assumption that there is no heteroclinic intersection between elements of $\mathcal{S}$. The proof is complete. $\qquad\square$

5. Diffeomorphisms with a mostly expanding center revisited

Throughout this section, we assume f to be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center. To make this article as self-contained as possible, we will provide a direct proof on the existence of physical measures for diffeomorphisms with a mostly expanding center. The proof is different from the original argument in [1] and is useful for the discussion in later sections.

One of the main difficulties in the study of diffeomorphisms with a mostly expanding center lies in the fact that the space $\mathrm{Gibbs}^u(f)$ (or $G^u(g)$ for nearby C^1 map g; see Definition 2.10) is 'too large', in the sense that it contains plenty of measures that are not physical.[2]

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[2]In comparison, if f has a *mostly contracting center*, then every measure in $\mathrm{Gibbs}^u(f)$ is a physical measure, and finiteness follows easily. See [26] and [32] for the discussion there.

We start solving this issue by introducing the following description for diffeomorphisms with a mostly expanding center, which turns out to be equivalent to Definition 1.2. The main advantage is that it gives a uniform estimate on the center Lyapunov exponents for measures in $\text{Gibbs}^u(f)$.

Proposition 5.1 ([52] [Proposition 6.1]). *Suppose that f has a mostly expanding center; then there is $N_0 \in \mathbb{N}$ and $b_0 > 0$ such that, for any $\tilde{\mu} \in Gibbs^u(f^{N_0})$,*

$$\int \log \|Df^{-N_0}|_{E^{cu}(x)}\| d\tilde{\mu}(x) < -b_0. \tag{6}$$

Remark 5.2. From now on, we assume $N_0 = 1$.

By the upper semi-continuity of the space $\text{G}^u(f)$ with respect to diffeomorphisms in C^1 topology (Proposition 2.13), we can extend this estimate to nearby C^1 maps.

Lemma 5.3. *There is a C^1 open neighbourhood $\mathcal{U}$ of f such that for any C^1 diffeomorphism $g \in \mathcal{U}$ and any $\mu \in \text{G}^u(g)$, we have*

$$\int \log \|Dg^{-1}|_{E_g^{cu}(x)}\| d\mu(x) < -b_0. \tag{7}$$

This is used in Subsection 5.1, where we show that for any C^1 diffeomorphism g in a small C^1 neighbourhood $\mathcal{U}$ of f, and for any $\mu \in \text{G}^u(g)$, μ typical points x have infinitely many *hyperbolic times* for the bundle E^{cu} in its orbit (see Lemma 5.8).

On the other hand, the space $\text{G}^{cu}(f)$ is also 'too large' because it may contain measures with negative center exponents. Such measures need not be a Gibbs u-state and thus not physical due to Proposition 2.1 (4). One way to solve this issue is to take the space of intersection, $\text{G}(f)$, which is a much smaller space to work with. However, this creates another problem: unlike the partial entropy, which is upper semi-continuous (which makes the space $\text{G}^u(f)$ upper semi-continuous in f), the metric entropy h_μ may not have such a property. This is dealt with in Subsection 5.2, as we introduce fake foliations for partially hyperbolic diffeomorphisms and show in Lemma 5.12 that the measures in $\text{G}^u(g)$ for $g \in \mathcal{U}$ are uniformly entropy expansive. As a consequence, in Subsection 5.3 it is shown (Corollary 5.16) that metric entropy, *when restricted to measures in* $\text{G}^u(g)$, varies in a upper semi-continuous fashion in weak-* topology and with respect the diffeomorphism $g \in \mathcal{U}$ in C^1 topology.

Finally, Subsection 5.4 contains the main result of this section: for any C^{1+} diffeomorphism $g \in \mathcal{U}$, every extreme element of $\text{G}(g)$ is an ergodic physical measure of g.

5.1. Hyperbolic times

Definition 5.4. Given $b > 0$, we say that n is a *b- hyperbolic time* for a point x if

$$\frac{1}{k} \sum_{j=n-k+1}^{n} \log \|Df^{-1}|_{E^{cu}(f^j(x))}\| \leq -b \text{ for any } 0 < k \leq n.$$

Let D be any C^1 disk; we use $d_D(\cdot,\cdot)$ to denote the distance between two points in the disk. Recall that for the dominated splitting $E^s \oplus E^{cu}$, one can define the center unstable cone field, which is invariant under forward iteration.

The next lemma states that if n is a hyperbolic time for x, then on the disk $f^n(D)$, one picks up a contraction by e^{-b} for each backward iteration.

Lemma 5.5 ([1] Lemma 2.7). *For any $b > 0$, there is $r > 0$ such that, given any C^1 disk D tangent to the center-unstable cone field, $x \in D$ and $n \geq 1$ a $b/2$-hyperbolic time for x, we have*

$$d_{f^{n-k}(D)}(f^{n-k}(y), f^{n-k}(x)) \leq e^{-kb/2} d_{f^n(D)}(f^n(x), f^n(y)),$$

for any point $y \in D$ with $d_{f^n(D)}(f^n(x), f^n(y)) \leq r$ and any $1 \leq k \leq n$.

Remark 5.6. For fixed $b_0/2 > 0$, we can take $r = r_1$ to be constant for the diffeomorphisms in a C^1 neighbourhood of f.

By Lemma 5.3 and Proposition 2.12, for any $g \in \mathcal{U}$, there is a full-volume subset Γ_g such that for any $x \in \Gamma_g$, any limit of the sequence $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{g^i(x)}$ belongs to $G(g)$. Thus, for any $x \in \Gamma_g$,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg^{-1}|_{E_g^{cu}(g^i(x))}\| = \limsup \int \log \|Dg^{-1}|_{E_g^{cu}(x)}\| d\frac{1}{n} \sum_{i=0}^{n-1} \delta_{g^i(x)} \tag{8}$$
$$< -b_0 < 0.$$

Define $H(b_0/2, x, g)$ to be the set of $b_0/2$-hyperbolic times for $x \in \Gamma_g$; that is, the set of times $m \geq 1$ such that

$$\frac{1}{k} \sum_{i=m-k+1}^{m} \log \|Dg^{-1}|_{E_g^{cu}(g^i(x))}\| \leq -b_0/2 \text{ for all } 1 \leq k \leq m. \tag{9}$$

By the Pliss lemma (see [1]), such hyperbolic times have positive density on the orbit segment from 0 to n: there exists $n_x \geq 1$ and $\delta_1 > 0$ such that

$$\#(H(b_0/2, x, g) \cap [1, n)) \geq n\delta_1 \text{ for all } n \geq n_x. \tag{10}$$

By Lemma 5.5 and Remark 5.6, there is $r_1 > 0$ that only depends on $\mathcal{U}$ and $b_0/2$, such that for any $x \in \Gamma_g$ and any disk D tangent to the center-unstable cone field, $x \in D$, $n \in H(b_0/2, x, g)$, we have

$$d_D(x, y) \leq e^{-nb_0/2} d_{g^n(D)}(g^n(x), f^n(y)), \tag{11}$$

for any $y \in D$ with $d_{g^n(D)}(g^n(x), g^n(y)) \leq r_1$. (We also assume that r_1 satisfies the condition (15) below, which depends only on the neighbourhood $\mathcal{U}$.) In particular, for $x \in \Gamma_g$ and disk D tangent to the center-unstable cone field containing x, $g^n(D)$ contains a smaller disk D_n with diameter r_1 for $n \in H(b_0/2, x, g)$ sufficiently large. Then $\cup_{z \in D_n} \mathcal{F}^s(z)$ contains an open ball with radius r_1.

Definition 5.7. Denote by $\mathcal{H}(b_0/2,g)$ the set of points x such that for any $k \geq 1$,

$$\frac{1}{k} \sum_{i=0}^{k-1} \log \|Dg^{-1}|_{E_g^{cu}(g^{-i}(x))}\| \leq -b_0/2 \text{ for all } k \geq 0. \tag{12}$$

In other words, for every $n > 0$, n is a hyperbolic time for the point $f^{-n}(x)$.

The next lemma shows that there are plenty of hyperbolic times on the forward orbit of x, every $\mu \in \mathrm{G}^u(g)$ and μ almost every x.

Lemma 5.8. *For any $g \in \mathcal{U}$ and any $\mu \in \mathrm{G}^u(g)$, we have*

$$\mu(\mathcal{H}(b_0/2,g)) \geq \delta_1, \tag{13}$$

where δ_1 is given in (10).

Proof. By Proposition 2.13, we may assume μ to be ergodic. By Birkhoff's theorem, we only need to show that for μ almost every x, $\liminf \frac{1}{n}\#\{1 \leq k \leq n; f^k(x) \in \mathcal{H}(b_0/2,g)\} \geq \delta_1$. It is equivalent to show that for some fixed m_x,

$$\liminf \frac{1}{n}\#\{1 \leq k \leq n; f^{k+m_x}(x) \in \mathcal{H}(b_0/2,g)\} \geq \delta_1. \tag{14}$$

By Lemma 5.3, take x be a typical point of μ, such that

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg^{-1}|_{E_g^{cu}(g^{-i}(x))}\| \leq -b_0.$$

We claim that there is $m > 0$ such that $g^{-m}(x) \in \mathcal{H}(b_0/2,g)$. Otherwise, for any $g^{-n}(x)$, there is $i_n > 0$ such that $\frac{1}{i_n}\sum_{i=0}^{i_n-1} \log\|Dg^{-1}|_{E_g^{cu}(g^{-i-n}(x))}\| \geq -b_0/2$. Recursively, we obtain a sequence of points: $n_1 = i_0$, $n_2 = n_1 + i_{n_1}, \cdots$; by induction, we have

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \log \|Dg^{-1}|_{E_g^{cu}(g^{-i}(x))}\| \geq -b_0/2.$$

This contradicts the choice of x.

Moreover, it is easy to see that for any $k \in H(b_0/2,g^{-m}(x),g)$, $g^{k-m}(x) \in \mathcal{H}(b_0/2,g)$. Then by (10) and taking $m_x = m$ in (14), we conclude the proof. $\square$

5.2. Fake foliations

In order to avoid assuming dynamical coherence of f, we use locally invariant (*fake*) foliations, a construction that follows Burns and Wilkinson [18] and goes back to Hirsch, Pugh and Shub [31]. We fix $\mathcal{U}$ a small C^1 neighbourhood of f provided by Lemma 5.3.

Lemma 5.9. *There are real numbers $\rho > r_0 > 0$ only depending on $\mathcal{U}$ with the following properties. For any $x \in M$, the neighbourhood $B(x,\rho)$ admits foliations $\hat{\mathcal{F}}_{g,x}^s$ and $\hat{\mathcal{F}}_{g,x}^{cu}$ such that for every $y \in B(x,r_0)$ and $* = \{s,cu\}$:*

(1) *the leaf $\hat{\mathcal{F}}_{g,x}^*(y)$ is C^1, and its tangent bundle $T_y(\hat{\mathcal{F}}_{g,x}^*(y))$ lies in a cone of $E^*(x)$;*

(2) *$g(\hat{\mathcal{F}}_{g,x}^s(y,r_0)) \subset \hat{\mathcal{F}}_{g,g(x)}^s(g(y))$ and $g^{-1}(\hat{\mathcal{F}}_{g,x}^{cu}(y,r_0)) \subset \hat{\mathcal{F}}_{g,f^{-1}(x)}^{cu}(g^{-1}(y))$;*

(3) *we have product structures inside $B(x, r_0)$; that is, for any $y, z \in B(x, r_0)$, there is a unique intersection between $\hat{\mathcal{F}}^s_{g,x}(y)$ with $\hat{\mathcal{F}}^{cu}_{g,x}(z)$, which we denote by $[y, z]$.*

For $g \in \mathcal{U}$ and any $x \in M$, we consider the following three types of Bowen balls:

- *finite Bowen ball*: $B_n(g, x, \varepsilon) = \{y \in M : d(g^i(x), g^i(y)) < \varepsilon, |i| < n\}$,
- *negative Bowen ball*: $B^-_\infty(g, x, \varepsilon) = \{y \in M : d(g^i(x), g^i(y)) < \varepsilon, i < 0\}$,
- *(two sided) infinite Bowen ball*:

$$B_\infty(g, x, \varepsilon) = \{y \in M : d(g^i(x), g^i(y)) < \varepsilon, i \in \mathbb{Z}\}.$$

The following was shown in the proof of [37], [Theorem 3.1].

Lemma 5.10. *For $\varepsilon < r_0/2$ and any $x \in M$, $B^-_\infty(g, x, \varepsilon) \subset \hat{\mathcal{F}}^{cu}_{g,x}(y, 2\varepsilon)$.*

We may take r_1 in the previous section to satisfy that

$$r_1 < r_0/2. \tag{15}$$

Then as a consequence of Lemma 5.5, we show that for every point in $\mathcal{H}(b_0/2, g)$, the unstable manifold has uniform size.

Lemma 5.11. *For any $x \in \mathcal{H}(b_0/2, g)$, $\hat{\mathcal{F}}^{cu}_{g,x}(x, r_1) \subset W^u_{\mathrm{loc}}(x)$. More precisely, for any $y \in \hat{\mathcal{F}}^{cu}_{g,x}(x, r_1)$,*

$$d_{\hat{\mathcal{F}}^{cu}_{g,g^{-n}(x)}(g^{-n}(x))}(g^{-n}(x), g^{-n}(y)) \leq e^{-nb_0/2} d_{\hat{\mathcal{F}}^{cu}_{g,x}(x)}(x, y).$$

The goal of this subsection is to show that the measures in $\mathrm{G}^u(g)$ for $g \in \mathcal{U}$ are uniformly entropy expansiveness.

Lemma 5.12. *For any $g \in \mathcal{U}$ and any measure $\mu \in \mathrm{G}^u(g)$, for μ almost every point x,*

$$B_\infty(g, x, r_1) = x.$$

Proof. By Lemma 5.10 and the choice of $r_1 \leq r_0/2$, we have

$$B_\infty(g, x, r_1) \subset B^-_\infty(g, x, r_1) \subset \hat{\mathcal{F}}^{cu}_{g,x}(x, r_1).$$

Let x be a μ typical point; by Lemma 5.8, we may assume that the forward orbit of x enters $\mathcal{H}(b_0/2, g)$ infinitely many times. Suppose that there is a distinct point $y \in B_\infty(g, x, r_1/2) \subset \hat{\mathcal{F}}^{cu}_{g,x}(x, r_1)$, we are going to prove by contradiction that x and y coincide with each other. Suppose that $f^n(x) \in \mathcal{H}(b_0/2, g)$, then $f^n(y) \in B_\infty(g, g^n(x), r_1/2) \in \hat{\mathcal{F}}^{cu}_{g,x}(x, r_1)$. By Lemma 5.11,

$$d_{\hat{\mathcal{F}}^{cu}_{g,x}(x)}(x, y) \leq e^{-nb_0/2} d_{\hat{\mathcal{F}}^{cu}_{g,g^n(x)}(g^n(x))}(g^n(x), g^n(y)) \leq e^{-nb_0/2} r_1.$$

Taking $n \to \infty$, we have $d_{\hat{\mathcal{F}}^{cu}_{g,x}(x)}(x, y) = 0$. Hence, $x = y$, a contradiction with the hypothesis that x and y are distinct. The proof is complete. $\square$

Remark 5.13. The classical definition of entropy expansive by Bowen requires the topological entropy of $B_\infty(g, x, r_1)$ to be vanishing for *every* $x \in M$. However, as observed in [37, Proposition 2.4], this is equivalent to having zero topological entropy for the infinite

Bowen ball for *every* invariant measure μ and μ *almost every x*. The statement of the previous lemma follows this approach.

Also note that this lemma does not immediately lead to the upper semi-continuity of h_μ as in the classical case, because we only have entropy expansive on a *subspace* of invariant measure. However, we will see in a moment that the upper semi-continuity holds for measures in G^u.

5.3. Upper semi-continuity of metric entropy

In this section, we are going to show that the metric entropy for measures in $G^u(\cdot)$ is upper semi-continuous, which is a consequence of the uniform entropy expansiveness for measures among $G^u(\cdot)$.

Define the ε-*tail entropy at x* by

$$h^*(g,x,\varepsilon) = h_{top}(g,B_\infty(g,x,\varepsilon)),$$

where we use Bowen's definition of the topological entropy [12]:

$$h_{top}(g,K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon,K);$$

here $r_n(\varepsilon,K)$ is the minimum cardinality of (n,ε)-spanning sets of K. For any probability measure μ of g, let $h^*(g,\mu,\varepsilon) = \int h^*(g,x,\varepsilon)d\mu(x)$.

As a direct consequence of Lemma 5.12, we get the following.

Lemma 5.14. *For any $g \in \mathcal{U}$ and any $\mu \in G^u(g)$, $h^*(g,\mu,r_1) = 0$.*

We also need the following lemma of [20], [Theorem 1.2].

Lemma 5.15. $h_\mu(g) - h_\mu(g,\mathcal{P}) \leq h^*(g,\mu,\rho)$ *for any finite measurable partition $\mathcal{P}$ with $diam(\mathcal{P}) \leq \rho$.*

By Lemma 5.14 and Lemma 5.15, we conclude that $h_\mu(g) = h_\mu(g,\mathcal{P})$ for any finite measurable partition $\mathcal{P}$ with $diam(\mathcal{P}) \leq r_1$. In particular, by a standard argument for the upper semi-continuity of metric entropy (see, for instance, [37], [Lemma 2.3]), we have the following.

Corollary 5.16. *Let g_n ($n \geq 0$) be a sequence of C^1 partially hyperbolic diffeomorphisms inside $\mathcal{U}$ and $\mu_n \in G^u(g_n)$. Suppose yjsy $g_n \to g_0$ in C^1 topology and $\mu_n \to \mu_0 \in G^u(g_0)$ in weak-* topology; then*

$$\limsup_{n \to \infty} h_{\mu_n}(g_n) \leq h_{\mu_0}(g_0).$$

5.4. Physical measures

In this section, we will provide a uniform treatment on the existence of physical measures for all C^{1+} diffeomorphisms in $\mathcal{U}$. For this purpose, let $r_1 > 0$ be given by Lemma 5.5 and (15).

Proposition 5.17. *Let g be any C^1 diffeomorphism of $\mathcal{U}$. Then $G(g)$ is compact and convex, and every extreme element of $G(g)$ is an ergodic measure. The map*

$\mathcal{G} : g \in \mathcal{U} \mapsto \mathrm{G}(g)$ *is upper semi-continuous with respect to diffeomorphisms in* $\mathcal{U}$ *under* C^1 *topology. Moreover, if g is* C^{1+}, *then* $\mathrm{G}(g)$ *has finitely many extreme points, each of which is a physical measure of g and vice versa. The basin of each physical measure of g contains Lebesgue almost every point of some ball with radius* r_1.

Proof. Recall that

$$\mathrm{G}^{cu}(g) = \left\{ \mu \in \mathcal{M}_{\mathrm{inv}}(f) : h_\mu(g) \geq \int \log(\det(Dg\,|_{E^{cu}(x)}))d\mu(x) \right\}.$$

Because the metric entropy function is affine, it follows that $\mathrm{G}^{cu}(f)$ is convex. By Proposition 2.13, $\mathrm{G}^u(g)$ is convex, and so is $\mathrm{G}(g) = \mathrm{G}^u(g) \bigcap \mathrm{G}^{cu}(g)$.

The compactness of $\mathrm{G}(g)$ follows from Corollary 5.16. More precisely, suppose that there is a sequence of invariant probabilities $\{\mu_n\}_{n=0}^\infty$ of g such that $\mu_n \in \mathrm{G}(g)$ and assume $\lim_{n\to\infty} \mu_n = \mu$. Because $\mu_n \in \mathrm{G}^{cu}(g)$, we have

$$h_{\mu_n}(g) \geq \int \log(\det(Dg\,|_{E^{cu}(x)}))d\mu_n(x).$$

Note that $\mu_n \in \mathrm{G}^u(g)$, and by Proposition 2.13, $\mathrm{G}^u(g)$ is compact; thus, we have $\mu \in \mathrm{G}^u(g)$. It then follows from Corollary 5.16 that $\limsup_{n\to\infty} h_{\mu_n}(g) \leq h_\mu(g)$, which implies

$$h_\mu(g) \geq \int \log(\det(Dg\,|_{E^{cu}(x)}))d\mu(x).$$

This means that $\mu \in \mathrm{G}^{cu}(g)$; thus, $\mu \in \mathrm{G}^u(g) \cap \mathrm{G}^{cu}(g) = \mathrm{G}(g)$.

Indeed, by Corollary 5.16 and a similar proof as above, for a sequence of C^1 maps $g_n \in \mathcal{U}, g_n \to g \in \mathcal{U}$ in C^1 topology and $\mu_n \in \mathrm{G}(g_n)$ converging to μ in weak-* topology, we have $\mu \in \mathrm{G}(g)$. Then the map $\mathcal{G}(\cdot)$ is upper semi-continuous, as claimed.

Suppose that μ is any extreme element of $\mathrm{G}(g)$; then it is contained in $\mathrm{G}^u(g)$. We claim the following.

Lemma 5.18. μ *is ergodic.*

Proof. Let $\tilde\mu$ be a typical ergodic component in the ergodic decomposition of μ. We are going to show that $\tilde\mu \in \mathrm{G}(g)$; this implies that $\tilde\mu$ is also an extreme element of $\mathrm{G}(g)$ and thus it coincides with μ.

By Proposition 2.13, $\tilde\mu \in \mathrm{G}^u(g)$. Thus, it suffices to show that $\tilde\mu \in \mathrm{G}^{cu}(g)$.

Because $g \in \mathcal{U}$, by Lemma 5.3, any measure $\nu \in \mathrm{G}^u(g)$ has positive center exponent. By Ruelle's inequality,

$$h_{\tilde\mu}(g) \leq \int \log(\det(Dg\,|_{E^{cu}(x)}))d\tilde\mu(x).$$

Because $\mu \in \mathrm{G}^{cu}(g)$,

$$h_\mu(g) \geq \int \log(\det(Dg\,|_{E^{cu}(x)}))d\mu(x).$$

Because entropy function is an affine functional with respect to invariant measures, we must have $h_{\tilde\mu}(g) = \int \log(\det(Dg\,|_{E^{cu}(x)}))d\tilde\mu(x)$ for typical ergodic components $\tilde\mu$ of μ. Thus, $\tilde\mu \in \mathrm{G}^{cu}(g)$. The proof is complete. $\qquad\square$

We continue the proof of Proposition 5.17. Assume that $g \in \mathcal{U}$ is a C^{1+} partially hyperbolic diffeomorphism. First we suppose that μ is an extreme element of $\mathrm{G}(g)$. Then by the discussion above, μ is ergodic with positive center exponents. Moreover, by Ruelle's inequality, we get

$$h_\mu(g) = \int \log(\det(Dg\,|_{E^{cu}(x)}))d\mu(x).$$

By the entropy formula of Ledrappier-Young [35], the disintegration of μ along the Pesin unstable manifold is equivalent to the Lebesgue measure on the leaves. This means, for μ almost every x, Lebesgue almost every point on the Pesin unstable manifold of x is a typical point of μ. Because the basin of μ is saturated by stable leaves (we use the fact that E^s is uniformly contracting) and the stable foliation is absolutely continuous, the union of the stable leaves of the previous full Lebesgue measure subset of $W^u(x)$ is contained in the basin of μ and has full volume inside a ball with center at x. Note, however, that such a ball may not have uniform radius r_1.

To obtain a ball with radius r_1 in the basin of μ, we apply Lemma 5.8 to obtain an $n > 0$ such that $g^n(x) \in \mathcal{H}(b_0/2, g)$. Then by Lemma 5.11, $W^u(g^n(x), g)$ contains a disk with radius r_1, where Lebesgue typical points in this disk are typical points of μ. By the uniform transversality between the bundles E^s and E^{cu}, the basin of μ contains Lebesgue almost every point of a ball at $g^n(x)$ with radius r_1, which we denote by $B_{g^n(x)}(r_1)$. It then follows that

$$\mu(B_{g^n(x)}(r_1)) > 0. \tag{16}$$

To simplify notation, we write any ball obtained in the above way by B_μ.

Because the basins of different physical measures are disjoint, $\mathrm{G}(g)$ has only finitely many extreme elements. We denote them by $\mu_1, \cdots, \mu_k$.

Now we prove that every physical measure μ of g is an extreme element of $\mathrm{G}(g)$. Because the basin of μ, $\mathcal{B}(\mu)$, has positive volume, by Proposition 2.12, Lebesgue almost every point $x \in \mathcal{B}(\mu)$ must satisfy $\mu = \lim \frac{1}{n} \sum_{i=0}^{n-1} \delta_{g^i(x)} \in \mathrm{G}(g)$. Thus, it only remains to show that μ is ergodic.

Because $\mathrm{G}(g)$ is convex with finitely many extreme elements, μ can be written as a combination:

$$\mu = a_1\mu_1 + \cdots + a_k\mu_k,$$

where $0 \leq a_1, \cdots, a_k \leq 1$ and $\sum_{i=1}^{k} a_k = 1$. There is $1 \leq j \leq k$ such that $a_j > 0$. Then by (16), $\mu(B_{\mu_j}) = a_j\mu_j(B_{\mu_j}) > 0$.

Thus, for every point $x \in \mathcal{B}(\mu)$ and n sufficiently large, $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{g^i(x)}(B_{\mu_j}) > 0$. In particular, there is $n_j > 0$ such that $g^{n_j}(x) \in B_{\mu_j}$.

Because $\mathrm{vol}(\mathcal{B}(\mu)) > 0$, take $x \in \mathcal{B}(\mu)$ a Lebesgue density point of $\mathcal{B}(\mu)$; that is, it satisfies

$$\lim_{r \to 0^+} \frac{\mathrm{vol}(B_x(r) \cap \mathcal{B}(\mu))}{\mathrm{vol}(B_x(r))} \to 1.$$

Then the above argument shows that $\mathcal{B}(\mu) \bigcap B_{\mu_j} = g^{n_j}(\mathcal{B}(\mu)) \bigcap B_{\mu_j}$ has positive Lebesgue measure.

Recall that B_{μ_j} is in the basin of μ_j. Therefore, the basin of μ and μ_j has nontrivial intersection. This implies $\mu = \mu_j$, and μ is ergodic. The proof of Proposition 5.17 is complete. $\square$

Remark 5.19. The C^{1+} regularity is used to

- show that the conditional measures of μ along unstable leaves are absolutely continuous; we need the work of Ledrappier and Young, which requires C^{1+}.
- show that the basin of μ contains Lebesgue almost every point in a ball; there we need the stable foliation to be absolutely continuous.

We will see in Section 7 that such regularity condition can be bypassed for generic C^1 diffeomorphisms in $\mathcal{U}$.

6. Proof of Theorem B and Corollary C

In this section, we provide the proof of Theorem B and Corollary C.

Throughout this section, we assume f to be a C^{1+} diffeomorphism with a mostly expanding center and $\mathcal{U}$ a sufficiently small C^1 neighbourhood of f. By Proposition 5.1, there is $b_0 > 0$ such that for any C^1 diffeomorphism $g \in \mathcal{U}$ and any $\mu \in \mathrm{G}^u(g)$,

$$\int \log \|Dg^{-1}\mid_{E^{cu}(x)}\| d\mu(x) < -b_0. \tag{17}$$

The structure of this section is as follows: In Subsection 6.1 we introduce Liao's shadowing lemma, which will be used in Subsection 6.2 to construct skeletons. For the discussion in Section 8, we will make the construction for every C^1 diffeomorphism $g \in \mathcal{U}$.

Then in Subsection 6.3, we will show that each element in $\mathcal{S}(g)$ is associated to a physical measure, assuming that f is C^{1+}. This concludes the proof of Theorem B. Finally, in Subsection 6.4 we provide the proof of Corollary C.

6.1. Liao's shadowing lemma

Definition 6.1. An orbit segment $(x, f(x), \cdots, f^n(x))$ is called λ-*quasi hyperbolic* if there exists $0 < \lambda < 1$ such that for $1 \le k \le n$,

$$\prod_{i=0}^{k-1} \|Df^{-1}\mid_{E^{cu}(f^{n-i}(x))}\| < \lambda^k, \tag{18}$$

$$\prod_{i=0}^{k-1} \|Df\mid_{E^s(f^i(x))}\| \le \lambda^k \tag{19}$$

and

$$\frac{\|Df\mid_{E^s(f^i(x))}\|}{m(Df\mid_{E^{cu}(f^i(x))})} \le \lambda^2, \tag{20}$$

where $m(\cdot)$ is the minimum norm.

Note that (19) and (20) are automatically satisfied due to the uniform contraction of Df on E^s and the domination between E^s and E^{cu}. In other words, $(x, f(x), \cdots, f^n(x))$ is λ-quasi-hyperbolic if n is a $(-\log \lambda)$-hyperbolic time for x. In this subsection we need the following shadowing lemma by Liao, which allows a quasi-hyperbolic, periodic pseudo-orbit to be shadowed by a periodic orbit with large unstable manifold.

Lemma 6.2 ([38, 27]). *For any $\lambda > 0$, there exist $\rho > 0$ and $L > 0$ such that for any λ-quasi-hyperbolic orbit $(x, f(x), \cdots, f^n(x))$ of f with $d(x, f^n(x)) \leq \rho$, there exists a hyperbolic periodic point $p \in M$ such that*

(a) *p is a hyperbolic periodic point with period n and with stable index i_s;*
(b) *$d(f^i(x), f^i(p)) \leq L d(x, f^n(x))$ for any $0 \leq i \leq n-1$;*
(c) *p has a uniformly sized unstable manifold: there is a constant $r > 0$ depending on λ such that the local unstable manifold of p contains a disk with radius r.*

Remark 6.3. The parameters in the previous lemma can be made uniform for diffeomorphisms in a C^1 neighbourhood $\mathcal{U}$ of f. Moreover, one can take δ sufficiently small; then $d(f^i(x), f^i(p)) \leq L d(x, f^n(x))$ is sufficiently small for any $0 \leq i \leq n-1$, and then

$$\prod_{i=0}^{k-1} \| Df^{-1} \mid_{E^{cu}(f^{n-i}(p))} \| \leq \lambda^k$$

for any $1 \leq k \leq n$. In particular, if one takes $\lambda = e^{-b_0/2}$, then the size of the unstable manifold of p can be chosen to be $r_1 > 0$, which is the constant given by Lemma 5.11.

Definition 6.4. A periodic point p of $g \in \mathcal{U}$ is called a λ-*hyperbolic periodic point* if it satisfies

$$\prod_{i=0}^{k-1} \| Df^{-1} \mid_{E^{cu}(f^{n-i}(p))} \| \leq \lambda^k \tag{21}$$

for any $1 \leq k \leq \pi(p)$.

By the previous discussion and Remark 5.6, we have shown the following.

Lemma 6.5. *For any $e^{-b_0/2}$-quasi-hyperbolic periodic point, its unstable manifold contains an r_1-ball inside the cu-fake leaf $\hat{\mathcal{F}}_{g,p}^{cu}(p, r_1)$*

6.2. Existence of skeleton

In this section, we will show that any C^1 diffeomorphism $g \in \mathcal{U}$ admits a skeleton. The main result of this section is Proposition 6.8.

In order to apply Liao's shadowing lemma, we need to establish the existence of orbit segments that are quasi-hyperbolic. This follows from Proposition 2.12 and (17).

Proposition 6.6. *Suppose $g \in \mathcal{U}$. There is a full-volume subset Γ_g such that for Lebesgue almost every point $x \in \Gamma_g$,*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg^{-1}|_{E^{cu}(g^{n-i}(x))}\| \leq -b_0.$$

By the Pliss lemma (see [1]), there exists $n_x \geq 1$ and $\delta_1 > 0$ such that

$$\#(H(b_0 3/4, x, g) \cap [1, n)) \geq n\delta_1 \text{ for all } n \geq n_x, \tag{22}$$

where $H(b_0 3/4, x, g)$ is the collection of $b_0 3/4$-hyperbolic times along the forward orbit of x.

Taking a sequence of integers $n_x \leq n_1 < n_2 < \cdots$ such that $n_i \in H(b_0 3/4, x, g))$, we may assume that $x_{n_i} = f^{n_i}(x)$ converges to a point x_0. For $\lambda = e^{-\frac{3b_0}{4}}$, ρ and L are obtained by Lemma 6.2. We may further assume that $\sup_{i,j}\{d(x_{n_i}, x_{n_j})\} \leq \rho_0 \leq \rho$, where ρ_0 satisfies that for any two points $y, z \in M$ with $d(y, z) \leq L\rho_0$, we have

$$|\log \|Dg^{-1}|_{E^{cu}(y)}\| - \log \|Dg^{-1}|_{E^{cu}(z)}\|| \leq b_0/4. \tag{23}$$

Because for any $i < j$, the pseudo-orbit $\{x_{n_i}, x_{n_i+1}, \cdots, x_{n_j-1}\}$ is $b_0 3/4$-quasi-hyperbolic, by Lemma 6.2, this pseudo-orbit is $Ld(x_{n_i}, x_{n_j}) \leq L\rho_0$ shadowed by a periodic orbit $p_{x,i,j}$. Because $x_{n_i} \to x_0$ as $i \to \infty$, all of the periodic points $p_{x,i,i+1}$ converge to x_0.

Moreover, by the choice of ρ_0 in (23), $p_{x,i,j}$ is a $e^{-b_0/2}$-quasi-hyperbolic periodic point. By Lemma 6.5, each periodic point $p_{x,i,j}$ has unstable manifold with size at least r_1. Their stable manifolds already have uniform size due to E^s being uniformly contracting (note that all $p_{x,i,j}$s have stable index i_s). Thus, there is m_x such that for any $i, j > m_x$, $p_{x,i,i+1}$ and $p_{x,j,j+1}$ are homoclinic related to each other and

$$\mathcal{F}^s_{\text{loc}}(x_i) \pitchfork W^u_{r_1}(p_{j,j+1}) \neq \emptyset.$$

Furthermore, $\mathcal{F}^s_{\text{loc}}(p_i)$ will intersect transversally with any disk center at x_j, tangent to the *cu* cone with radius at least r_1.

To simplify notation, we will write $p_{x,i,i+1} = p_{x,i}$.

Lemma 6.7. *For any $i > m_x$, $x \in Cl(\mathcal{F}^s(Orb(p_{x,i})))$.*

Proof. Let U be any small neighbourhood of x. Because for any $i > m_x$, all of the hyperbolic periodic points p_i are homoclinic related to each other, we only need to show that there is $i > m_x$ such that $\mathcal{F}^s(\text{Orb}(p_i)) \cap U \neq \emptyset$; then the lemma will follow from the inclination lemma.

We take $\varepsilon > 0$ small enough such that $\hat{\mathcal{F}}^{cu}_{g,x}(x, \varepsilon) \subset U$. By Lemma 5.5, for $i > m_x$ sufficiently large, $g^i(\hat{\mathcal{F}}^{cu}_{g,x}(x, \varepsilon)) \supset \hat{\mathcal{F}}^{cu}_{g,x_{n_i}}(x_{n_i}, r_1)$, where the latter is a disk tangent to a *cu* cone with uniform diameter. This means that when i is sufficiently large,

$$g^{n_i}(\hat{\mathcal{F}}^u_{g,x}(x, \varepsilon)) \pitchfork \mathcal{F}^s_{\text{loc}}(p_i) \neq \emptyset.$$

By the invariance of the stable manifold under the iteration of g, we have $U \cap \mathcal{F}^s(\text{Orb}(p_i)) \neq \emptyset$.

The proof is complete. $\qquad\square$

Now we are ready to construct the skeleton for $g \in \mathcal{U}$. By Proposition 6.6, for each $x \in \Gamma_g$, we fix any one of $p_{x,i}$ for $i > m_x$ and denote it by p_x. Then by the previous lemma, the union $\bigcup_{x \in \Gamma} \mathcal{F}^s(\mathrm{Orb}(p_x))$ is dense in the manifold M.

Moreover, because each periodic point p_x has stable and unstable manifold with size at least r_1, there are only finitely many of them that are not homoclinically related to each other, with number uniformly bounded from above. Take $\{p_1, \cdots, p_k\}$ a subset of $\{p_x\}_{x \in \Gamma}$ that are not homoclinic related and have maximal cardinality.

We claim that $\bigcup_{i=1,\cdots,k} \mathcal{F}^s(\mathrm{Orb}(p_i))$ is dense in the manifold M. Assume that this is not the case; then we can take p_x for $x \in M \setminus \bigcup_{i=1,\cdots,k} \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i)))$. By the choice of $\{p_1, \cdots, p_k\}$, p_x must be homoclinically related to some p_i. However, this means that $\mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i))) = \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_x)))$ by the inclination lemma. Lemma 6.7 then shows that $x \in \mathrm{Cl}(\mathcal{F}^s(\mathrm{Orb}(p_i)))$, which is a contradiction.

Thus, $\{p_1, \cdots, p_k\}$ forms a pre-skeleton. By Lemma 4.5, we have shown the following.

Proposition 6.8. *Every C^1 diffeomorphism $g \in \mathcal{U}$ admits a skeleton $\mathcal{S}(g) = \{p_1, \cdots, p_k\}$, such that for any $1 \leq i \leq k$, $W^u(p_i)$ contains a ball in the fake cu leaf with center at p_i and radius r_1.*

From now on, we fix $\mathcal{S}(g) = \{p_1, \cdots, p_k\}$ a skeleton obtained as above.

6.3. Skeleton and measures

In this section we assume $g \in \mathcal{U}$ to be C^{1+}; then by Lemma 5.3, g has a mostly expanding center. We will establish a one-to-one correspondence between elements of $\mathcal{S}(g)$ and the physical measures of g.

By Proposition 5.17, g has only finitely many physical measures $\{\mu_1, \cdots, \mu_l\}$. Moreover, from Lemma 5.8 and Proposition 5.17, there is $r_1 > 0$ only depending on $\mathcal{U}$ and b_0 such that, for any physical measure μ_j of g, there is a μ_j regular point x_j such that

(a) $x_j \in \mathcal{H}(b_0/2, g)$ and thus has Pesin unstable manifold with size larger than r_1;

(b) μ regular points consist of Lebesgue almost every point on the Pesin unstable manifold of x_j.

The main result of this section is the following.

Proposition 6.9. *The number of physical measures and the number of elements of the skeleton of g are the same; that is, $k = l$. Indeed, there is a bijective map $j \rightarrow i(j)$ such that for any physical measure μ_j of g, there is $p_{i(j)} \in \mathcal{S}(g)$ such that $\mathrm{supp}(\mu_j) = Cl(W^u(Orb(p_i), g))$ and Lebesgue almost every point on $W^u(Orb(p_i), g)$ belongs to the basin of μ_j. Moreover, the closure of $\mathcal{F}^s(Orb(p_i))$ coincides with the closure of $\mathcal{B}(\mu_j)$.*

Proof. Fix any physical measure μ_j of g. By (a) above, there is $p_i \in \mathcal{S}(g)$ such that $\mathcal{F}^s(\mathrm{Orb}(p_i)) \pitchfork W^u_{r_1}(x_j, g) \neq \emptyset$. By the inclination lemma, $g^n(W^u_{r_1}(x_j, g))$ converges to $W^u(\mathrm{Orb}(p_i), g)$. Because $W^u_{r_1}(x_j, g) \subset \mathrm{supp}(\mu_j)$ by (b) above, we have $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i), g)) \subset \mathrm{supp}(\mu_j)$.

To show the reversed inclusion, note that for n large enough, by the inclination lemma, $g^n(W^u_{r_1}(x, g))$ approaches $W^u_{\mathrm{loc}}(p)$ in the following sense: there is a stable holonomy map

from $W^u_{\mathrm{loc}}(p)$ to $g^n(W^u_{r_1}(x,g))$ induced by the stable foliation. Because the set of μ_j typical points is invariant under iteration, Lebesgue almost every point of $g^n(W^u_{r_1}(x,g))$ is also typical for μ_j. Because stable foliation is absolutely continuous and the basin of μ_j is s-saturated, it follows that Lebesgue almost every point of $W^u(\mathrm{Orb}(p_i),g)$ belongs to the basin of μ_j.

Take any point $y \in W^u(p_i) \cap \mathcal{B}(\mu_j)$. Because $g^n(y) \in W^u(\ \mathrm{Orb}(p_i))$ for any $n \geq 0$, $\mu_j = \lim \frac{1}{n}\sum \delta_{g^i(y)}$ is supported on $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i),g))$. As a conclusion,

$$\mathrm{supp}(\mu_j) = \mathrm{Cl}(W^u(\mathrm{Orb}(p_i),g)). \tag{24}$$

Because Lebesgue almost every point on $W^u(\mathrm{Orb}(p_i),g)$ belongs to the basin of μ_j, the map $j \to i(j)$ is injective; in particular, we have $k \geq l$. After reordering the periodic points of $\mathcal{S}(g)$, we may assume that $i(j) = j$ for $j = 1, \cdots, l$.

In order to prove $k = l$, we only need to show that $\{p_1, \cdots, p_l\}$ is a pre-skeleton; that is, $\bigcup^l_{i=1} \mathcal{F}^s(\mathrm{Orb}(p_i))$ is dense in the manifold M. By Proposition 5.17, the union of basins of physical measures has full volume; thus, it suffices to prove that for each $1 \leq i \leq l$, the closure of $\mathcal{F}^s(\mathrm{Orb}(p_i))$ coincides with the closure of $\mathcal{B}(\mu_i)$.

By (24) we have $p_i \in \mathrm{supp}(\mu_i)$. Take $r > 0$ sufficiently small such that $\mu_i(B_r(p_i)) > 0$ and $B_r(p_i) \subset \mathcal{O}_i = \bigcup_{y \in W^u(\mathrm{Orb}(p_i),g)} \mathcal{F}^s(y)$. For any $x \in \mathcal{B}(\mu_i)$, because we have $\frac{1}{n}\sum^{n-1}_{i=0} \delta_{f^i(x)} \to \mu_i$, there is n sufficiently large such that $\frac{1}{n}\sum^{n-1}_{i=0} \delta_{f^i(x)}(B_r(p_i)) > 0$. This shows that there is $m > 0$ such that $f^m(x) \in B_r(p_i) \subset \mathcal{O}_i$. By (ii) of Proposition 4.2, $f^m(x)$ is accumulated by $\mathcal{F}^s(\ \mathrm{Orb}(p_i))$ and so is x. Thus, we have shown that the basin of μ_i is contained in the closure of $\mathcal{F}^s(\mathrm{Orb}(p_i))$, and the reversed inclusion follows immediately from the u-saturation of $\mathrm{supp}(\mu_i)$. We now conclude that $k = l$.

The proof is complete. $\qquad\square$

Proof of Theorem B. By Proposition 6.8, f admits an index i_s skeleton. Let $\mathcal{S} = \{p_1, \cdots, p_k\}$ be any index i_s skeleton of f. By Proposition 6.9, the number of physical measures is precisely $k = \#\mathcal{S}$, and for each $p_i \in \mathcal{S}$ there exists a distinct physical measure μ_i such that

(1) the closure of $W^u(Orb(p_i))$ coincides with $\mathrm{supp}(\mu_i)$ and by (ii) of Proposition 4.2, they also coincide with the homoclinic class of the orbit $\mathrm{Orb}(p_i)$.

(2) the closure of $\mathcal{F}^s(\mathrm{Orb}(p_i))$ coincides with the closure of the basin of the measure μ_i.

Moreover, by (ii) of Proposition 4.2,

$$\mathrm{Int}(\mathrm{Cl}(\mathcal{B}(\mu_i))) \cap \mathrm{Int}(\mathrm{Cl}(\mathcal{B}(\mu_j))) = \emptyset$$

for $1 \leq i \neq j \leq k$. The proof is finished. $\qquad\square$

6.4. Proof of Corollary C

We finish this section with the proof of Corollary C.

Proof of Corollary C. Let f be C^{1+}. For any $n > 0$ and ν an ergodic Gibbs u-state of f^n, by Lemma 2.3, $\mu_n = \frac{1}{n}\sum^{n-1}_{i=0} f^i_*(\nu)$ is an invariant Gibbs u-state of f. It is easy to

see that for ν typical point x, its center exponents with respect to f^n are n times of the corresponding exponents with respect to f. In particular, the center exponents of every Gibbs u-state of f^n are positive. Thus, f^n has a mostly expanding center as well.

Because $\{p_1, \cdots, p_k\}$ is an index i_s skeleton of f, $\bigcup_i \bigcup_{q \in \mathrm{Orb}(p_i)} \mathcal{F}^s(q)$ is dense in the manifold M, which means that $\mathcal{S} = \{q \in \mathrm{Orb}(p_i), i = 1, \ldots, k\}$ is a pre-skeleton of f^n for every $n \geq 0$. By Lemma 4.5, it has a subset that is a skeleton of f^n. It follows from Theorem B that f^n has finitely many physical measures with number bounded by $P = \prod_{i=1}^{k} \pi(p_i) = \#S$.

Moreover, because elements of $\mathcal{S}$ are all distinct fixed points of f^{nP} for any $n > 0$, it is a skeleton for f^{nP}, $n > 0$. Then by Theorem B, f^{nP} have the same number of physical measures for every $n > 0$. Let μ be a physical measure of f^P. By Proposition 5.17, μ is ergodic, and its conditional measures along the Pesin unstable manifolds are equivalent to the Lebesgue measure on the leaves. Below we will show that μ is ergodic for f^{nP} for all $n > 0$.

To this end, let $\tilde{\mu}$ be any ergodic component of μ with respect to f^{nP}; then the conditional measures of $\tilde{\mu}$ along the Pesin unstable manifolds are still equivalent to the Lebesgue measure on the leaves. It then follows from the argument of Proposition 5.17 that $\tilde{\mu}$ is a physical measure of f^{nP}. Because the number of physical measures of f^{nP} is constant, $\tilde{\mu}$ must be the only ergodic component of μ with respect to f^{nP}. It then follows that $\mu = \tilde{\mu}$, which is ergodic for f^{nP}.

Then, by the classical work of Ornstein and Weiss [41], every physical measure of f^P is a Bernoulli measure. $\square$

7. Proof of Theorem E

In this section, we study the robustness of the skeleton and physical measures under C^1 topology among C^{1+} diffeomorphisms and prove Theorem E.

For this purpose, we assume that $f : M \to M$ is a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center and $\mathcal{U}$ a C^1 neighbourhood of f satisfying Lemma 5.3 and Proposition 5.17. Let b_0 be given in Lemma 5.3 and r_1 be given by Proposition 5.17. We take $\mathcal{S}(f) = \{p_1, \cdots, p_k\}$ a skeleton of f. Because $\bigcup_{i=1}^{k} \mathcal{F}^s(\mathrm{Orb}(p_i(f)), f)$ is dense in the manifold M, by the continuity of stable foliation with respect to diffeomorphisms in C^1 topology, we may assume that $\mathcal{U}$ is sufficiently small such that for any C^1 diffeomorphism $g \in \mathcal{U}$, the continuation of $\mathcal{S}(f)$ given by the continuation of hyperbolic saddles $\mathcal{S}(g) = \{p_1(g), \cdots, p_k(g)\}$ satisfies that $\bigcup_{i=1}^{k} \mathcal{F}^s(\mathrm{Orb}(p_i(g)), g)$ is r_1 dense; that is, for any $x \in \mathcal{H}(b_0/2, g)$,

$$\bigcup_{i=1}^{k} \mathcal{F}^s(\mathrm{Orb}(p_i(g)), g) \pitchfork W_{r_1}^u(x, g) \neq \emptyset,$$

where $W_{r_1}^u(x, g)$ is given by Lemma 5.11.

Note that $\mathcal{S}(g)$ may not be a skeleton. In the following, we will show the relation between skeletons of diffeomorphisms in $\mathcal{U}$. For the discussion in the next section, we will state the following lemma for C^1 diffeomorphisms in $\mathcal{U}$.

Lemma 7.1. *For C^1 diffeomorphisms in $\mathcal{U}$, the number of elements of a skeleton varies upper semi-continuously. More precisely, for $g \in \mathcal{U}$:*

(1) $\mathcal{S}(g) = \{p_1(g), \cdots, p_k(g)\}$ *is a pre-skeleton of g; thus, it contains a subset that is a skeleton of g;*

(2) *suppose that $\{q_1(g), \cdots, q_l(g)\}$ is a skeleton of g; then there is a C^1 neighbourhood $\mathcal{V}$ of g such that for any $h \in \mathcal{V}$, $\{q_1(h), \cdots, q_l(h)\}$ is a pre-skeleton of h.*

Proof. By Proposition 6.8, g admits a skeleton $\{q_1(g), \cdots, q_l(g)\}$ such that each $q_j(g)$ $(j = 1, \cdots, l)$ has unstable manifold with size r_1. Then by the previous assumption on $\mathcal{U}$, for every $1 \le j \le l$, there is a $1 \le i \le k$ such that $\mathcal{F}^s(\mathrm{Orb}(p_i(g)), g) \pitchfork W_{r_1}^u(q_j(g), g) \ne \emptyset$. Thus, by the inclination lemma, $\mathcal{F}^s(\mathrm{Orb}(q_j(g)), g)$ is accumulated by $\mathcal{F}^s(\mathrm{Orb}(p_i(g)), g)$, which implies that $\cup_{i=1}^k \mathcal{F}^s(\mathrm{Orb}(p_i(g)))$ is dense in the manifold M. This finishes the proof of (1).

The proof of (2) is quite similar. Take $\mathcal{V}$ sufficiently small such that for any C^1 diffeomorphism $h \in \mathcal{V}$, the continuation $\{q_1(h), \cdots, q_l(h)\}$ satisfies the condition that $\cup_{i=1}^k \mathcal{F}^s(\mathrm{Orb}(q_i(h)), h)$ is r_1 dense. By Proposition 6.8, every $h \in \mathcal{V} \subset \mathcal{U}$ admits a skeleton $\{q_1'(h), \cdots, q_t'(h)\}$, such that each $q_i'(h)$ has unstable manifold with size r_1. Then for every $1 \le j \le t$, there is a $1 \le i \le l$ such that $\mathcal{F}^s(\mathrm{Orb}(q_i(h)), h) \pitchfork W_{r_1}^u(q_j'(h), h) \ne \emptyset$. Thus, by the inclination lemma, $\mathcal{F}^s(\mathrm{Orb}(q_j'(h)), h)$ is accumulated by $\mathcal{F}^s(\mathrm{Orb}(q_i(h)), h)$, which implies that $\cup_{i=1}^k \mathcal{F}^s(\mathrm{Orb}(q_i(h)), h)$ is dense in M. This finishes the proof of (2). □

Thus, by Lemma 4.4, the number of elements of the skeleton of g is bounded from above by $k = \#\mathcal{S}(f)$. It follows that, restricted to a C^1 open and dense subset $\mathcal{U}^\circ \subset \mathcal{U}$, the number of elements of a skeleton for diffeomorphisms of $\mathcal{U}^\circ$ is locally constant. More precisely, for any $1 \le i \le k$, we write

$$\mathcal{U}_i = \{g \in \mathcal{U}; \text{skeleton of } g \text{ has less than } i \text{ number of elements.}\}$$

Then $\mathcal{U}_i$ is an open set, and $\mathcal{U}^\circ = \mathcal{U}_1 \bigcup_{2 \le i \le k} (\mathcal{U}_i \setminus \mathrm{Cl}(\mathcal{U}_{i-1}))$ satisfies our requirement.

By Theorem B, the number of physical measures for C^{1+} diffeomorphisms in $\mathcal{U}^\circ$ is locally constant.

Suppose that $f_n \in \mathcal{U}^\circ$ is a sequence of C^{1+} diffeomorphisms such that $f_n \to f_0 \in \mathcal{U}^\circ$. We assume that all f_n have $m \le k$ physical measures. By the previous argument, all diffeomorphisms f_n and f_0 have the same number of elements in their skeletons. In particular, by Lemma 7.1, we may take a skeleton $\mathcal{S}(f_0) = \{p_1(f_0), \cdots, p_m(f_0)\}$ of f_0 such that its continuation $\mathcal{S}(f_n) = \{p_1(f_n), \cdots, p_m(f_n)\}$ is a skeleton of f_n. For f_n $(n \ge 0)$, denote by $\mu_{n,1}, \cdots, \mu_{n,m}$ the physical measures of f_n associated with the periodic point $p_j(f_n)$ as explained in Theorem B. Next we are going to show the following.

Lemma 7.2. $\mu_{n,i} \overset{weak*}{\longrightarrow} \mu_{0,i}$.

Proof. For simplicity, we will only prove it for $i = 1$. We prove by contradiction and assume (after taking subsequence if necessary) that $\mu_{n,1} \overset{weak*}{\longrightarrow} \mu \ne \mu_{0,1}$.

By Proposition 5.17, the space $\mathrm{G}(\cdot)$ is compact and convex; extreme elements of $\mathrm{G}(\cdot)$ are precisely those physical measures, and it varies in a upper semi-continuous fashion with

respect to diffeomorphisms in $\mathcal{U}$ under C^1 topology. Thus, $\mu_{n,1} \in \mathrm{G}(f_n)$ and $\mu \in \mathrm{G}(f_0)$. Moreover, μ can be written as a combination of the physical measures of f_0:

$$\mu = a_1 \mu_{0,1} + \cdots a_m \mu_{0,m}.$$

By our assumption, $a_1 \neq 1$; thus, there is $1 < i \leq m$ such that $a_i > 0$. We will show that this implies heteroclinic intersection between $p_1(f_n)$ and $p_i(f_n)$, which is a contradiction.

Take $r > 0$ sufficiently small, such that $B_r(p_i(f_0)) \subset \cup_{x \in W^u(p_i(f_0), f_0)} \mathcal{F}^s(x, f_0)$. Then by the continuity of unstable manifolds of $p_i(\cdot)$ and the continuity of stable foliation with respect to diffeomorphisms, there is n_0 such that for any $n > n_0$, any point $x \in B_r(p_i(f_n))$,

$$\mathcal{F}^s_{\mathrm{loc}}(x, f_n) \pitchfork W^u(p_i(f_n), f_n) \neq \emptyset. \tag{25}$$

By Theorem B, $p_i(f_0) \in \mathrm{supp}(\mu_{0,i})$ and $\mu_{0,i}(B_r(p_i(f_0))) > 0$. Because $\mu_{n,1} \to \mu$, which also assigns positive measure to $B_r(p_i(f_0))$, there is $n > n_0$ such that $\mu_{n,1}(B_r(p_i(f_0))) > 0$. In particular, we have $\mathrm{supp}(\mu_{n,1}) \cap B_r(p_i(f_0)) \neq \emptyset$. Again by Theorem B, $\mathrm{supp}(\mu_{n,1}) = H(p_1(f_n), f_n)$; thus, $\mathcal{F}^s(\,\mathrm{Orb}(p_1(f_n)), f_n) \cap B_r(p_i(f_0)) \neq \emptyset$. By (25),

$$\mathcal{F}^s(\mathrm{Orb}(p_1(f_n))) \pitchfork W^u(p_i(f_n), f_n) \neq \emptyset,$$

which contradicts the fact that $\{p_1(f_n), \cdots, p_k(f_n)\}$ is a skeleton of f_n and thus by Lemma 4.1[(1)] there is no heteroclinic intersection between $p_i(f_n)$ and $p_j(f_n)$ for $1 \leq i \neq j \leq k$. $\qquad\square$

To prove Theorem E, it remains to show that for diffeomorphisms in $\mathrm{Diff}^{1+}(M) \cap \mathcal{U}^\circ$, the supports of corresponding physical measures and the closures of their basins vary in a lower semi-continuous fashion, both in the sense of the Hausdorff topology.

Indeed, by the unstable manifold theorem of fixed saddle, for each $R > 0$, the local invariant manifolds $W^u_R(\mathrm{Orb}(p_i(g), g))$ vary continuously with $g \in \mathcal{U}$; moreover, the stable foliation varies continuously with respect to g. Thus, the closures of $W^u(\mathrm{Orb}(p_i(g), g))$ and $\bigcup_{x \in W^u(\,\mathrm{Orb}_{(p_i(g)), g})} \mathcal{F}^s(x, g)$ both vary in a lower semi-continuous fashion with g, relative to the Hausdorff topology. By Theorem B, this means that the supports and the closures of the basins of the physical measures vary lower semi-continuously with the dynamics. The proof of Theorem E is now complete.

8. Proof of Theorem F

In this section we will generalise the result of Theorem E to C^1 generic diffeomorphisms in $\mathcal{U}$. The proof is similar to [32, Theorem B]. The key observations are as follows:

- C^{1+} diffeomorphisms are dense in C^1 topology.
- Skeletons are upper semi-continuous in $\mathcal{U}$.
- The support of physical measures for $C^{1+}g \in \mathcal{U}$ are homoclinic classes, which are (generically) Lyapunov stable and lower semi-continuous with the dynamics.
- The candidate space of physical measures, $\mathrm{G}(\cdot)$, is upper semi-continuous.

These properties will allow us to find a residual subset of $\mathcal{U}$, consisting of continuity points of $H(p_i(\cdot), \cdot)$ and $\mathcal{G}(\cdot)$. We will prove Theorem F on this residual subset of $\mathcal{U}$.

Throughout this section, let $f : M \to M$ be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center, $\mathcal{S}(f) = \{p_1, \cdots, p_k\}$ be a skeleton of f and $\mathcal{U}$ be the C^1 neighbourhood of f provided by Theorem E. Recall that by Lemma 7.1, the cardinality of skeleton varies in an upper semi-continuous way; we may choose a C^1 open and dense subset $\mathcal{U}^\circ \subset \mathcal{U}$ such that the cardinality of skeleton is C^1 locally constant for diffeomorphisms in $\mathcal{U}^\circ$.

Take any C^{1+} diffeomorphism $g \in \mathcal{U}^\circ$; then g has $l \leq k$ physical measures due to Theorem E. Furthermore, there is a subset of the continuation $\mathcal{S}(g) = \{p_1(g), \cdots, p_k(g)\}$ that forms a skeleton of g. After reordering, we may assume $\{p_1(g), \cdots, p_l(g)\}$ to be a skeleton of g. Then by Lemma 7.1[(2)], there is a C^1 neighbourhood $\mathcal{V} \subset \mathcal{U}^\circ$ of g such that for any C^1 diffeomorphism $h \in \mathcal{V}$, $\{p_1(h), \cdots, p_l(h)\}$ forms a skeleton of h.

Then by Lemma 4.1[(2)], for any C^1 diffeomorphism $h \in \mathcal{V}$ and any $1 \leq i \neq j \leq l$, $W^u(\mathrm{Orb}(p_i(h)), h) \cap \mathcal{F}^s(\mathrm{Orb}(p_j(h)), h) = \emptyset$. Using Bonatti and Crovisier's connecting lemma ([8]), we see that for any diffeomorphism $h' \in \mathcal{V}$ and any $1 \leq i \neq j \leq l$,

$$\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h')), h')) \cap \mathrm{Cl}(W^s(\mathrm{Orb}(p_j(h')), h')) = \emptyset,$$

because otherwise one can create a nontrivial intersection between $W^u(\mathrm{Orb}(p_i(\cdot)), \cdot)$ and $\mathcal{F}^s(\mathrm{Orb}(p_j(\cdot)), \cdot)$.

By Proposition 4.2,

$$\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h')), h')) = H(p_i(h'), h') \subset \mathrm{Cl}(W^s(\mathrm{Orb}(p_j(h')), h')).$$

Thus, we have

$$H(p_i(h'), h') \cap H(p_j(h'), h') = \emptyset, \text{ and} \tag{26}$$

$$\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h'), h'))) \cap \mathrm{Cl}(W^u(\mathrm{Orb}(p_j(h')), h')) = \emptyset. \tag{27}$$

We need the following generic property proved by Morales and Pacifico [17].

Proposition 8.1. *For every h that belongs to a C^1 residual subset of diffeomorphisms $\mathcal{R}_0$ and every periodic point p of h, the set $\mathrm{Cl}(W^u(\mathrm{Orb}(p), h))$ is Lyapunov stable.*

Recall that the map $\mathcal{G}$ that maps a diffeomorphism $h \in \mathcal{V}$ to $\mathrm{G}(h)$ is upper semi-continuous by Proposition 5.17. Let $\mathcal{R}_1 \subset \mathcal{V}$ be the residual subset of diffeomorphisms that are continuity points of the map $\mathcal{G}$. For each $1 \leq i \leq l$, also consider the map $\mathcal{I}_i$ from $\mathcal{V}$ to compact subsets of M:

$$\mathcal{I}_i(h) = H(p_i(h), h).$$

Because homoclinic classes vary lower semi-continuously with respect to diffeomorphisms (because they contain hyperbolic horseshoes), there is a residual subset of diffeomorphisms $\mathcal{R}_2 \subset \mathcal{V}$ that consists of the continuity points of $\mathcal{I}_i$ for every $1 \leq i \leq l$. Now let us take $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2 \subset \mathcal{V}$. We are going to show that the residual set $\mathcal{R}$ satisfies the conditions we need.

Proposition 8.2. *Every C^1 diffeomorphism $h \in \mathcal{R}$ has exactly l physical measures, each of which is supported on $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h)), h))$ for some $i = 1, \cdots, k$. Furthermore, the*

basin of each physical measure covers a full-volume subset within a neighbourhood of its support.

Proof. For any C^{1+} diffeomorphism $h' \in \mathcal{V}$, denote by $\mu_{h',1}, \cdots, \mu_{h',l}$ the ergodic physical measures of h'. Then by Proposition 5.17, $G(h')$ is the simplex generated by $\{\mu_{h',1}, \cdots, \mu_{h',l}\}$. For any $h \in \mathcal{R}$, by the continuity of the map $\mathcal{G}$ at h, we see that $G(h) = \mathcal{G}(h)$ is a simplex of dimension $m_h \leq l$. In particular, the number of extreme elements of $G(h)$ is at most l. Below we will show that it is in fact l.

Denote the extreme points of $G(h)$ by $\mu_{h,1}, \cdots, \mu_{h,m_h}$. Let h_n be a sequence of C^{1+} diffeomorphisms converging to h in C^1 topology. By continuity of $\mathcal{G}(\cdot)$ and relabelling if necessary, we may assume that $\lim \mu_{h_n,i} = \mu_{h,i}$ for $i = 1, \cdots, m_h$. Note that $\mu_{h,i}$ is supported on $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h)), h))$. This is because by Theorem B, $\mu_{h_n,i}$ is supported on $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h_n)), h_n)) = H(p_i(h_n), h_n)$, and h is a continuity point of the map $\Gamma_i(\cdot)$, so we must have $\lim_n H(p_i(h_n), h_n) = H(p_i(h), h)$.

Next, we claim that $m_h = l$. Assume that this is not the case. Then we take $m_h < j \leq l$ and take a weak-$*$ limit $\mu_h = \lim_n \mu_{h_n,j}$. Note that $\mu_h \in G(h)$ is supported on $\mathrm{Cl}(W^u(\mathrm{Orb}(p_j(h)), h))$ by the discussion above. Take any ergodic component $\tilde{\mu}_h$ of μ_h; then $\tilde{\mu}_h \in G(h)$ by Lemma 5.18 and is still supported on $\mathrm{Cl}(W^u(\mathrm{Orb}(p_j(h))), h)$. Thus, by (26), $\tilde{\mu}_h \neq \mu_{h,i}$ for every $i = 1, \cdots, m_h$. We have thus created a new extreme point of $G(h)$, which is a contradiction.

To finish the proof, we have to show that each $\mu_{h,i}$ is a physical measure. Because $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h)), h))$ is Lyapunov stable, we can take $U_i \supset V_i$ open neighbourhoods for each $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h)), h))$, such that $\{U_i\}_{i=1,\cdots,l}$ are disjoint and for each i and any $n > 0$, $h^n(V_i) \subset U_i$. By Proposition 2.12, there is a full-volume subset $\Gamma_i \subset V_i$ such that for any $x \in \Gamma_i$, any limit μ of the sequence $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{h^i(x)}$ belongs to $G(h)$. Note that because $x \in V_i$, we have $h^n(x) \in U_i$ for all $n \geq 1$. As a result, μ is supported on U_i. On the other hand, $\mu_{h,i}$ is the only ergodic measure in $G(h)$ that is supported on U_i. It follows that $\mu = \mu_{h,i}$. This implies that Lebesgue almost every point of $x \in V_i$ belongs to the basin of $\mu_{h,i}$. The proof is complete. $\square$

We conclude the proof of Theorem F with the following lemma.

Lemma 8.3. *The basin of $\mu_{h,i}$ for $i = 1, \cdots, l$ covers a full-volume set.*

Proof. Let Γ be the full-volume subset given by Proposition 2.12. We are going to show that $\mathrm{vol}(\Gamma \setminus \bigcup_{i=1}^{l} \mathcal{B}(\mu_{h,i})) = 0$.

We prove by contradiction. Write $\Lambda = \Gamma \setminus \bigcup_{i=1}^{l} \mathcal{B}(\mu_{h,i})$ and suppose that $\mathrm{vol}(\Lambda) > 0$. Let $x \in \Lambda$ be a Lebesgue density point of Λ, which means that for any $r > 0$, we have $\mathrm{vol}(B_r(x) \cap \Lambda) > 0$. Let μ be any limit point of the sequence $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{h^i(x)}$. Because $\mu \in G(h)$, μ can be written as a combination of $\mu_{h,i}$:

$$\mu = a_1 \mu_{h,1} + \cdots + a_l \mu_{h,l},$$

where $a_1 + \cdots + a_k = 1$.

Suppose without loss of generality that $a_1 > 0$; then $\mu(V_1) > 0$, where V_1 is the neighbourhood of $\mathrm{Cl}(W^u(\mathrm{Orb}(p_i(h)), h))$ in the proof of the previous proposition. Thus, there is $n > 0$ such that $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{h^i(x)}(V_1) > 0$. In particular, there is $0 \leq m \leq n-1$

such that $h^m(x) \in V_1$. Take $\varepsilon > 0$ sufficiently small; then we have $h^m(B_\varepsilon(x)) \subset V_1$. By Proposition 8.2, $f^m(B_\varepsilon(x) \cap \Lambda)$ intersects with the basin of $\mu_{h,1}$ on a positive volume set. Because the basin of a measure is invariant under iteration of h and h^{-1}, we have $\text{vol}(\Lambda \cap \mathcal{B}(\mu_{h,1})) > 0$, which contradicts the choice of Λ. $\qquad\square$

9. Gibbs-Markov-Young structure

To study statistical properties of some nonuniformly hyperbolic systems, in [53] Young constructed Markov towers, which are Markov partitions with infinitely many symbols and certain recurrence property. In particular, she used towers to study statistical properties of these nonuniformly hyperbolic systems, including the existence of physical measures, exponential decay of correlations and the validity of the central limit theorem for the physical measure. These structures have some properties which address to Gibbs states and they are nowadays commonly called as GMY structures.

In [2], Alves and Li obtained GMY structures for partially hyperbolic attractors and they managed to prove the exponential decay of correlations: if the lack of expansion of the system at time n in the center-unstable direction is exponentially small, then the system has some GMY structure for physical measures with exponential decay of recurrence times. In this section we will show that their criterion is satisfied for any physical measures of any C^{1+} diffeomorphisms with a mostly expanding center.

As before, we assume f to be a C^{1+} partially hyperbolic diffeomorphism with a mostly expanding center, $\{p_1, \cdots, p_k\}$ is a skeleton of f and $\mu_1, \cdots, \mu_k$ are the corresponding physical measures of f in the sense of Theorem B. Recall that $P = \prod_{i=1}^{k} \pi(p_i)$.

By Corollary C, $\{f^{nP}\}_{n>0}$ also have a mostly expanding center, and they share the same physical measures and skeletons. Therefore, to simply notation, we may assume that $\{p_i\}_{i=1}^{k}$ are all fixed points and $P = 1$. Moreover, by Proposition 5.1, we may assume that there is $b_0 > 0$ such that for any $\mu \in \mathrm{G}^u(f)$,

$$\int \log \|Df^{-1}|_{E^{cu}(x)}\| d\mu(x) < -b_0. \tag{28}$$

The notation below was used by Alves and Li [2] and clearly resembles our definition of hyperbolic times.

Definition 9.1. Given $b > 0$, we say that f is b *nonuniformly expanding* (b-NUE) at a point x in the central-unstable direction if

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}(f^j(x))}\| < -b. \tag{29}$$

If f satisfies (b-NUE) at some point x, then the *expansion time function* at x,

$$\mathcal{E}_b(x) = \min\left\{ N \geq 1 : \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-1}|_{E^{cu}(f^i(x))}\| < -b/2 \ \text{ for any } n \geq N \right\}, \tag{30}$$

is defined and finite. We call $\{x : \mathcal{E}_b(x) > n\}$ the *tail of $b/2$-hyperbolic times* (at time n).

We need the following two propositions from [2], which play a key role in the proof of decay of correlations and the central limit theorem.

Proposition 9.2 ([2]). *Assume for $b > 0$ that there is a local unstable disk D of f and constants $0 < \tau \leq 1$, $c > 0$ such that*

$$\mathrm{vol}_D(\mathcal{E}_b > n) = O(e^{-cn^\tau}).$$

Then some power f^l has an physical measure μ and there is $d > 0$ such that

$$C_\mu(\phi, \psi \circ f^{ln}) = O(e^{-dn^\tau})$$

for Holder continuous $\phi : M \to \mathbb{R}$ and $\psi \in L^\infty(\mu)$.

Proposition 9.3 ([2]). *Assume for $b > 0$ that there is a local unstable disk D of f and constants $0 < \tau \leq 1$, $c > 0$ such that*

$$\mathrm{vol}_D(\mathcal{E}_b > n) = O(e^{-cn^\tau}).$$

Then some power f^l has an physical measure μ; moreover, given any Hölder continuous function ϕ, the following limit exists:

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \int \left(\sum_{j=0}^{n-1} \phi \circ f^{jl} - n \int \phi d\mu \right)^2 d\mu.$$

Furthermore, if $\sigma^2 > 0$, then there is a rate function $c(\varepsilon) > 0$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \mu \left(\left| \sum_{j=0}^{n-1} \phi \circ f^{jl} - n \int \phi d\mu \right| \geq \varepsilon \right) = -c(\varepsilon).$$

Remark 9.4. From the proof, the physical measure coincides with the limit of

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{vol}_{f^i(\Lambda)},$$

where $\Lambda \subset D$ is some subset with positive volume.

With this notation, we are ready to prove Theorem G and Corollary H. It suffices for us prove only for physical measures μ_1: Take $D = W_r^u(p_1)$. We will show in the end of this section that D satisfies the following property.

Lemma 9.5. *There are constants $0 < \tau \leq 1$ and $c > 0$ such that*

$$\mathrm{vol}_D(\mathcal{E}_{b_0} > n) = O(e^{-cn^\tau}).$$

Then we may apply Proposition 9.2 and Proposition 9.3 on some physical measure μ for some power f^l of f. Moreover, by Proposition 6.9, Lebesgue almost every point belongs

to the basin of μ_1, and thus by Remark 9.4, for any subset $\Lambda \subset D$ with positive volume, we have

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \text{vol}_{f^i(\Lambda)} = \mu_1.$$

Thus, we conclude the proof of Theorem G and Corollary H.

It remains to show the proof of Lemma 9.5.

Proof. We need the following result, which generalises [24, Theorem 1].

Proposition 9.6 ([23, Theorem D']). *Let $\mathcal{B}$ be any foliation box for the unstable foliation $\mathcal{F}^u$ of f, A be any Hölder function and $I_A = \{\int A d\mu\}_{\mu \in \text{Gibbs}^u(g)}$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$, $C > 0$ such that for any plaque L of $\mathcal{F}^u \,|\, \mathcal{B}$,*

$$\text{vol}_L \left(\left\{ x : d\left(\frac{1}{n} S_n(A)(x), I_A \right) \geq \varepsilon \right\} \right) \leq C e^{-\delta n},$$

where $S_n(A) = \sum_{i=1}^{n} A(f^i(x))$.

Fix $\mathcal{B}$ to be any foliation box for the unstable foliation $\mathcal{F}^u$ such that $D \subset \mathcal{B}$. By (28), for $A = \log \|Df^{-1}|_{E^{cu}(x)}\|$, $I_A \subset (\infty, -b_0)$. Applying the previous proposition with $\varepsilon = b_0/2$, we obtain $C > 0, \delta > 0$ such that for any plaque L of $\mathcal{F}^u \,|\, \mathcal{B}$,

$$\text{vol}_L \left(\left\{ x : \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-1}|_{E^{cu}(f^i(x))}\| \geq -b_0/2 \right\} \right) \leq C e^{-\delta n}. \tag{31}$$

Note that

$$\{x : \mathcal{E}_{b_0} > n\} \subset \bigcup_{m \geq n} \left\{ x : \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-1}|_{E^{cu}(f^i(x))}\| \geq -b_0/2 \right\}.$$

Thus, by (31), there are C' and δ' such that for any unstable plaque $L \subset \mathcal{B}$,

$$\text{vol}_L(\mathcal{E}_{b_0} > n) \leq C' e^{-\delta' n}.$$

Because D is the local unstable manifold at p_1, $\mathcal{F}^u$ also induces a subfoliation of D (note that $\dim D = \dim E^{cu}$). It is well known that $\mathcal{F}^u$ is absolutely continuous, and so is the subfoliation of D. Then the previous inequality implies that there is $C_0 > 0$ such that

$$\text{vol}_D(\mathcal{E}_{b_0} > n) \leq C_0 e^{-\delta' n}. \tag{32}$$

Then Lemma 9.5 follows with $\tau = 1$. □

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