ANZIAM J. 50(2008), 143–176
doi:10.1017/S1446181109000108

SPACE–TIME STRUCTURE AND SPINOR GEOMETRY

GEORGE SZEKERES[1] and LINDSAY PETERS[✉ 2]

(Received 11 March, 2009)

Abstract

The structure of space–time is examined by extending the standard Lorentz connection group to its complex covering group, operating on a 16-dimensional "spinor" frame. A Hamiltonian variation principle is used to derive the field equations for the spinor connection. The result is a complete set of field equations which allow the sources of the gravitational and electromagnetic fields, and the intrinsic spin of a particle, to appear as a manifestation of the space–time structure. A cosmological solution and a simple particle solution are examined. Further extensions to the connection group are proposed.

2000 *Mathematics subject classification*: primary 83C05; secondary 83E99.

Keywords and phrases: space–time structure, spinor geometry, gravitational field, electromagnetic field.

1. Introduction

The geometric structure of space–time has long been regarded as settled by general relativity: a four-dimensional pseudo-Riemannian manifold with signature 3+1, equipped with appropriate field equations. Connection and parallel displacement have played a comparatively minor role in this development. The significance of affine connection for space–time geometry was recognized quite early by Weyl [13], but the development of the theory remained largely unaffected by it. Among the few attempts to attribute a greater independent role to affine connection perhaps the best known is the Palatini [8] suggestion (see also Schrödinger [10, Chapter 12]) to treat affine connection as the fundamental field quantity of the geometry and to derive its relation to the (nonsymmetric) metric tensor from a variation principle. The pseudo-Riemannian character of space–time appears somewhat artificially however, through a metric tensor supplied by the symmetric part of the Ricci tensor.

In more recent times there has been renewed interest in affine connections with torsion, notably in the U4 theory of Sciama [11], Kibble [4] and Hehl *et al.* [3].

[1]School of Mathematics, University of New South Wales, Sydney 2052, Australia (deceased).
[2]Pacific Knowledge Systems, Australian Technology Park, Sydney 1430, Australia;
e-mail: l.peters@pks.com.au.

The appearance of torsion in the connection is motivated by physical rather than geometrical necessity: the spin angular momentum tensor of the matter field generates torsion and through it spin itself becomes the source of the gravitational field. The electromagnetic field forms no part of the affine connection and U4 theory makes no claim to be a "unified" field theory. The principal gain derived from the theory is that it definitively establishes the form in which torsion and spin angular momentum must appear in both the gravitational equations and in Dirac's equation. There is an excellent account of that theory in [3], including an extensive list of references.

Our purpose here is to examine the structure of space–time from a predominantly geometric point of view. In spite of the remarkable success of general relativity as a physical theory, Einstein's model of space–time is not wholly satisfactory. The field equations of nonempty space–time require a matter tensor which is not an inherent part of the geometric structure but injected from the outside so to speak into the geometrical framework. In this respect we are no better off than in Newton–Galilei space–time where the geometric structure is also a mere framework for the phenomena of the physical world. The gravitational field itself appears as a curious hybrid between "geometry" and "matter".

A second, even more disturbing shortcoming of the Einstein model is that the group of connection of the underlying pseudo-Riemannian manifold is the real pseudo-orthogonal group with signature 3+1 (the Lorentz group) and therefore inherently incapable of accommodating particles with spin as manifestations of the space–time structure. For that we need (at least) the covering group of the Lorentz group $\mathcal{L}$, that is, the complex spin representation of $\mathcal{L}$, as the group of connection. The ensuing geometrical structure is of course far more complex than a pseudo-Riemannian geometry, but the return is also far greater. In carrying out the program we need to work with complex representation spaces upon which the spin representation can act. As a result, two distinct types of field quantities will appear in the description of the geometry. The first are structure entities, that is, complex-valued fields over a base manifold $\mathcal{M}$ with appropriate tensorial properties with respect to certain abstract spaces. The second are proper geometric quantities, that is, real fields with tensorial properties with respect to the underlying 4-manifold $\mathcal{M}$ (the space-like manifold) with coordinates x^μ, $\mu = 1, 2, 3, 4$ in some local neighbourhood of $\mathcal{M}$. Only the geometric quantities will represent physically observable entities.

As it turns out, quantities of the first type (the structure entities) are not really tensor fields in the ordinary sense but classes of equivalent tensor fields which, however, produce the same real tensor fields in the underlying space–time manifold. No individually selected representation of the equivalence class can therefore be considered as an "observable" field quantity. A good analogue is the electromagnetic potential in Maxwell's theory which is only defined modulo an arbitrary gradient field.

Two fibre bundles over $\mathcal{M}$ will describe the structure entities: one termed the "metric bundle" (Section 2) which serves as a messenger between structure and geometry (and on the way supplies the pseudo-Riemannian metric tensor for $\mathcal{M}$), and the "spinor bundle" (Section 3) which carries the spin representation of an appropriate extension of $\mathcal{L}$. The first of these replaces Einstein's Vierbein device using Minkowski's representation of the Lorentz group with three real space coordinates and one imaginary time coordinate. The second bundle is a spinor version of the Vierbein, namely a tetrad of orthonormal Dirac spinor fields which arise naturally in this representation of spinors by a method that goes back to Eddington [2]. The fibres of this bundle will be an equivalence class of fibres.

The quantities which describe spinor connection (Section 4), when converted into real geometric fields, represent both the electromagnetic field and a form of Hehl's torsion field. The physical role of these tensor fields emerges when we postulate a variation principle in Section 5 which supplies the field equations of the geometry. Dirac's equation (in a general relativistic form) appears as a constraint in the Hamiltonian of the variation integral, and a Lagrangian factor in the constraint (actually, its reciprocal) can be interpreted as "cosmic time" which, contrary to the usual coordinate time, is here a scalar physical quantity.

The theory is "unified" in so far as it provides the sources of the electromagnetic and gravitational fields, as well as being able to represent particles with spin. A further interesting result is that separate equations arise for the material and intensity components of the electromagnetic field, compared to the classical Maxwell equations where these components are not separated.

Finally, we provide here just two simple examples of the types of solutions possible. In Section 6 we describe a cosmological solution to the field equations with a Robertson–Walker line element and nonzero spinor field tetrad, corresponding to the Einstein–de Sitter model.

In Section 7 we describe one of the simplest possible elementary particle solutions, namely, a neutral spherical symmetric (spinless) solution with mass. Whether or not this represents a real particle, it is illustrative in that the spinor and gravitational fields generate an internal structure which is highly nontrivial. Of further interest is the fact that the magnitude of the spinor field, and hence the mass, is "quantized" by the requirement that the solution be singularity-free.

The example solutions presented here are torsion-free; however, more complex particle solutions have also been examined by the authors [9]. These include a spherically symmetric charged particle ("meson-like"), an axially symmetric charged particle with spin ("electron-like"), and a magnetic monopole particle with charge. These solutions illustrate how charge and spin arise naturally from the geometry, but have required extensive computer-based numerical integration and the results are not reproduced here.

Lynch, an early collaborator with Szekeres, has also explored particle solutions using this type of approach and has attempted to make more definite identifications with known particles. (See [7] for an electron/positron solution and [6] for a neutrino solution.)

The discussion throughout is in an explicit coordinatized (nonaxiomatic) form, in the spirit of Descartes rather than that of Hilbert.

2. The metric bundle

To describe the metric (or Vierbein) bundle we introduce the symbols

$$j_1 = j_2 = j_3 = i, \quad j_4 = 1$$

and define $\varepsilon_n = j_n^2$, so that

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1, \quad \varepsilon_4 = 1.$$

The Minkowski representation space of the Lorentz group $\mathcal{L} \equiv O_+(3, 1)$ can then be described as a 4-vector space $\mathcal{V}$ with elements $v = (v_1, v_2, v_3, v_4)$ with reality conditions $j_n v_n$ real, $n = 1, 2, 3, 4$. The effect of $\sigma \in \mathcal{L}$ upon $v \in \mathcal{V}$ is then described by

$$(\sigma v)_m = \sum_{n=1}^{4} M_{mn} v_n,$$

where $M = (M_{mn})$ is a complex orthogonal matrix satisfying

$$\sum_k M_{mk} M_{nk} = \sum_k M_{km} M_{kn} = \delta_{mn} \quad \text{(the Kronecker symbol)}$$

with reality conditions $j_m j_n M_{mn}$ real, $\det M = 1$.

The advantage of using the Minkowski instead of the standard Lorentz representation of $\mathcal{L}$ is that there is no need to distinguish (in terms of components) between vectors and their duals: the action of $\sigma \in \mathcal{L}$ upon $u^* = (u_n) \in \mathcal{V}^*$ (the dual space of $\mathcal{V}$) is

$$(u^* \sigma^{-1})_n = \sum_k u_k \breve{M}_{kn} = \sum_k M_{nk} u_k, \quad (\breve{M}_{kn}) = M^{-1}.$$

The inner product $(u^*, v) = \sum_n u_n v_n$ is invariant under the action of $\mathcal{L}$ and will be denoted $(u, v) = u_n v_n$. We have used here the summation convention for repeated Roman indices, even though both appear in the lower position (they always will be). This summation convention will be used forthwith, except in conjunction with the symbols j_n, ε_n (such as stating reality condition for vectors or matrices). It is understood that k, l, m, n, p, q and so on run from 1 to 4. Later on the Roman suffices a, b, c, will also appear; they will run from 0 to 5.

The fibre of the metric (or Vierbein) bundle is expressed now by the matrix $(g_{n\mu})$, $1 \le n, \mu \le 4$ where the index n indicates that for fixed μ, $(g_{n\mu})$ is a vector in $\mathcal{V}$ and for fixed n, $(g_{n\mu})$ is a covector with respect to a change of coordinates in the underlying base manifold $\mathcal{M}$, that is, $g_{n\mu}$ transforms into $g_{n\mu} \partial x^\mu / \partial x^\nu$. We assume that

$$\underline{\underline{g}} = i \det(g_{n\mu}) \neq 0.$$

Clearly $\underline{\underline{g}}$ is real and transforms into $\underline{\underline{g}}\ \partial(x)/\partial(y)$ under a change of coordinates in $\mathcal{M}$. Hence $\underline{\underline{g}}$ is what Weyl and Schrödinger call a *density* over $\mathcal{M}$, the first instance of a real geometric quantity. Under a transformation of the basis in $\mathcal{V}$ by (M_{mn}), $\underline{\underline{g}}$ gets multiplied by $\det M = 1$, that is, $\underline{\underline{g}}$ is invariant with respect to the choice of basis in $\mathcal{V}$.

https://doi.org/10.1017/S1446181109000108 Published online by Cambridge University Press

We shall refer to such transformations of the $\mathcal{V}$-base as anholonomic transformations; true $\mathcal{M}$-vectors and tensors must be invariant to them.

Our second example of a real tensor field over $\mathcal{M}$ is

$$g_{\mu\nu} = g_{m\mu} g_{m\nu} = g_{\nu\mu}$$

which is a real symmetric covariant tensor of rank two. Here $g = (g_{n\mu})$ can be interpreted as a nonsingular linear mapping from the tangent vector space at $p \in \mathcal{M}$ to $\mathcal{V}$ and *vice versa*. It converts complex vectors in $\mathcal{V}$ into real tangent vectors in $\mathcal{M}$. Associated with g is a dual mapping

$$g^* = (g_n^\mu), \quad j_n g_n^\mu \text{ real}$$

from the cotangent vector space at p to $\mathcal{V}$ with the property that

$$g_m^\mu g_{m\nu} = \delta_\nu^\mu, \quad g_m^\mu g_{n\mu} = \delta_{mn}$$

that is, (g_m^μ) is the inverse of the matrix $(g_{m\mu})$. Here of course the Einstein summation convention on repeated upper and lower Greek indices is used and the equations express the fact that $(g_{m\mu})$, $\mu = 1, 2, 3, 4$, represents a Vierbein of four orthonormal vectors over $\mathcal{M}$.

One feature of the metric bundle $g = (g_{m\mu})$ is that its components are *not* uniquely determined by $g_{\mu\nu}$. In fact if $M(\underline{x}) = (M_{mn}(\underline{x}))$ is a field of Minkowski orthogonal matrices (with the appropriate reality conditions) then $g^*_{m\mu} = (M_{mn} g_{n\mu})$ is equivalent to $g_{n\mu}$ in the sense that it produces the same metric tensor and density $\underline{\underline{g}}$. We shall refer to such a Minkowski transformation as a reorientation of the Vierbein frame and we may regard the fibre of the metric bundle as the equivalence class of the $(g_{n\mu})$ under reorientation.

We conclude this section with a remark on the space $\mathcal{W}$ of skew tensors generated by the wedge products $u \wedge v$ of vectors in $\mathcal{V}$ which will play an important part in the following sections. If $w = (w_{mn}) \in \mathcal{W}$ then $w_{nm} = -w_{mn}$ with the reality conditions $j_m j_n w_{mn}$ real. The effect of $\sigma \in \mathcal{L}$ on w is

$$(\sigma w \sigma^{-1})_{mn} = M_{mp} M_{nq} w_{pq}.$$

3. The spinor bundle

A spinor tetrad has an obvious representation by means of 4×4 complex matrices in which each column represents a Dirac spin vector. To express the relevant algebraic transformation properties of the tetrad it is more convenient to use an alternative model which goes back to Eddington (see also Benn [1]) and which utilizes the equivalence of 4×4 matrices and a 16-dimensional complex Clifford algebra Ω with identity element I.

To obtain a convenient vector basis for Ω we introduce the Dirac symbols $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ satisfying

$$\Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2\delta_{mn} I.$$

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We also introduce the 15 symbols $\Gamma_{ab} = -\Gamma_{ba}$, $0 \le a < b \le 5$, by

$$\Gamma_{0m} = -\Gamma_{m0} = \Gamma_m, \quad \Gamma_{05} = -\Gamma_{50} = -\Gamma_1\Gamma_2\Gamma_3\Gamma_4,$$
$$\Gamma_{ab} = -\Gamma_{ba} = -i\Gamma_{0a}\Gamma_{0b} \quad 0 \le a < b \le 5,$$

with multiplication rules

$$\begin{aligned} \Gamma_{ab}\Gamma_{ab} = I, \quad \Gamma_{ba} = -\Gamma_{ab}, \\ \Gamma_{ab}\Gamma_{ac} = i\Gamma_{bc} \quad a \ne b \ne c \ne a, \\ \Gamma_{ab}\Gamma_{cd}\Gamma_{ef} = I \quad (abcdef) \text{ an even permutation of } (012345). \end{aligned} \tag{3.1}$$

See Appendix B for a correspondence between the Dirac gamma matrices and elements of the Γ_{ab}.

These multiplication rules can be concisely expressed by

$$\Gamma_{ab}\Gamma_{cd} = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})I + \tfrac{1}{2}E_{abcdef}\Gamma_{ef}$$
$$+ i(\delta_{ac}\Gamma_{bd} + \delta_{bd}\Gamma_{ac} - \delta_{ad}\Gamma_{bc} - \delta_{bc}\Gamma_{ad}).$$

I and the Γ_{ab} form a complex vector space over $\mathbb{C}$ for Ω, and every $\omega \in \Omega$ has a unique representation

$$\omega = \alpha I + \tfrac{1}{2}\gamma_{ab}\Gamma_{ab}, \quad \alpha, \gamma_{ab} \in \mathbb{C}, \quad \gamma_{ba} = -\gamma_{ab}.$$

The algebra Ω is equipped with an involutary anti-isomorphism $\dagger : \Omega \to \Omega$ with the following properties

$$\begin{aligned} (\omega^\dagger)^\dagger = \omega, \quad (\omega_1 + \omega_2)^\dagger = \omega_1^\dagger + \omega_2^\dagger, \quad (\omega_1\omega_2)^\dagger = \omega_2^\dagger\omega_1^\dagger \\ (c\omega)^\dagger = c^*\omega^\dagger \quad c \in \mathbb{C}, \ c^* \text{ the complex conjugate.} \end{aligned} \tag{3.2}$$

To express the $\dagger$ mapping in terms of the Γ_{ab} we first define the complex conjugate of $\omega = \alpha I + (1/2)\gamma_{ab}\Gamma_{ab}$ as $\omega^* = \alpha^* I + (1/2)\gamma_{ab}^*\Gamma_{ab}$ and set

$$\omega^\dagger = \Gamma_{04}\omega^*\Gamma_{04}. \tag{3.3}$$

With this definition, the properties (3.2) readily follow from (3.1). We call $\omega^\dagger$ the adjoint of $\omega \in \Omega$, and ω is said to be self-adjoint if $\omega^\dagger = \omega$, and skew-adjoint if $\omega^\dagger = -\omega$. The self-adjoint elements of Ω form a 16-dimensional real vector space spanned by I and the elements

$$j_a j_b \Gamma_{ab} \quad 0 \le a < b \le 5$$

(no summation for the symbols j_a), where

$$j_0 = 1, \quad j_1 = j_2 = j_3 = i, \quad j_4 = 1, \quad j_5 = i.$$

The skew-adjoint elements

$$\frac{i}{2} j_a j_b \Gamma_{ab}, \quad 0 \le a < b \le 5 \tag{3.4}$$

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form the basis of a 15-dimensional Lie algebra $\mathbf{k}$ under the usual Lie product, $[A, B] = AB - BA$, isomorphic to the Lie algebra of a six-dimensional proper pseudo-orthogonal group with signature $(+, -, -, -, +, -)$. The Lie group $\mathcal{K}$ generated by $\mathbf{k}$ is the covering group of this pseudo-orthogonal group; indeed $\exp(2\pi(i/2)\Gamma_{mn})$ for $1 \leq m < n \leq 4$ and $\exp(2\pi(i/2)\Gamma_{05})$ are both $-I$, not I.

Since the elements of $\mathbf{k}$ are skew-adjoint it follows that $\sigma^{\dagger} = \sigma^{-1}$, for all $\sigma \in \mathcal{K}$. Extending the basis (3.4) by iI we obtain a 16-dimensional Lie algebra $\mathbf{k}_0$ and a Lie group $\mathcal{K}_0$ which will ultimately serve as the group of spinor connection of the geometry.

The elements

$$\frac{i}{2} j_m j_n \Gamma_{mn}, \quad 1 \leq m \leq n \leq 4 \tag{3.5}$$

span a six-dimensional Lie subalgebra $\mathbf{l}$ of $\mathbf{k}$, generating the covering group $\hat{\mathcal{L}}$ of the proper Lorentz group $\mathcal{L}$.

Under conjugation by elements of $\hat{\mathcal{L}}$, Ω as a complex vector space is the direct sum of five invariant subspaces

$$\Omega_I, \Omega_{\hat{I}}, \Omega_{\mathcal{V}}, \Omega_{\hat{\mathcal{V}}}, \Omega_{\mathcal{W}}$$

of complex dimensions one, one, four, four, six respectively, and spanned by the following basis elements of Ω:

$$\begin{gathered} \Omega_I : \{I\}, \quad \Omega_{\hat{I}} : \{\Gamma_{05}\}, \\ \Omega_{\mathcal{V}} : \{\Gamma_{n5}\}, \ 1 \leq n \leq 4, \quad \Omega_{\hat{\mathcal{V}}} : \{\Gamma_{0n}\}, \ 1 \leq n \leq 4, \\ \Omega_{\mathcal{W}} : \{\Gamma_{mn}\}, \ 1 \leq m < n \leq 4. \end{gathered}$$

This is easily seen by conjugating with the infinitesimal generators of $\hat{\mathcal{L}}$, using the commutator rules:

$$[\Omega_{\mathcal{W}}, \Omega_{\mathcal{W}}] = \Omega_{\mathcal{W}}, \quad [\Omega_{\mathcal{V}}, \Omega_{\mathcal{W}}] = \Omega_{\mathcal{V}}, \quad [\Omega_{\hat{\mathcal{V}}}, \Omega_{\mathcal{W}}] = \Omega_{\hat{\mathcal{V}}}, \quad [\Omega_{\hat{I}}, \Omega_{\mathcal{W}}] = 0.$$

The rest of the commutator rules (easily checked) are:

$$\begin{gathered} [\Omega_{\mathcal{V}}, \Omega_{\mathcal{V}}] = \Omega_{\mathcal{W}}, \quad [\Omega_{\hat{\mathcal{V}}}, \Omega_{\hat{\mathcal{V}}}] = \Omega_{\mathcal{W}}, \quad [\Omega_{\mathcal{V}} \Omega_{\hat{\mathcal{V}}}] = \Omega_{\hat{I}}, \\ [\Omega_{\hat{I}}, \Omega_{\mathcal{V}}] = \Omega_{\hat{\mathcal{V}}}, \quad [\Omega_{\hat{I}}, \Omega_{\hat{\mathcal{V}}}] = \Omega_{\mathcal{V}}. \end{gathered}$$

In particular, $\Omega_0 = \Omega_I \oplus \Omega_{\hat{I}} \oplus \Omega_{\mathcal{W}}$ is an eight-dimensional (associative) subalgebra over $\mathbb{C}$ with centre $\Omega_I \oplus \Omega_{\hat{I}}$, and the Lie group $\hat{\mathcal{L}}$ lies in Ω_0. More specifically, it lies in the eight-dimensional subalgebra over $\mathbb{R}$ spanned by the elements $I, i\Gamma_{05}, ij_m j_n \Gamma_{mn}, 1 \leq m \leq 4$.

The algebra Ω admits $\hat{\mathcal{L}}$ as a group of operators, the action of $\sigma \in \hat{\mathcal{L}}$ on $\omega \in \Omega$ being the ring product $\sigma\omega\sigma^{-1}$, and

$$\Omega = \Omega_I \oplus \Omega_{\hat{I}} \oplus \Omega_{\mathcal{V}} \oplus \Omega_{\hat{\mathcal{V}}} \oplus \Omega_{\mathcal{W}} \tag{3.6}$$

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is a decomposition into irreducibles under this action. To express the action in coordinatized form, let us define the $\mathbb{R}$-linear mappings

$$\phi_{\mathcal{V}} : \mathcal{V} \to \Omega_{\mathcal{V}}, \quad \phi_{\hat{\mathcal{V}}} : \hat{\mathcal{V}} \to \Omega_{\hat{\mathcal{V}}}, \quad \phi_{\mathcal{W}} : \mathcal{W} \to \Omega_{\mathcal{W}},$$

by

$$\phi_{\mathcal{V}} u = u_n \Gamma_{n5}, \quad \phi_{\hat{\mathcal{V}}} v = v_n \Gamma_{0n}, \quad \phi_{\mathcal{W}} w = \tfrac{1}{2} w_{mn} \Gamma_{mn},$$

where $u \in \mathcal{V}$, $v \in \hat{\mathcal{V}}$, $w \in \mathcal{W}$. Then for $\sigma \in \hat{\mathcal{L}}$

$$\begin{gathered} \sigma(\phi_{\mathcal{V}} u)\sigma^{-1} = \phi_{\mathcal{V}}(\bar{\sigma} u), \quad \sigma(\phi_{\hat{\mathcal{V}}} v)\sigma^{-1} = \phi_{\hat{\mathcal{V}}}(v\bar{\sigma}^{-1}), \\ \sigma(\phi_{\mathcal{W}} w)\sigma^{-1} = \phi_{\mathcal{W}}(\bar{\sigma} w \bar{\sigma}^{-1}), \end{gathered} \tag{3.7}$$

where on the left we have ring multiplication in Ω, and on the right $\bar{\sigma}$ denotes the map of σ in $\mathcal{L}$. Indeed if

$$\sigma = \exp\left(\frac{i}{4} P_{mn} \Gamma_{mn}\right), \qquad j_m j_n P_{mn} \text{ real},$$

and if $M = \exp(P)$, $M = (M_{mn})$, $P = (P_{mn})$, then M is orthogonal with $j_m j_n M_{mn}$ real, and we have to verify

$$\begin{aligned} &\sigma\big(v_n \Gamma_{0n} + u_n \Gamma_{n5} + \tfrac{1}{2} w_{mn} \Gamma_{mn}\big)\sigma^{-1} \\ &\qquad = (M_{np} v_p)\Gamma_{0n} + (M_{np} u_p)\Gamma_{n5} + \tfrac{1}{2}(M_{mp} M_{nq} w_{pq})\Gamma_{mn}. \end{aligned} \tag{3.8}$$

It is sufficient to check (3.8) for infinitesimal operators $\sigma = I + (i/4)\epsilon P_{mn} \Gamma_{mn}$ using (3.1).

Ω as a complex vector space also admits $\hat{\mathcal{L}}$, and in fact the whole of Ω, as a left operator. Denote by Ω^l this 16-dimensional (left) representation space, the action $\omega \in \Omega$ on $\psi \in \Omega^l$ being the ring product $\omega\psi$. Similarly, define Ω^r, the complex vector space Ω with Ω as a right operator domain. The action of $\omega \in \Omega$ upon $\psi^\dagger \in \Omega^r$ is the ring product $\psi^\dagger \omega^\dagger$ where $\omega^\dagger$ is the adjoint of ω.

The dagger mapping is now extended to

$$\dagger : \Omega^l \to \Omega^r \quad \text{and} \quad \dagger : \Omega^r \to \Omega^l$$

according to the definition

$$\psi^\dagger = \psi^* \, \Gamma_{04},$$

where ψ^* is the Hermitian conjugate of ψ. Note that for $\omega \in \Omega$, $\psi \in \Omega^l$ we have

$$(\omega\psi)^\dagger = \psi^\dagger \omega^\dagger \quad \text{but} \quad (\psi\omega)^\dagger = \omega^* \psi^\dagger.$$

Next we define an inner product $\langle \psi_1 \psi_2 \rangle$ for $\psi_1, \psi_2 \in \Omega^l$. Denote by $qs(\omega)$ the I-component of ω in the decomposition (3.6), that is

$$qs\big(\alpha I + \tfrac{1}{2}\gamma_{ab}\Gamma_{ab}\big) = \alpha.$$

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The notation comes from the fact that the Γ_{ab} have a trace-free standard matrix representation (see Appendix A). Now define the inner product

$$\langle\psi_1\psi_2\rangle = qs(\psi_1^\dagger\psi_2). \tag{3.9}$$

This inner product has the obvious properties

$$\langle(\psi_1+\psi_2)\psi\rangle = \langle\psi_1\psi\rangle + \langle\psi_2\psi\rangle, \quad \langle\psi(\psi_1+\psi_2)\rangle = \langle\psi\psi_1\rangle + \langle\psi\psi_2\rangle,$$
$$\langle\psi_1(c\psi_2)\rangle = c\langle\psi_1\psi_2\rangle = \langle(c^*\psi_1)\psi_2\rangle,$$
$$\langle\psi_2\psi_1\rangle = \langle\psi_1\psi_2\rangle^*, \quad \langle\psi_1(\omega\psi_2)\rangle = \langle(\omega^\dagger\psi_1)\psi_2\rangle,$$

for any ψ, ψ_1, $\psi_2 \in \Omega^l$, $\omega \in \Omega$. The last expression will be denoted $\langle\psi_1\omega\psi_2\rangle$. Note that $\langle\psi\omega\psi\rangle$ is real if ω is self-adjoint, and imaginary if ω is anti-adjoint. Also note that if $\sigma \in \mathcal{K}$ then

$$\langle(\sigma\psi_1)(\sigma\psi_2)\rangle = \langle\psi_1\sigma^{-1}\sigma\psi_2\rangle = \langle\psi_1\psi_2\rangle,$$

that is, the inner product is invariant under the action of $\mathcal{K}$.

In (3.7) we have established, through the ϕ-mappings, an association between vectors of $\mathcal{V}$, $\hat{\mathcal{V}}$, or $\mathcal{W}$ and elements of Ω. The mappings were expressed in terms of the components of vectors, therefore the resulting elements of Ω depended on the orthonormal basis in $\mathcal{V}$. If this basis is changed by the transformation $\sigma \in \mathcal{L}$, the elements $\omega \in \Omega$ associated with u, v, w by means of (3.7) transform into $\sigma\omega\sigma^{-1}$. Consequently $\psi \in \Omega^l$ must transform into $\sigma\psi$, and $\psi^\dagger \in \Omega^r$ into $\psi^\dagger\sigma^\dagger = \psi^\dagger\sigma^{-1}$. This association of the anholonomic Vierbein frame transformations with "spinor frame transformations" $\psi \to \sigma\psi$ achieves that inner products $\langle\psi_1\omega\psi_2\rangle$ are independent of the basis of the anholonomic coordinates.

We are now in the position to define spinor tetrads. First note that Ω^l decomposes into the 4 minimal left ideals,

$$\Omega^l = \Omega^{(++)} \oplus \Omega^{(+-)} \oplus \Omega^{(-+)} \oplus \Omega^{(--)}, \tag{3.10}$$

where for $\eta = \pm$, $\zeta = \pm$, each $\Omega^{(\eta\zeta)}$ is spanned by

$$Y_1^{(\eta\zeta)} = \tfrac{1}{4}(\Gamma_{01} + i\eta\Gamma_{15} + i\zeta\Gamma_{02} - \eta\zeta\Gamma_{25}),$$
$$Y_2^{(\eta\zeta)} = \tfrac{1}{4}(\Gamma_{03} + i\eta\Gamma_{35} - \zeta\Gamma_{45} + i\eta\zeta\Gamma_{04}),$$
$$Y_3^{(\eta\zeta)} = \tfrac{1}{4}(\Gamma_{23} + \eta\Gamma_{14} + i\zeta\Gamma_{31} + i\eta\zeta\Gamma_{24}),$$
$$Y_4^{(\eta\zeta)} = \tfrac{1}{4}(\Gamma_{12} + \eta\Gamma_{34} - \zeta I - \eta\zeta\Gamma_{05}).$$

A multiplication table for the $\Gamma_{ab}Y_m$ is given in Appendix C. It can be easily verified that the ideals $\Omega^{(\eta\zeta)}$ are mutually orthogonal under the inner product (3.9).

A Dirac spinor $\psi^{(\eta\zeta)}$ is an element of one of these ideals, and for fixed η and ζ

$$\psi^{(\eta\zeta)} = u_n^{(\eta\zeta)} Y_n^{(\eta\zeta)} \quad \text{(summing for n)}$$

where $u_n^{(\eta\zeta)}$ is a complex function on $\mathcal{M}$.

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Dirac spinors are complex 4-vectors and with this basis have the representation:

$$\psi^{++} = \begin{pmatrix} 0 & u_1 & -u_1 & 0 \\ 0 & u_2 & -u_2 & 0 \\ 0 & u_3 & -u_3 & 0 \\ 0 & u_4 & -u_4 & 0 \end{pmatrix}, \quad \psi^{+-} = \begin{pmatrix} 0 & u_1 & u_1 & 0 \\ 0 & u_2 & u_2 & 0 \\ 0 & u_3 & u_3 & 0 \\ 0 & u_4 & u_4 & 0 \end{pmatrix},$$

$$\psi^{-+} = \begin{pmatrix} u_1 & 0 & 0 & u_1 \\ u_2 & 0 & 0 & u_2 \\ u_3 & 0 & 0 & u_3 \\ u_4 & 0 & 0 & u_4 \end{pmatrix}, \quad \psi^{--} = \begin{pmatrix} u_1 & 0 & 0 & -u_1 \\ u_2 & 0 & 0 & -u_2 \\ u_3 & 0 & 0 & -u_3 \\ u_4 & 0 & 0 & -u_4 \end{pmatrix},$$

with each $u_n = u_n^{(\eta\zeta)} \in \mathbb{C}$.

A spinor tetrad ψ is simply an element of Ω^l. It is uniquely written as the sum of the four Dirac spinors

$$\psi = \sum_{\eta\zeta} \psi^{(\eta\zeta)} = \sum_{\eta\zeta} u_n^{(\eta\zeta)} Y_n^{(\eta\zeta)}. \tag{3.11}$$

If ψ is given by (3.11), and

$$\phi = \sum_{\eta\zeta} v_n^{(\eta\zeta)} Y_n^{(\eta\zeta)} \quad \text{each } v_n = v_n^{(\eta\zeta)} \in \mathbb{C},$$

then

$$\langle \psi\phi \rangle = \frac{i}{4} \sum_{\eta\zeta} \eta (u_1^* v_3 - u_3^* v_1 + u_2^* v_4 - u_4^* v_2).$$

In particular,

$$|\psi|^2 \equiv \langle \psi \Gamma_{04} \psi \rangle = \frac{1}{4} \sum_{\eta\zeta} \sum_{n} |u_n^{(\eta\zeta)}|^2. \tag{3.12}$$

A table of all relevant inner products is given in Appendix D.

The decomposition (3.10) into minimal left ideals is of course not unique, not even if orthogonality of the members is required. A right transformation $\psi \to \psi\tau$ by some invertible element $\tau \in \Omega$ carries the decomposition (3.10) into a new one provided that $\langle (\psi_1 \tau)(\psi_2 \tau) \rangle = \langle \psi_1 \psi_2 \rangle$, for all $\psi_1, \psi_2 \in \Omega^l$. This will be so if τ is *unitary*, that is, if $\tau\tau^{(*)} = 1$.

The transformations

$$\tau = e^{i\lambda\Gamma_{05}} \quad \lambda \in \mathbb{R} \tag{3.13}$$

form a subgroup $\mathcal{H}$ of unitary transformations, and have the effect of multiplying ψ by a phase factor $e^{i\eta\lambda}$.

Finally, we note for later reference that

$$\psi^{(\eta\zeta)} \Gamma_{05} = \eta \psi^{(\eta\zeta)}, \quad \psi^{(\eta\zeta)} \Gamma_{12} = -\zeta \psi^{(\eta\zeta)}, \quad \psi^{(\eta\zeta)} \Gamma_{34} = -\eta\zeta \psi^{(\eta\zeta)}, \tag{3.14}$$

hence the ideals of the decomposition admit multiplication from the right by Γ_{05}, Γ_{12} and Γ_{34}.

https://doi.org/10.1017/S1446181109000108 Published online by Cambridge University Press

TABLE 1. Possible charge and spin properties for various types of spinor field tetrad combinations.

Ideals	Charge	Spin	Form of the tetrad
4	Neutral	Spinless	$\psi = \psi^{++} + \psi^{+-} + \psi^{-+} + \psi^{--}$
2	Positive	Spinless	$\psi = \psi^{++} + \psi^{+-}$
	Negative	Spinless	$\psi = \psi^{-+} + \psi^{--}$
2	Neutral	Spinless	$\psi = \psi^{++} + \psi^{--}$
	Neutral	Spinless	$\psi = \psi^{-+} + \psi^{+-}$
2	Neutral	Up	$\psi = \psi^{++} + \psi^{-+}$
	Neutral	Down	$\psi = \psi^{+-} + \psi^{--}$
1	Positive	Up	$\psi = \psi^{++}$
	Negative	Up	$\psi = \psi^{-+}$
1	Positive	Down	$\psi = \psi^{+-}$
	Negative	Down	$\psi = \psi^{--}$

Considering now just the action of Γ_{05} (whose connection field we will later associate with the electromagnetic field), the fact that two of the ideals receive a phase factor of the opposite sign means that both neutral and oppositely charged pair solutions can arise from a single geometry. For example, a 2-ideal solution can be used to give the oppositely charged pair $\psi = \psi^{++} + \psi^{+-}$ ($\psi\Gamma_{05} = \psi$) and $\psi = \psi^{-+} + \psi^{--}$ ($\psi\Gamma_{05} = -\psi$).

We note also that spin-up/spin-down pairs can be constructed from 1-ideal or 2-ideal combinations. In an appropriate coordinate system, the z-component of the spin angular momentum operator $J_z = -i\hbar(\partial/\partial\phi) + (\hbar/2)\Gamma_{12}$. Provided the coefficients u_n have the appropriate $e^{i\phi}$ factor, and using Appendix C, we can achieve $J_z\psi = (\hbar/2)\psi$ (spin-up) from $\psi = \psi^{++} + \psi^{-+}$, and $J_z\psi = -(\hbar/2)\psi$ (spin-down) from $\psi = \psi^{+-} + \psi^{--}$. Similarly for the 1-ideal tetrads $\psi = \psi^{\eta+}$ and $\psi = \psi^{\eta-}$ for fixed η.

The possible charge and spin properties for various types of spinor field tetrad combinations are listed in Table 1. Note that either a 4-ideal or 2-ideal tetrad could be used for a neutral, spinless solution.

4. Spinor connection

Under parallel displacement of a spinor tetrad, both a left and right connection group are admitted. For the left connection group we take not just the covering group $\hat{\mathcal{L}}$ of the Lorentz group, but the full conformal group which is Ω itself. We therefore admit the infinitesimal transformations

$$\begin{gathered} \frac{i}{4} j_m j_n \Gamma_{mn} \quad 1 \leq m < n \leq 4, \qquad \frac{i}{2} j_n \Gamma_{05} \quad 1 \leq n \leq 4, \\ \frac{1}{2} j_n \Gamma_{n5} \quad 1 \leq n \leq 4, \qquad \frac{1}{2}\Gamma_{05}, \end{gathered} \tag{4.1}$$

in the virtual displacement of the spinor field.

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It is important to note that the only transformations of the connection group Ω that are assumed to have real counterparts are those of the reorientation group $\hat{\mathcal{L}}$, due to its linkage with the Vierbein frame. (See an earlier paper by Szekeres [12] for a description of the problems that one otherwise encounters.) In particular, we do not admit transformations of the left connection group that would lead to nonmetricity of the generated 4-vector covariant derivative. See Leuhr *et al.* [5] for a classification of general spinor connections and their resulting 4-vector connections, indicating that transformations (4.1) may still not be the most general, even allowing for the metricity constraint.

For the right connection group we take the one-parameter Lie group of unitary transformations $\mathcal{H}$ with Lie algebra generator $i\Gamma_{05}$ defined in (3.13). We will see that this introduces a four-potential that can be identified with the electromagnetic potential.

From (3.14), the right connection group could be extended to include the generators $i\Gamma_{12}$ and $i\Gamma_{34}$, in fact the full eight-parameter group SU(3), which could then possibly represent the potentials of the “strong” and “weak” forces. For simplicity, this extension is not considered here. See [6] for an example of the use of this extension to represent the strong charge.

The simplification of the field equations that results from ignoring these extensions will not however affect the two simple solutions presented in Sections 6 and 7.

Accordingly, the covariant derivative of a spinor field tetrad is given by

$$\begin{aligned}\psi_{/\mu} = \psi_{,\mu} - \frac{i}{4}S_{mn\mu}\Gamma_{mn}\psi - \frac{i}{2}B_{n\mu}\Gamma_{0n}\psi - \frac{1}{2}C_{n\mu}\Gamma_{n5}\psi \\ - \frac{1}{2}H_{\mu}\Gamma_{05}\psi - \frac{i}{2}K_{\mu}\psi\Gamma_{05},\end{aligned} \tag{4.2}$$

where for fixed m and n, $j_m j_n S_{mn\mu}$, $j_n B_{n\mu}$, $j_n C_{n\mu}$, H_μ, K_μ are real, $S_{nm\mu} = -S_{mn\mu}$ and $\psi_{,\mu}$ stands for $\partial_\mu\psi = \partial\psi/\partial x^\mu$.

For the adjoint tetrad we have

$$\begin{aligned}\psi^{\dagger}_{/\mu} = \psi^{\dagger}_{,\mu} + \frac{i}{4}S_{mn\mu}\psi^{\dagger}\Gamma_{mn} + \frac{i}{2}B_{n\mu}\psi^{\dagger}\Gamma_{0n} \\ + \frac{1}{2}C_{n\mu}\psi^{\dagger}\Gamma_{n5} + \frac{1}{2}H_{\mu}\psi^{\dagger}\Gamma_{05} + \frac{i}{2}K_{\mu}\Gamma_{05}\psi^{\dagger}.\end{aligned}$$

Spin curvature tensors are obtained from

$$\begin{aligned}\psi_{/\mu/\nu} - \psi_{/\nu/\mu} = -\frac{i}{4}R_{mn\mu\nu}\Gamma_{mn}\psi - \frac{i}{2}R_{0m\mu\nu}\Gamma_{0m}\psi - \frac{1}{2}R_{05\mu\nu}\Gamma_{05}\psi \\ - \frac{1}{2}R_{m5\mu\nu}\Gamma_{m5}\psi - \frac{i}{2}P_{\mu\nu}\psi\Gamma_{05},\end{aligned}$$

where

$$\begin{aligned}R_{mn\mu\nu} = S_{mn\mu,\nu} - S_{mn\nu,\mu} + S_{mp\mu}S_{pn\nu} - S_{mp\nu}S_{pn\mu} - B_{m\mu}B_{n\nu} \\ + B_{m\nu}B_{n\mu} + C_{m\mu}C_{n\nu} - C_{m\nu}C_{n\mu}\end{aligned} \tag{4.3}$$

and

$$R_{0m\mu\nu} = B_{m\mu,\nu} - B_{m\nu,\mu} + B_{p\mu}S_{pm\nu} - B_{p\nu}S_{pm\mu} - C_{m\mu}H_\nu + C_{m\nu}H_\mu,$$
$$R_{m5\mu\nu} = C_{m\mu,\nu} - C_{m\nu,\mu} + C_{p\mu}S_{pm\nu} - C_{p\nu}S_{pm\mu} - B_{m\mu}H_\nu + B_{m\nu}H_\mu,$$
$$R_{05\mu\nu} = H_{\mu,\nu} - H_{\nu,\mu} - B_{n\nu}C_{n\mu} + B_{n\mu}C_{n\nu} \quad \text{and}$$
$$P_{\mu\nu} = K_{\mu,\nu} - K_{\nu,\mu}.$$

For fixed m and n, $j_m j_n S_{mn\mu}$, $j_n B_{n\mu}$, $j_n C_{n\mu}$, H_μ, and K_μ are covectors of $\mathcal{M}$. Let us examine their transformation under a change of the anholonomic (Vierbein) base in $\mathcal{V}$ and the associated spin transformation in Ω^l.

A *re-orientation* of the Vierbein and spinor frames is a field of group elements $\sigma : \mathcal{M} \to \hat{\mathcal{L}}$,

$$\sigma(\underline{x}) = e^{\pi(\underline{x})}, \qquad \pi(\underline{x}) \in \mathbf{l}$$

($\mathbf{l}$ defined in (3.5)), representing these changes of spin frames over a region of $\mathcal{M}$. The transformed spinor field tetrad is $\tilde{\psi}(\underline{x}) = \sigma(\underline{x})\psi(\underline{x})$, and for this to remain a spinor field tetrad we must have

$$\tilde{\psi}_{/\mu} = \sigma(\psi_{/\mu}),$$

that is

$$\begin{aligned}(\sigma\psi)_{,\mu} &- \frac{i}{4}\tilde{S}_{mn\mu}\Gamma_{mn}\sigma\psi - \frac{i}{2}\tilde{B}_{n\mu}\Gamma_{0n}\sigma\psi - \frac{1}{2}\tilde{C}_{n\mu}\Gamma_{n5}\sigma\psi \\ &- \frac{1}{2}\tilde{H}_\mu\Gamma_{05}\sigma\psi - \frac{i}{2}\tilde{K}_\mu\sigma\psi\Gamma_{05} \\ &= \sigma\bigg(\psi_{,\mu} - \frac{i}{4}S_{mn\mu}\Gamma_{mn}\psi - \frac{i}{2}B_{n\mu}\Gamma_{0n}\psi - \frac{1}{2}C_{n\mu}\Gamma_{n5}\psi \\ &\quad - \frac{1}{2}H_\mu\Gamma_{05}\psi - \frac{i}{2}K_\mu\psi\Gamma_{05}\bigg)\end{aligned}$$

for arbitrary ψ, where $\tilde{S}_{mn\mu}$, $\tilde{B}_{n\mu}$, $\tilde{C}_{n\mu}$, $\tilde{H}_\mu$, $\tilde{K}_\mu$ are the components after reorientation. This gives immediately $\tilde{H}_\mu = H_\mu$, $\tilde{K}_\mu = K_\mu$, that is, neither are affected by re-orientation. The transformation law for the left connection is well known: if

$$\sigma = \exp\left(\frac{i}{4}P_{mn}\Gamma_{mn}\right), \qquad j_m j_n P_{mn} \text{ real}, \qquad P_{nm} = -P_{mn}$$

and if $P = (P_{mn})$, $M = e^P$, then

$$\tilde{S}_{mn\mu} = M_{mp}M_{nq}S_{pq\mu} + M_{mp,\mu}M_{np}$$

and similarly $B_{n\mu}$ and $C_{n\mu}$ transform as vectors. We also get, using the orthogonality of M

$$\tilde{R}_{mn\mu\nu} = M_{mp}M_{nq}R_{pq\mu\nu}.$$

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This shows that for given μ, ν, $(R_{mn\mu\nu}) \in \mathcal{W}$. It follows that $R_{\mu\nu\rho\sigma} = g_{m\mu} g_{n\nu} R_{mn\rho\sigma}$ is a real tensor, with the symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}.$$

For convenience we call it the curvature tensor although it has no such geometrical role as far as tangent vectors and tensors of the manifold are concerned. Only the metric (Riemannian) curvature tensor has such a role.

Next we consider changes in the connection under a phase transformation field

$$\tau(\underline{x}) = \exp\left\{\frac{i}{2}\lambda(\underline{x})\Gamma_{05}\right\}, \quad \lambda(\underline{x}) \text{ real},$$

acting from the right on the spinor tetrad. We require

$$(\psi\tau)_{,\mu} - \frac{i}{2}\tilde{K}_\mu \psi\tau\Gamma_{05} = \left(\psi_{,\mu} - \frac{i}{2}K_\mu\psi\Gamma_{05}\right)\tau$$

which gives

$$\tilde{K}_\mu \Gamma_{05} = (K_\mu + \lambda_{,\mu})\Gamma_{05}$$

noting that $\Gamma_{05}\tau = \tau\Gamma_{05}$. Hence

$$\tilde{K}_\mu = K_\mu + \lambda_{,\mu}. \tag{4.4}$$

This shows that K_μ is not an ordinary covector field but an equivalence class of such fields under the transformation (4.4) (a gauge transformation). This is precisely the behaviour of an electromagnetic four-potential. Note that $S_{mn\mu}$, $B_{n\mu}$, $C_{n\mu}$ and H_μ are invariant under phase transformations.

A "metric" connection can now be defined for spinor tetrads. Define

$$N_{mn\nu} \equiv g_m^\rho g_{n\mu} \{^{\mu}_{\nu\rho}\} - g_m^\lambda g_{n\lambda,\nu} \tag{4.5}$$

where

$$\{^{\mu}_{\nu\rho}\} = \tfrac{1}{2} g^{\lambda\mu}(g_{\lambda\rho,\nu} + g_{\lambda\nu,\rho} - g_{\rho\nu,\lambda})$$

is the usual Riemann–Christoffel affinity pertaining to $g_{\mu\nu}$.

It is straightforward to check that $N_{mn\mu}$ is a covector for each m and n, and that $N_{nm\nu} = -N_{mn\nu}$. From (4.5) it can also be seen that under reorientation, $N_{mn\nu}$ transforms like a (left) connection. Hence $N_{mn\nu}$ can be regarded as the metric part of $S_{mn\nu}$, although it has no such geometrical role, and separating it off from $S_{mn\nu}$ is just a computational convenience. Indeed, if we define

$$T_{mn\nu} = S_{mn\nu} - N_{mn\nu} \tag{4.6}$$

then $T_{mn\nu}$ (for fixed ν) transforms as a skew tensor in $\mathcal{W}$, with $j_m j_n T_{mn\nu}$ real. Hence $T_{\mu\nu\rho} = g_{m\mu} g_{n\nu} T_{mn\rho}$ is a real contortion tensor with the skew symmetry

$$T_{\nu\mu\rho} = -T_{\mu\nu\rho}. \tag{4.7}$$

We can also define a metric derivative for spinor fields, namely

$$\psi_{;\mu} = \psi_{,\mu} - \frac{i}{4} N_{mn\mu} \Gamma_{mn} \psi. \tag{4.8}$$

Although it is a legitimate spinor field with respect to reorientations of the spinor frame, it does not behave like a spinor field tetrad with respect to (right) phase transformations.

The tensor $T_{\mu\nu\rho}$ can be used to express the relation between $R_{\mu\nu\rho\sigma}$ and the Riemannian curvature tensor

$$G^{\mu}_{\nu\rho\sigma} = -\{^{\mu}_{\nu\rho}\}_{,\sigma} + \{^{\mu}_{\nu\sigma}\}_{,\rho} + \{^{\mu}_{\lambda\rho}\}\{^{\lambda}_{\nu\sigma}\} - \{^{\mu}_{\lambda\sigma}\}\{^{\lambda}_{\nu\rho}\}$$

namely, using (4.3),

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= G_{\mu\nu\rho\sigma} + T_{\mu\nu\rho;\sigma} - T_{\mu\nu\sigma;\rho} + T_{\mu\lambda\rho} T^{\lambda}_{\nu\sigma} - T_{\mu\lambda\sigma} T^{\lambda}_{\nu\rho} - B_{\mu\rho} B_{\nu\sigma} \\ &\quad + B_{\mu\sigma} B_{\nu\rho} + C_{\mu\rho} C_{\nu\sigma} - C_{\mu\sigma} C_{\nu\rho}, \end{aligned} \tag{4.9}$$

where semicolon denotes covariant differentiation with respect to the Christoffel affinity. That is,

$$A^{\alpha}_{\beta;\gamma} = A^{\alpha}_{\beta,\gamma} + \{^{\alpha}_{\sigma\gamma}\} A^{\sigma}_{\beta} - \{^{\sigma}_{\beta\gamma}\} A^{\alpha}_{\sigma}$$

is the covariant derivative of a tensor field $A^{\alpha}_{\beta\gamma}$. The Ricci tensor is given by

$$G_{\mu\nu} = -\{^{\lambda}_{\mu\nu}\}_{,\lambda} + \{^{\lambda}_{\mu\lambda}\}_{,\nu} + \{^{\rho}_{\lambda\nu}\}\{^{\lambda}_{\mu\rho}\} - \{^{\lambda}_{\rho\lambda}\}\{^{\rho}_{\mu\nu}\}$$

with the gravitational scalar

$$G = g^{\mu\nu} G_{\mu\nu}.$$

Finally we define the Dirac (scalar) differential of $\psi \in \Omega^{l}$

$$D\psi = \Gamma^{\mu} \psi_{/\mu} \quad \text{where } \Gamma^{\mu} = g^{\mu}_{n} \Gamma_{0n}. \tag{4.10}$$

It is clearly a spinor tetrad, and will play a crucial role in the field equations in the following section.

To facilitate the transition from a purely geometrical to a physical description of the field equations, it is useful to introduce the notion of a *metric index*. We note that for a given coordinate system, the metric tensor $g_{\mu\nu}$ is only determined up to a scaling factor, depending on the "unit of length" chosen for the manifold. Suppose we change the unit of length by a factor of λ, without changing the coordinate system. Since $g_{\mu\nu}\, dx^{\mu} dx^{\nu}$ is the square of a length, we must multiply $g_{\mu\nu}$ by λ^{-2} so that the same length is expressed in the new, rescaled unit. We express this by saying that $g_{\mu\nu}$ has a metric index of 2. Similarly, the metric indexes of $g_{n\mu}$ and g^{μ}_{n} are 1 and -1 respectively. Hence note from (4.10) that the metric index of D is also -1. Since any equation expressing a law of nature must be valid independently of the unit of length chosen, we shall provide the appropriate factor to ensure that all terms appearing in such an equation have the same metric index.

https://doi.org/10.1017/S1446181109000108 Published online by Cambridge University Press

5. Field equations

The field equations of the geometry are obtained from a Hamiltonian variation principle, applied to suitable action integrals and linked together in a single world Lagrangian. We assume that the 4-manifold $\mathcal{M}$ is equipped with a metric field, a connection field and a spinor field tetrad. The field quantities to be varied independently are the components of the metric and connection fields, namely, $g_{n\mu}$, $S_{mn\mu}$, $B_{n\mu}$, $C_{n\mu}$, K_μ and H_μ. The spinor field tetrad enters the variation principle only in the form of a homogeneous constraint, expressing the condition that the Dirac differential of an admissible ψ should satisfy the generalized Dirac equation

$$D\psi + \Lambda^{-1}\Gamma_{05}\psi = 0. \tag{5.1}$$

Here Λ is a positive constant, denoting the unit length chosen for the manifold, and required as the metric index of D is -1.

To incorporate this constraint into the variation principle, we construct the real scalar density

$$\underline{\underline{L}} = \frac{i}{2}\,\underline{\underline{g}}\big\langle (D\psi + \Lambda^{-1}\Gamma_{05}\psi)\psi - \psi(D\psi + \Lambda^{-1}\Gamma_{05}\psi)\big\rangle. \tag{5.2}$$

Using (4.2), this expands to

$$\begin{aligned}\underline{\underline{L}} = {} & \frac{i}{2}\,\underline{\underline{g}}\,\langle \psi_{,\mu}\Gamma^\mu\psi - \psi\Gamma^\mu\psi_{,\mu}\rangle - \frac{1}{2}\,\underline{\underline{g}}\,K_\mu\langle\psi\Gamma^\mu\psi\Gamma_{05}\rangle \\ & - \frac{1}{8}S_{mn\mu}\,\underline{\underline{g}}\,\langle\psi(\Gamma_{mn}\Gamma^\mu + \Gamma^\mu\Gamma_{mn})\psi\rangle - \frac{1}{4}B_{n\mu}\,\underline{\underline{g}}\,\langle\psi(\Gamma_{0n}\Gamma^\mu + \Gamma^\mu\Gamma_{0n})\psi\rangle \\ & - \frac{i}{4}C_{n\mu}\,\underline{\underline{g}}\,\langle\psi(\Gamma_{n5}\Gamma^\mu + \Gamma^\mu\Gamma_{n5})\psi\rangle - i\Lambda^{-1}\,\underline{\underline{g}}\,\langle\psi\Gamma_{05}\psi\rangle.\end{aligned} \tag{5.3}$$

(Note that there is no H_μ term.) It is not difficult to check that variation of (5.2) with respect to ψ gives (5.1), provided that the symmetry conditions,

$$\begin{gathered} S_{mn\mu}(\Gamma^\mu\Gamma_{mn} - \Gamma_{mn}\Gamma^\mu) = 0, \quad B_{m\mu}(\Gamma^\mu\Gamma_{0m} - \Gamma_{0m}\Gamma^\mu) = 0, \\ C_{m\mu}(\Gamma^\mu\Gamma_{m5} - \Gamma_{m5}\Gamma^\mu) = 0, \end{gathered} \tag{5.4}$$

are satisfied. Conversely, if (5.1) is satisfied, then clearly $\underline{\underline{L}} = 0$. To obtain the field equations for the connection fields, we first vary $S_{mn\mu}$ freely in

$$\int\left(\underline{\underline{R}} + \frac{\Lambda^2}{T}\,\underline{\underline{L}}\right)d^4\underline{x} \tag{5.5}$$

where

$$R = -R^{\mu\nu}_{\mu\nu} = -g^\mu_m g^\nu_n R_{mn\mu\nu}$$

is the curvature scalar formed from the curvature tensor (4.3) and $\underline{\underline{R}} \equiv \underline{\underline{g}}\,R$ is the corresponding density. The Lagrangian factor $(1/T)\Lambda^2$ is not necessarily a constant, but may be a function of the cosmological epoch, and we take this into account in

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Section 6. The term Λ^2 is introduced to make T dimensionless. Although T is independent of the unit of length, its value depends on the normalization of the ψ fields which at this stage is wholly arbitrary. As $\underline{\underline{L}}$ is homogeneous of degree two in ψ, it is in fact $(1/\sqrt{T})\psi$ which is able to be determined from the variational principle.

Using (5.3), the relevant terms in (5.5) for the variation of $S_{mn\mu}$ are

$$-g^{\mu}_{m} g^{\nu}_{n} (S_{mn\mu,\nu} - S_{mn\nu,\mu} + S_{mp\mu} S_{pn\nu} - S_{mp\nu} S_{pn\mu})$$
$$- \frac{\Lambda^2}{8T} S_{mn\mu} \langle \psi (\Gamma_{mn} \Gamma^{\mu} + \Gamma^{\mu} \Gamma_{mn}) \psi \rangle.$$

Variation of $S_{mn\lambda}$ gives, when converted to real tensors using (4.6),

$$g_{\lambda\mu} T^{\rho}_{\nu\rho} - g_{\lambda\nu} T^{\rho}_{\mu\rho} + T_{\lambda\mu\nu} - T_{\lambda\nu\mu} = \frac{\Lambda^2}{4T} E_{\lambda\mu\nu\sigma} \langle \psi \Gamma_{05} \Gamma^{\sigma} \psi \rangle, \tag{5.6}$$

where

$$E_{\lambda\mu\nu\sigma} = i\, g_{k\lambda}\, g_{m\mu}\, g_{n\nu}\, g_{p\sigma}\, E_{0kmnp5} = \underline{\underline{g}}\ \mathrm{sig}(\lambda\mu\nu\sigma)$$

is the real alternating tensor. Contraction of (5.6) with respect to λ and μ gives

$$T^{\rho}_{\nu\rho} = 0$$

which means that the first of the symmetry conditions (5.4) is satisfied. Equation (5.6) simplifies to

$$T_{\lambda\mu\nu} - T_{\lambda\nu\mu} = \frac{\Lambda^2}{4T} E_{\lambda\mu\nu\sigma} \langle \psi \Gamma_{05} \Gamma^{\sigma} \psi \rangle$$

and finally, from the symmetry (4.7),

$$T_{\lambda\mu\nu} = \frac{\Lambda^2}{8T} E_{\lambda\mu\nu\sigma} \langle \psi \Gamma_{05} \Gamma^{\sigma} \psi \rangle, \tag{5.7}$$

that is, the contortion tensor is fully skew-symmetric.

Note that although the contortion tensor appears in the general form of these field equations, the right-hand side of (5.7) may vanish identically depending on the assumptions made about the form of the metric and the spinor tetrad for a given solution. That type of solution will then be torsion-free. The simple solutions given in Sections 6 and 7 are examples of this.

Variation of $B_{n\mu}$ in (5.5) gives

$$B_{\mu\nu} = \frac{\Lambda^2}{12T} g_{\mu\nu} \langle \psi\psi \rangle,$$

which shows that $B_{\mu\nu}$ is symmetric, satisfies the second condition of (5.4), and has the simple contraction

$$B \equiv \Lambda B^{\rho}_{\rho} = \frac{\Lambda^3}{3T} \langle \psi\psi \rangle,$$

where the additional Λ factor makes B scale-free.

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Variation of $C_{n\mu}$ in (5.5) gives

$$C_{\mu\nu} = i\frac{\Lambda^2}{8T}E_{\mu\nu\lambda\sigma}\langle\psi\Gamma^\lambda\Gamma^\sigma\psi\rangle$$

which shows that $C_{\mu\nu}$ is skew-symmetric, and satisfies the last of the conditions (5.4).

Unlike the contortion tensor, we do not hazard any "physical" interpretation of the $B_{\mu\nu}$ or $C_{\mu\nu}$ tensors.

The generalized Dirac equation (5.1) now takes the form

$$\begin{aligned}
0 = \Gamma^\mu\psi_{;\mu} &- \frac{i}{4}T^{\alpha\beta\gamma}E_{\alpha\beta\gamma\mu}\Gamma_{05}\Gamma^\mu\psi && (\Gamma_{n5}\text{ term})\\
&- \frac{i}{4}C^{\alpha\beta}E_{\alpha\beta\gamma\mu}\Gamma^\gamma\Gamma^\mu\psi && (\Gamma_{mn}\text{ term})\\
&- \frac{i}{2}K_\mu\Gamma^\mu\psi\Gamma_{05} && (\Gamma_{0n}\text{ term})\\
&- \frac{i}{2}\Lambda^{-1}B\psi && (I\text{ term})\\
&+ \Lambda^{-1}\Gamma_{05}\psi && (\Gamma_{05}\text{ term}).
\end{aligned} \tag{5.8}$$

Note that each type of element of Ω appears as a left operator, and this is the motivation for the Γ_{05} operator with the "mass" term $\Lambda^{-1}\psi$ in the Dirac equation (5.1). Also note that H_μ does not enter the equation explicitly at all. Defining the skew-symmetric tensors

$$Q_{\mu\nu} = H_{\mu,\nu} - H_{\nu,\mu}$$

and

$$P_{\mu\nu} = K_{\mu,\nu} - K_{\nu,\mu}$$

the electromagnetic Lagrangian is taken to be

$$\underline{\underline{P}} = -\underline{\underline{g}}\, P_{\mu\nu}Q^{\mu\nu}. \tag{5.9}$$

As before, we vary K_μ and H_μ in

$$\int\left(\underline{\underline{P}} + \frac{1}{b}\underline{\underline{L}}\right)d^4\underline{x} \tag{5.10}$$

where the Lagrangian factor $1/b$ is dimensionless. The value of b again depends on the normalization of ψ, but b/T is independent of spinor normalization and is likely to be a function of the cosmological epoch, as we shall see later.

The variation gives the electromagnetic field equations with an explicit source term, namely,

$$J^\mu \equiv Q^{\mu\nu}_{;\nu} = \frac{1}{4b}\langle\psi\Gamma^\mu\psi\Gamma_{05}\rangle \tag{5.11}$$

and

$$P^{\mu\nu}_{;\nu} = 0. \tag{5.12}$$

The form of the electromagnetic equations suggests that $Q^{\mu\nu}$ be interpreted as a "density" or "material" field, and $P^{\mu\nu}$ as a pure "intensity" field. These electromagnetic equations have a very satisfactory appearance in the sense that they address one of the crucial problems of unified field theories. Consider Maxwell's equations

$$\text{curl}\,\underline{H} - \dot{\underline{D}} = \underline{I}, \quad \text{div}\underline{D} = \rho, \quad \text{curl}\,\underline{E} - \dot{\underline{B}} = \underline{0}, \quad \text{div}\underline{B} = 0,$$

where $\underline{I}$ is the current, ρ is the charge, $\underline{H}$ and $\underline{D}$ are the magnetic and displacement ("material") fields, and $\underline{E}$ and $\underline{B}$ are the electric and magnetic ("intensity") fields.

Instead of the usual identifications

$$\underline{H} = \underline{B}, \quad \underline{D} = \underline{E} \tag{5.13}$$

(apart from proportionality constants), which cannot be justified by geometrical considerations alone (see [10, Page 25]), (5.11) and (5.12) provide separate equations for the material and intensity fields respectively. The Maxwell identifications (5.13) are only possible in a region where the spinor field is negligible, that is remote from the centre of a particle, and correspond to

$$K_\mu = H_\mu. \tag{5.14}$$

This does *not* follow from the geometry itself, and it is only by considering particular solutions to the field equations that the boundary conditions (5.14) could possibly be enforced.

There is also a striking geometric difference between the two connection vectors K_μ and H_μ constituting the electromagnetic potential. The unitary group $\mathcal{H}$ of right connection with Lie generator $(i/2)\Gamma_{05}$ pertaining to K_μ is compact, and its members are indeed phase transformations (see (3.13)). However the one-parameter left connection group with Lie generator $(1/2)\Gamma_{05}$ pertaining to H_μ is noncompact. It is similar in effect to Weyl's original gauge group of "length curvature", that is scale transformations. It is interesting to note how both versions of Weyl's gauge group appear here in a very natural fashion, one representing the electromagnetic intensity, the other material electromagnetic field density.

Finally, to derive the field equations for the metric field, we first note that by contracting the curvature tensor (4.9) we obtain

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G + T_{\alpha\beta\mu} T_\nu^{\alpha\beta} - \tfrac{1}{2} g_{\mu\nu} T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} \\ - \tfrac{3}{4}\Lambda^{-1} B_{\mu\nu} B - C_{\mu\rho} C_\nu^\rho + \tfrac{1}{2} g_{\mu\nu} C^{\rho\sigma} C_{\rho\sigma}. \end{aligned} \tag{5.15}$$

We now vary g_m^ν in

$$\int \left(\underline{\underline{R}} + \frac{\Lambda^2}{T} \left(\underline{\underline{L}} + b\, \underline{\underline{P}} \right) \right) d^4 \underline{x},$$

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where the world Lagrangian is obtained from the Lagrangians (5.5) and (5.10). The variation is particularly simple to perform (in contrast to ordinary relativity) since g_m^ν appears only algebraically in the expression. We get

$$\begin{aligned} G_{\mu\nu} &- \frac{1}{2} g_{\mu\nu} G \\ &= \frac{b\Lambda^2}{T}\left[g^{\rho\sigma}(P_{\mu\rho}Q_{\nu\sigma} + P_{\nu\rho}Q_{\mu\sigma}) - \frac{1}{2} g_{\mu\nu} P_{\rho\sigma} Q^{\rho\sigma}\right] \\ &\quad + \frac{b\Lambda^2}{2T}[K_\mu J_\nu + K_\nu J_\mu] + \frac{3}{8} g_{\mu\nu}\Lambda^{-2}B^2 + \frac{1}{2} g_{\mu\nu} T^{\lambda\rho\sigma}T_{\lambda\rho\sigma} \\ &\quad - \frac{1}{2} g_{\mu\nu} C^{\lambda\rho}C_{\lambda\rho} - \frac{i\Lambda^2}{8T}\langle \psi_{;\mu}\Gamma_\nu\psi + \psi_{;\nu}\Gamma_\mu\psi - \psi\Gamma_\mu\psi_{;\nu} - \psi\Gamma_\nu\psi_{;\mu}\rangle. \end{aligned} \tag{5.16}$$

The right-hand side of this equation is the "energy–momentum" tensor in a perfectly explicit form, comprising of electromagnetic, torsion and matter components. Note that of the ten equations defined by (5.16), only six are independent. The rest follow from the contracted Bianchi identities

$$\left(G_\mu^\nu - \tfrac{1}{2} g_\mu^\nu G\right)_{;\nu} = 0.$$

Taking the inner product of ψ with (5.8), one finds

$$\begin{aligned} i\langle \psi_{;\mu}\Gamma^\mu\psi\rangle &= 2T\Lambda^{-2}T^{\lambda\rho\sigma}T_{\lambda\rho\sigma} + 2bK_\mu J^\mu + \tfrac{3}{2} T\Lambda^{-4}B^2 \\ &\quad - 2T\Lambda^{-2}C^{\lambda\rho}C_{\lambda\rho} + i\Lambda^{-1}\langle\psi\Gamma_{05}\psi\rangle. \end{aligned}$$

Contracting (5.16) and (5.15) one finds the gravitational and curvature scalars

$$G = -T^{\lambda\rho\sigma}T_{\lambda\rho\sigma} - \frac{3}{4}\Lambda^{-2}B^2 + C^{\lambda\rho}C_{\lambda\rho} + \frac{i\Lambda}{2T}\langle\psi\Gamma_{05}\psi\rangle, \quad R = \frac{i\Lambda}{2T}\langle\psi\Gamma_{05}\psi\rangle. \tag{5.17}$$

We now have the complete set of field equations, namely, the generalized Dirac equation (5.8), the electromagnetic equations (5.11) and (5.12), and the gravitational equations (5.16). To get an idea of the actual values of b, T and Λ, compare with Dirac's

$$-i\hbar c\Gamma^\mu\psi_{,\mu} - qA_\mu\Gamma^\mu\psi + Mc^2\psi = 0, \tag{5.18}$$

where $\hbar$ is Planck's constant, c is the speed of light, q is the elementary charge, and M is the mass of a particle represented by this equation. Comparison with (5.8) gives

$$K_\mu = \frac{2q}{\hbar c}A_\mu, \quad P_{\mu\nu} = \frac{2q}{\hbar c}F_{\mu\nu}, \tag{5.19}$$

where A_μ is the electromagnetic four-potential and $F_{\mu\nu}$ the electromagnetic field tensor using Lorentz–Heaviside units. Furthermore

$$\Lambda = \frac{\hbar}{Mc} \approx \frac{0.352\times 10^{-42}}{M} \tag{5.20}$$

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(Λ in m, M in kg), is the Compton wavelength associated with a particle of mass M. For example, if M is of the order of the proton mass (10^{-27} kg), then Λ is of order 10^{-15} m.

Although presumably (5.19) is the correct conversion between our geometric quantities and electromagnetic intensities, the analogy with Dirac's (5.18) is largely formal. For instance, K_μ here denotes the four-potential of the total electromagnetic field, including the particle itself, and not merely the potential of the external field. Similarly, there is no need to regard ψ as a probability amplitude.

Comparison of (5.16) with Einstein's gravitational equations for pure electromagnetic fields, namely

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G = \frac{8\pi K}{c^4}\left(g^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)$$

gives

$$\frac{T}{b} = \frac{\Lambda^2 q^2 c^2}{\pi \hbar^2 K}$$

where $K \approx 6.674 \times 10^{-11}\mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$ is Newton's constant of gravitation. In Lorentz–Heaviside units

$$q^2 = 4\pi\hbar c\alpha,$$

where $\alpha \approx 1/137.036$ is the fine structure constant. Hence

$$\frac{T}{b} = \frac{4\Lambda^2\alpha c^3}{\hbar K} \approx 1.118 \times 10^{68}\Lambda^2. \tag{5.21}$$

If Λ is of order 10^{-15} m for example, then T/b is of order 10^{38}. The age of the universe is about 4×10^{17} s which in absolute units of time Λc^{-1} is of order 10^{41}. This suggests that T/b may be related to the cosmological epoch.

The value of b depends on the normalization of the spinor field. Compare (5.11) with Maxwell–Dirac's

$$F^{\mu\nu}_{,\nu} = q\langle\psi\Gamma^\mu\psi\rangle, \tag{5.22}$$

which holds for an particle with elementary charge q if ψ consists of just one member of the tetrad and a conventional normalization for ψ is used. Provided we can describe the particle field in a stationary coordinate system in which "space" and "time" are separated, and using (3.12), the requirement is that

$$\int \underline{\underline{g}}^{\,*}|\psi|^2\, d^3\underline{x} = \int \underline{\underline{g}}^{\,*}\langle\psi\Gamma_{04}\psi\rangle\, d^3\underline{x} = 1 \tag{5.23}$$

where the integral is extended over a 3-space region outside which the metric field of the particle is flat. Here $\underline{\underline{g}}^{\,*}$ is the 3-space coordinate density. From (5.11), (5.22) and (5.19) we obtain

$$b = \frac{\hbar c}{8q^2} = \frac{1}{32\pi\alpha} \approx 1.363 \tag{5.24}$$

which can be regarded as a defining equation for the (local) conventional normalization of the spinor field tetrad in terms of the fine structure constant α. We shall refer to (5.23) as the *normalization hypothesis* which all charged particles (perhaps also neutral particles) are supposed to satisfy when b is given by (5.24). There is no hope to "prove" the normalization hypothesis as long as only isolated particle solutions are considered; any such proof must come from the study of interactions between particles.

With the normalization of ψ induced by (5.24) we obtain from (5.20) and (5.21)

$$T = \frac{\Lambda^2 c^3}{8\pi \hbar K} = \frac{\Lambda c^2}{8\pi M K}. \tag{5.25}$$

If the unit of length (that is, the metre) is fixed in terms of Λ, the unit of time (the second) fixed so that c should have the value $\approx 3 \times 10^8$ m s^{-1}, and the unit of mass (kg) fixed so that Planck's constant should have the value $\approx 6.626 \times 10^{-34}$ kg m s^{-1}, then (5.25) tells us that the gravitational constant is proportional to $1/T$. This is the precise meaning of the statement that "the gravitational constant is decreasing in time". If T has settled down to a practically constant value in the latter stages of the evolution of the universe, then so did the gravitational constant.

We also note from (5.25) that the gravitational radius ν of a particle with mass μM is given by

$$\nu = \mu M \frac{K}{c^2} = \frac{\mu \Lambda}{8\pi T}. \tag{5.26}$$

We now have a complete correspondence between the geometrical objects and the physical quantities that they represent. Further determinations of the values of these quantities must come from solutions of the field equations, specifically from particle solutions with certain symmetries which could be identified with known particles.

6. A cosmological solution

We consider a solution with a simple cosmological line element of the form

$$\Lambda^2 dt^2 - \Lambda^2 f(t)^2 (dx_1^2 + dx_2^2 + dx_3^2) \tag{6.1}$$

and we seek to determine $f(t)$ in the presence of a global cosmological spinor field. The factor Λ^2 makes the t-coordinate independent of the unit of length chosen, and the unit on the t-axis corresponds to Λ/c seconds. The coordinate on the x-axes can be fixed for example by the condition

$$f(t_0) = 1 \tag{6.2}$$

for the current epoch $t = t_0$. The simplest orientation of the Vierbein metric field $g_{m\beta}$ generating line element (6.1) is

$$g_{m\beta} = i\Lambda f \delta_{m\beta} \quad 1 \le m,\ \beta \le 3, \quad g_{4\tau} = \Lambda \quad \tau = 4.$$

Assuming ψ only depends on t, from (4.8) we have

$$\psi_{;\beta} = -\tfrac{1}{2}\delta_{m\beta} f'(t) \Gamma_{m4}\psi \quad 1 \le \beta \le 3, \quad \psi_{;\tau} = \psi_{,\tau}. \tag{6.3}$$

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The Einstein–Ricci tensor components are

$$-G_{\alpha\beta}+\frac{1}{2}g_{\alpha\beta}G=-\delta_{\alpha\beta}f^2\left(2\frac{f''}{f}+\left(\frac{f'}{f}\right)^2\right)\quad \alpha,\beta=1,2,3,$$
$$-G_{\tau\tau}+\frac{1}{2}g_{\tau\tau}G=3\left(\frac{f'}{f}\right)^2. \tag{6.4}$$

Requiring an electrically neutral ($K_\mu=H_\mu=0$), torsion-free ($T_{\lambda\mu\nu}=0$) field, from (5.7) and (5.11) we have

$$\langle\psi\Gamma_{0m}\psi\Gamma_{05}\rangle=\langle\psi\Gamma_{m5}\psi\rangle=0\quad m=1,2,3,4 \tag{6.5}$$

and the gravitational equations (5.16) become

$$-G_{\mu\nu}+\frac{1}{2}g_{\mu\nu}G=\frac{i\Lambda^2}{8T}\langle\psi_{;\mu}\Gamma_\nu\psi+\psi_{;\nu}\Gamma_\mu\psi-\psi\Gamma_\mu\psi_{;\nu}-\psi\Gamma_\nu\psi_{;\mu}\rangle. \tag{6.6}$$

Allowing for T to be a function of t on the cosmological scale, which we did not allow for in (5.5), the variation of ψ in

$$\int\frac{\Lambda^2}{T}\underline{\underline{L}}\;d^4\underline{x}$$

adds a cosmological term to the generalized Dirac equation (5.1), that is,

$$D\psi+\Lambda^{-1}\Gamma_{05}\psi-\Lambda^{-1}\frac{T'}{2T}\Gamma_{04}\psi=0$$

which gives using (6.3)

$$\Gamma_{04}\psi_{,\tau}+\left(\frac{3}{2}\frac{f'}{f}-\frac{1}{2}\frac{T'}{T}\right)\Gamma_{04}\psi+\Gamma_{05}\psi=0. \tag{6.7}$$

It is convenient here to use a spinor tetrad with components in all four ideals (see Table 1), that is,

$$\psi=\Lambda^{-3/2}T^{1/2}f^{-3/2}\psi_0\quad\text{where }\psi_0=e^{i\epsilon t}\sum_{\eta,\zeta}u_m^{(\eta,\zeta)}Y_m^{(\eta,\zeta)}$$

and where ϵ and the u_m are scale-free constants.

Equations (6.5) and (6.7) are satisfied if

$$u_1=u_2=-u_3=-u_4=U\neq0\quad\forall u_k=u_k^{(\eta,\zeta)},\ \epsilon^2=1,$$

where U is a positive constant. The only surviving term on the right-hand side of (6.6) is

$$\frac{i\Lambda^2}{4T}\langle\psi_{;\tau}\Gamma_\tau\psi-\psi\Gamma_\tau\psi_{;\tau}\rangle=\frac{\epsilon\Lambda^3}{2T}\langle\psi\Gamma_{04}\psi\rangle=2\epsilon U^2f^{-3} \tag{6.8}$$

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which gives, from (6.4),

$$2\frac{f''}{f} + \left(\frac{f'}{f}\right)^2 = 0 \tag{6.9}$$

and

$$3\left(\frac{f'}{f}\right)^2 = 2\epsilon U^2 f^{-3}, \tag{6.10}$$

which forces $\epsilon = 1$.

Equation (6.9) with the scaling condition (6.2) gives

$$f(t) = \left(\frac{t}{t_0}\right)^{2/3},$$

which is the well known zero-pressure Einstein–de Sitter model. This is a "pure" model in the sense that all contribution to the energy–momentum tensor comes from the spinor field, and no account is taken of the effect of local disturbances such as galaxies or radiation which would require additional matter tensors. These would of course interact with the cosmological background (spinor) field which thus replaces Mach's principle in a concrete form. If Hubble's constant is about 2×10^{-4} per megaparsec, then $t_0 \approx 10^{26}\Lambda^{-1}$.

Equation (6.10) gives

$$U^2 = \tfrac{2}{3} t_0^{-2}$$

and so we obtain

$$\frac{1}{\sqrt{T}}\psi = \Lambda^{-3/2}\left(\frac{2}{3}\right)^{1/2} t^{-1} e^{it} \sum_{\eta,\zeta} \left(Y_1^{(\eta,\zeta)} + Y_2^{(\eta,\zeta)} - Y_3^{(\eta,\zeta)} - Y_4^{(\eta,\zeta)}\right).$$

Note that neither T nor ψ can be determined from the equation, only $(1/\sqrt{T})\psi$ as indicated in Section 5.

It is interesting to consider a particle ensemble source for the line element (6.1). Suppose that the line element is produced on the average by neutral particles of mass μM. The energy in a region occupied by just one of these particles is simply $\mu M c^2$.

However, this equals the integral over that region of the energy density given by the contracted energy–momentum tensor $\underline{\underline{S}}{}^{\tau}_{\tau}$, as follows:

$$\begin{aligned} \mu M c^2 &= \int \underline{\underline{S}}{}^{\tau}_{\tau}\, d^3\underline{x} \\ &= \int -\frac{c^4}{8\pi K}\Lambda^{-1}\, \underline{\underline{g}}\left(G^{\tau}_{\tau} - \frac{1}{2} g^{\tau}_{\tau} G\right) d^3\underline{x} \\ &= \int \frac{c^4\Lambda^4}{16\pi K T}\langle \psi \Gamma_{04} \psi\rangle\, d^3\underline{x} \quad \text{from (6.8)} \\ &= \frac{c^4\Lambda}{16\pi K T} \quad \text{assuming (5.23).} \end{aligned} \tag{6.11}$$

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Hence

$$\mu = \frac{c^2\Lambda}{16\pi KTM} = \frac{1}{2}$$

from (5.25). That is, the normalization hypothesis (5.23) would imply a simple constraint on the mass of particles in such an ensemble.

7. A neutral spherically symmetric particle solution

There are two types of neutral spherically symmetric solutions of interest: one representing an elementary particle and the other a massive body. We shall only consider here a particle-like "pure" solution in which the energy–momentum tensor is given wholly by the right-hand side of the gravitational equations (5.16). A massive solution would clearly require an additional matter tensor coming from the averaged-out field of matter.

Using spherical coordinates, the following standard line element will be assumed:

$$\Lambda^2 g(r)^2\, dt^2 - \Lambda^2\big(f(r)^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\big). \tag{7.1}$$

The simplest orientation of the Vierbein metric field generating this line element is

$$g_{1r} = i\Lambda f, \quad g_{2\theta} = i\Lambda r, \quad g_{3\phi} = i\Lambda r \sin\theta, \quad g_{4\tau} = \Lambda g,$$

with all other components zero. The indices r, θ, ϕ, τ stand for 1, 2, 3, 4 respectively. As in (6.1), the Λ factor has again been introduced to make the coordinates independent of the unit of length chosen. With this metric, the nonzero components of the Einstein–Ricci tensor are

$$\begin{aligned}
-G_{rr} + \frac{1}{2}g_{rr}G &= \frac{2}{r}\frac{g'}{g} - \frac{1}{r^2}(f^2-1),\\
-\frac{f^2}{r^2}\left(G_{\theta\theta} - \frac{1}{2}g_{\theta\theta}G\right) &= \frac{g''}{g} - \frac{f'g'}{fg} + \frac{1}{r}\left(\frac{g'}{g} - \frac{f'}{f}\right),\\
G_{\phi\phi} - \frac{1}{2}g_{\phi\phi}G &= \left(G_{\theta\theta} - \frac{1}{2}g_{\theta\theta}G\right)\sin^2\theta,\\
-\frac{f^2}{g^2}\left(G_{\tau\tau} - \frac{1}{2}g_{\tau\tau}G\right) &= \frac{2}{r}\frac{f'}{f} + \frac{1}{r^2}(f^2-1).
\end{aligned} \tag{7.2}$$

A neutral, spinless solution requires a spinor field tetrad with components in at least two of the minimal left ideals (see Table 1). However, the simplest such solution is without torsion, again requiring the vanishing of the inner products (6.5), and so leading to the four-ideal tetrad

$$\Lambda^{3/2}\psi = \frac{1}{\sqrt{g}}e^{i\epsilon t}\sum_{\eta,\zeta} e^{(i/2)\zeta\phi}u_m^{(\eta,\zeta)}Y_m^{(\eta,\zeta)},$$

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with

$$u_1 = \eta\zeta P e^{(i/2)\zeta\theta} - iQe^{-(i/2)\zeta\theta}, \quad u_2 = -\eta\zeta P e^{-(i/2)\zeta\theta} + iQe^{(i/2)\zeta\theta},$$
$$u_3 = -\eta\zeta P e^{(i/2)\zeta\theta} - iQe^{-(i/2)\zeta\theta}, \quad u_4 = \eta\zeta P e^{-(i/2)\zeta\theta} + iQe^{(i/2)\zeta\theta},$$

where P and Q are real functions of r, and $\epsilon^2 = 1$. We anticipate $\epsilon = +1$ so that the solution can merge into the cosmological background field, but do not assume this at this stage. The only relevant nonzero inner products are, using (3.12) and Appendix D,

$$\langle\psi\Gamma_{05}\psi\rangle = \frac{4i}{g}\Lambda^{-3}(Q^2 - P^2) \quad \text{and} \tag{7.3}$$

$$\langle\psi\Gamma_{04}\psi\rangle = \frac{4}{g}\Lambda^{-3}(Q^2 + P^2). \tag{7.4}$$

The particular form of the tetrad has also been designed to satisfy the Dirac equation (5.1), which here takes the simple form

$$\Gamma^{\mu}\left(\psi_{,\mu} - \frac{i}{4}N_{mn\mu}\Gamma_{mn}\psi\right) + \Lambda^{-1}\Gamma_{05}\psi = 0. \tag{7.5}$$

Using Appendix C and the nonzero $N_{mn\mu}$ values, which are

$$N_{12\theta} = \frac{1}{f}, \quad N_{13\phi} = \frac{1}{f}\sin\theta, \quad N_{23\phi} = \cos\theta, \quad N_{14\tau} = -i\frac{g'}{f},$$

(7.5) is expanded. Equating the coefficients of each of the four ideals, we get four sets of identical equations each defining P and Q, which are

$$P' = \frac{1}{r}(f-1)P - \epsilon\frac{f}{g}Q - fQ \quad \text{and} \quad Q' = -\frac{1}{r}(f+1)Q + \epsilon\frac{f}{g}P - fP. \tag{7.6}$$

Finally, the gravitational equations

$$G_{\mu\mu} - \frac{1}{2}g_{\mu\mu}G = \frac{i}{4T}\Lambda^2\langle\psi\Gamma_{\mu}\psi_{;\mu} - \psi_{;\mu}\Gamma_{\mu}\psi\rangle \quad \mu = r, \theta, \phi, \tau$$

give, from (7.2),

$$\begin{aligned} \frac{2}{r}\frac{g'}{g} - \frac{1}{r^2}(f^2 - 1) &= \frac{2}{T}\frac{f^2}{g^2}\left(\epsilon(P^2 + Q^2) + g(Q^2 - P^2) - \frac{2}{r}gPQ\right), \\ \frac{2}{r}\frac{f'}{f} + \frac{1}{r^2}(f^2 - 1) &= \frac{2\epsilon}{T}\frac{f^2}{g^2}(P^2 + Q^2). \end{aligned} \tag{7.7}$$

The second-order equation in (7.2), coming from the $G_{\theta\theta}$ and $G_{\phi\phi}$ components, can be shown to be a consequence of the other two (that is, a Bianchi identity).

Equations (7.6) replace the usual equations of state for the matter and pressure density distribution; they represent, with (7.7), a "pure" or particle-like field in which

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there is no separate matter tensor in Einstein's equations to account for the averaged-out distribution of matter.

The right-hand side of the second equation in (7.7) represents a matter density. Taking the cosmologically determined value of ϵ, namely $\epsilon = 1$, the matter density will be positive as it should be. That is, the cosmological background spinor field not only supplies the local inertial frames but also imposes positivity of the mass of the particle.

At large distances where we can ignore the right-hand sides of (7.7), we can set $f \approx g^{-1} \approx 1 + \nu/r$ as in the Schwarzschild solution. For a particle solution, the gravitational radius ν is expected to be extremely small. Indeed if the mass is of the same order as the elementary mass M, then according to (5.26), ν is of order $1/T$ in units of Λ.

Equations (7.6) have a solution as $r \to \infty$ in which P and Q tend to zero of order at least $1/r$. For small r, but still large relative to ν, Q grows like $1/r^2$ and totally dominates P. The critical behaviour occurs near the Schwarzschild radius 2ν, and it is convenient to rescale r and Q according to

$$r = \nu y, \qquad Q = \frac{\tilde{Q}}{\nu}. \tag{7.8}$$

We then have from (7.6) and (7.7) with sufficient approximation

$$\frac{dP}{dy} = \frac{1}{y}(f-1)P - \frac{f}{g}(1+g)\tilde{Q}, \qquad \frac{d\tilde{Q}}{dy} = -\frac{1}{y}(f+1)\tilde{Q}, \tag{7.9}$$

$$\frac{df}{dy} + \frac{f}{2y}(f^2-1) = 0, \qquad \frac{dg}{dy} - \frac{g}{2y}(f^2-1) = 0. \tag{7.10}$$

Equations (7.10) are of course satisfied by Schwarzschild's $f^{-1} = g = \gamma = \sqrt{(1-2/y)}$. The second equation of (7.9) is then solved by

$$\tilde{Q} = \frac{q_0}{y}\frac{1-\gamma}{1+\gamma}, \qquad q_0 \text{ constant}. \tag{7.11}$$

We may assume $q_0 > 0$. As y approaches the critical value 2, $\tilde{Q}$ approaches $q_0/2$ and P grows like $2q_0 \log(1/\gamma)$. P can therefore be neglected in comparison with Q provided that $g = \gamma$ is large compared with $e^{-1/\nu}$. On the other hand, the right-hand terms in (7.7) become significant and the equations are replaced by

$$\frac{df}{dy} = -\frac{1}{4}f^3\left(1 - \frac{1}{T_1 g^2}\right) \quad \text{and} \quad \frac{dg}{dy} = \frac{1}{4}f^2 g\left(1 + \frac{1}{T_1 g^2}\right),$$

where we have defined

$$T_1 = \frac{T}{2q_0^2}. \tag{7.12}$$

Assuming that T_1 is large, these give

$$f = \frac{gT_1}{g^2 T_1 + 1} \tag{7.13}$$

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and

$$g^2T_1 - 1 + \log(g^2T_1) = \tfrac{1}{2}T_1(y-2). \tag{7.14}$$

In (7.14), the constant of integration has been chosen so that f should reach its maximum value of $\sqrt{T_1}/2$ at $y=2$, corresponding to $g = 1/\sqrt{T_1}$. This maximum is quite sharp, and f crashes down steeply inside the critical shell of width $\Delta y = 1/T_1$, (which replaces the Schwarzschild singular sphere). To see this, note that as we cross into the interior of the critical shell, but still close to $y=2$, f/g becomes very nearly T_1 by (7.13), and we get from (7.7)

$$\frac{1}{f}\frac{df}{dy} + \frac{1}{4}(f^2-1) = \frac{1}{4}T_1$$

which is solved (to order T_1) by

$$f = \frac{\sqrt{T_1}}{\sqrt{2(1+e^{T_1(2-y)/2})}}.$$

Hence when $2 - y \gg \log T_1/T_1$, both f and g become very small. Neglecting f against 1 and setting

$$u = \nu\frac{f}{g}, \quad P = \frac{1}{y\nu}p, \quad Q = \frac{1}{y\nu}q, \tag{7.15}$$

(7.6) and (7.7) can be replaced in the interior by

$$p' = -uq, \quad q' = up, \tag{7.16}$$

$$y\frac{f'}{f} = \frac{1}{2} + \frac{1}{T\nu^2}u^2(p^2+q^2), \tag{7.17}$$

$$y\frac{u'}{u} = 1. \tag{7.18}$$

Noting that $u = \nu T$ at $y=2$, (7.18) implies

$$u = \frac{\nu}{2}T_1 y. \tag{7.19}$$

Now (7.16) gives $pp' + qq' = 0$ hence $p^2 + q^2 =$ constant. However, at $y=2$,

$$p^2 + q^2 = 4\nu^2Q^2 = \frac{T}{2T_1} \tag{7.20}$$

by (7.8). Equation (7.17) then gives

$$\frac{1}{y}\frac{f'}{f} = \frac{1}{2y^2} + \frac{T_1}{8}$$

which is solved by

$$f = \frac{1}{2}\sqrt{\frac{T_1 y}{2}}e^{-(T_1/4)(1-y^2/4)}.$$

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From (7.16) and (7.20) we get

$$p = \sqrt{\frac{T}{2T_1}}\,\sin\left(\frac{\nu T_1}{4}(4-y^2)\right), \quad q = \sqrt{\frac{T}{2T_1}}\,\cos\left(\frac{\nu T_1}{4}(4-y^2)\right). \tag{7.21}$$

Finally, from (7.15) and (7.19) we have $g = 2f/yT$ and so

$$g = \frac{1}{\sqrt{T_1 y}} e^{-(T_1/4)(1-y^2/4)}.$$

This shows that in the immediate neighbourhood of the origin, g behaves like $c_0/\sqrt{y}$ for an exceedingly small c_0. Nevertheless it indicates a singular behaviour of the metric tensor at the origin, at least in the coordinates of the line element (7.1). To check whether this is a true space–time singularity, we need to see if the curvature scalar remains finite at $y = 0$. From (5.17) and (7.3)

$$\Lambda^2 G = \frac{2}{gr^2T}(p^2 - q^2).$$

From (7.21)

$$\frac{1}{T}(p^2 - q^2) = -\frac{1}{2T_1}\cos\left(\frac{\nu T_1}{2}(4-y^2)\right),$$

which vanishes, to the order y^2, provided that

$$2\nu T_1 = \left(k + \frac{1}{2}\right)\pi$$

for an integer k. From (7.12) the condition therefore becomes

$$q_0^2 = \frac{2\nu T}{(2k+1)\pi}.$$

Thus q_0 can only take certain discrete values, given the Schwarzschild mass ν, a phenomenon not unlike quantization. The lowest value $k = 0$ is distinguished by the property that the curvature scalar does not change sign (remains negative) throughout the solution. Accepting this criterion for the solution, q_0 is uniquely determined by

$$q_0^2 = \frac{2\nu}{\pi}T. \tag{7.22}$$

To determine the mass, we must calculate the normalization integral

$$\begin{aligned} I_0 &= \int \underline{\underline{g}}^{*}\langle\psi\Gamma_{04}\psi\rangle\, d^3\underline{x} \\ &= 4\pi\Lambda^3\int_0^\infty \langle\psi\Gamma_{04}\psi\rangle f r^2\, dr \\ &= 16\pi\int_0^\infty (P^2 + Q^2)\frac{f}{g}r^2\, dr, \quad \text{using (7.4)}. \end{aligned} \tag{7.23}$$

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Outside the critical shell there is only a negligible contribution of order $1/T$, hence from (7.15) we have

$$
\begin{aligned}
I_0 &= 16\pi \int_0^2 (p^2+q^2) u \, dy \\
&= 8\pi q_0^2 \nu T \int_0^2 y \, dy \\
&= 8\pi \nu T, \quad \text{using (7.12) and (7.19).}
\end{aligned} \tag{7.24}
$$

Suppose the mass is μM, and noting that the gravitational radius ν is in units of Λ, then by (5.26)

$$I_0 = \mu. \tag{7.25}$$

The mass of the particle is therefore $I_0 M = M$ if we accept the normalization hypothesis (5.23). Note that this is twice the value obtained for the cosmological particle. As in (6.11), the energy density is given by

$$\underset{=}{S}{}^{\tau}_{\tau} \equiv -\frac{c^4}{8\pi K} \Lambda^{-1} \underset{=}{g} \left(G^{\tau}_{\tau} - \frac{1}{2} g^{\tau}_{\tau} G \right).$$

and from (7.2) we have

$$
\begin{aligned}
-\underset{=}{g} \left(G^{\tau}_{\tau} - \frac{1}{2} g^{\tau}_{\tau} G \right) &= \Lambda^2 \frac{g}{f} \left(\frac{2}{r} \frac{f'}{f} + \frac{1}{r^2}(f^2-1) \right) r^2 \sin\theta \\
&= \Lambda^2 \frac{2}{T} \frac{f}{g} (P^2+Q^2) r^2 \sin\theta.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\int \underset{=}{S}{}^{\tau}_{\tau} \, d^3\underline{x} &= \frac{c^4 \Lambda}{KT} \int \frac{f}{g} (P^2+Q^2) r^2 \, dr \\
&= \frac{c^4 \Lambda I_0}{16\pi K T} \quad \text{from (7.23)} \\
&= \frac{1}{2} M I_0 c^2 \quad \text{from (5.25)} \\
&= \frac{1}{2} \mu M c^2 \quad \text{from (7.25).}
\end{aligned}
$$

This shows that the field energy–momentum tensor supplies exactly half of the total energy of the particle, the other half supplied of course by the energy–momentum pseudo-tensor.

With the normalization hypothesis $I_0 = 1$, the mass of the particle is M, giving $\nu T = 1/8\pi$ from (7.24) and $q_0 = 1/2\pi$ from (7.22). Using (7.8) and (7.11) hence

$$Q \approx \frac{q_0 \nu}{2r^2} = \frac{1}{32\pi^2 r^2 T}$$

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for $r \approx 1$. Therefore P and Q reach in this region the intensity of the cosmological background field which then takes over from both. Hence the effective radius of the particle is of order Λ.

In some respect the solution has simpler properties than Schwarzschild space–time, in particular, it is free from the weird properties associated with the singular sphere. For example, in contrast to "black hole" solutions, the particle is transparent. The coordinate time required for a light signal to reach the critical shell from the centre is

$$t_1 = \int_0^{2\nu} \frac{f}{g}\, dr = \int_0^2 u\, dy = \nu T_1 = \frac{\pi}{4}$$

and to cross the shell

$$t_2 = \int_{2\nu}^{\Lambda} \frac{T_1}{1 + g^2 T_1}\, dr = 2\nu \int_{1/T_1}^{T_1} \frac{d\tau}{\tau} = 4\nu \log \frac{\pi}{4\nu}$$

as $\tau = g^2 T_1$. That is, the crossing time is negligible and so it requires $\pi\Lambda/2c$ seconds to completely traverse the particle.

However the main interest of the solution, irrespective of whether or not it represents a realistic particle, is that it provides an example of a particle-like solution with a highly nontrivial and singularity-free internal structure. It is noteworthy and of no small significance that both the gravitational and spinor fields play a role in producing this structure.

Appendix A. Matrix representation of the Γ_{ab}

The standard Hermitian matrix representation of the the Γ_{ab} is conveniently given in terms of the Pauli spin matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The following table lists the standard Hermitian matrix representation of the Γ_{ab}; they establish an isomorphism between $\mathcal{W}$ and the algebra of 4×4 matrices:

$$\Gamma_{01} = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}, \quad \Gamma_{02} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \Gamma_{03} = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix},$$

$$\Gamma_{04} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}, \quad \Gamma_{05} = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}, \quad \Gamma_{12} = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix},$$

$$\Gamma_{13} = \begin{bmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}, \quad \Gamma_{14} = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, \quad \Gamma_{15} = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix},$$

$$\Gamma_{23} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \Gamma_{24} = \begin{bmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{bmatrix}, \quad \Gamma_{25} = \begin{bmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{bmatrix},$$

$$\Gamma_{34} = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad \Gamma_{35} = \begin{bmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}, \quad \Gamma_{45} = \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}.$$

Appendix B. Correspondence of Gamma matrices to Γ_{ab}

The standard Gamma matrices (shown below in the covariant Dirac representation) correspond to the following Γ_{ab}:

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \Gamma_{23}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = -i\Gamma_{24},$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = i\Gamma_{12}, \quad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = i\Gamma_{25}.$$

Appendix C. Multiplication table for $\Gamma_{ab} Y_m^{(\eta\zeta)}$

	Y_1	Y_2	Y_3	Y_4
Γ_{01}	$-\zeta Y_4$	$-\zeta Y_3$	$-\zeta Y_2$	$-\zeta Y_1$
Γ_{02}	$-iY_4$	iY_3	$-iY_2$	iY_1
Γ_{03}	ζY_3	$-\zeta Y_4$	ζY_1	$-\zeta Y_2$
Γ_{04}	$-i\eta Y_3$	$-i\eta Y_4$	$i\eta Y_1$	$i\eta Y_2$
Γ_{05}	$-\eta Y_1$	$-\eta Y_2$	ηY_3	ηY_4
Γ_{15}	$-i\eta\zeta Y_4$	$-i\eta\zeta Y_3$	$i\eta\zeta Y_2$	$i\eta\zeta Y_1$
Γ_{25}	ηY_4	$-\eta Y_3$	$-\eta Y_2$	ηY_1
Γ_{35}	$i\eta\zeta Y_3$	$-i\eta\zeta Y_4$	$-i\eta\zeta Y_1$	$i\eta\zeta Y_2$
Γ_{45}	Y_3	Y_4	Y_1	Y_2
Γ_{12}	ζY_1	$-\zeta Y_2$	ζY_3	$-\zeta Y_4$
Γ_{13}	iY_2	$-iY_1$	iY_4	$-iY_3$
Γ_{14}	$\eta\zeta Y_2$	$\eta\zeta Y_1$	$-\eta\zeta Y_4$	$-\eta\zeta Y_3$
Γ_{23}	$-\zeta Y_2$	$-\zeta Y_1$	$-\zeta Y_4$	$-\zeta Y_3$
Γ_{24}	$i\eta Y_2$	$-i\eta Y_1$	$-i\eta Y_4$	$i\eta Y_3$
Γ_{34}	$-\eta\zeta Y_1$	$\eta\zeta Y_2$	$\eta\zeta Y_3$	$-\eta\zeta Y_4$

Appendix D. Inner products

$$\langle \psi\Gamma_{01}\psi\Gamma_{05}\rangle = \frac{1}{4}\sum_{\eta\zeta} i\zeta(-u_1^*u_2 - u_2^*u_1 + u_3^*u_4 + u_4^*u_3)$$

$$\langle \psi\Gamma_{02}\psi\Gamma_{05}\rangle = \frac{1}{4}\sum_{\eta\zeta} (-u_1^*u_2 + u_2^*u_1 + u_3^*u_4 - u_4^*u_3)$$

$$\langle \psi\Gamma_{03}\psi\Gamma_{05}\rangle = \frac{1}{4}\sum_{\eta\zeta} i\zeta(u_1^*u_1 - u_2^*u_2 - u_3^*u_3 + u_4^*u_4)$$

$$\langle\psi\Gamma_{04}\psi\Gamma_{05}\rangle = \frac{1}{4}\sum_{\eta\zeta}\eta(u_1^*u_1 + u_2^*u_2 + u_3^*u_3 + u_4^*u_4)$$

$$\langle\psi\Gamma_{04}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}(u_1^*u_1 + u_2^*u_2 + u_3^*u_3 + u_4^*u_4)$$

$$\langle\psi\Gamma_{05}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i(u_1^*u_3 + u_3^*u_1 + u_2^*u_4 + u_4^*u_2)$$

$$\langle\psi\Gamma_{15}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}\zeta(u_1^*u_2 + u_2^*u_1 + u_3^*u_4 + u_4^*u_3)$$

$$\langle\psi\Gamma_{25}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i(-u_1^*u_2 + u_2^*u_1 - u_3^*u_4 + u_4^*u_3)$$

$$\langle\psi\Gamma_{35}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}\zeta(-u_1^*u_1 + u_2^*u_2 - u_3^*u_3 + u_4^*u_4)$$

$$\langle\psi\Gamma_{45}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i\eta(u_1^*u_1 + u_2^*u_2 - u_3^*u_3 - u_4^*u_4)$$

$$\langle\psi\Gamma_{14}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i\zeta(u_1^*u_4 + u_4^*u_1 + u_3^*u_2 + u_2^*u_3)$$

$$\langle\psi\Gamma_{24}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}(-u_1^*u_4 + u_4^*u_1 + u_2^*u_3 - u_3^*u_2)$$

$$\langle\psi\Gamma_{34}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i\zeta(u_1^*u_3 + u_3^*u_1 - u_2^*u_4 - u_4^*u_2)$$

$$\langle\psi\Gamma_{13}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}\eta(u_1^*u_4 + u_4^*u_1 - u_2^*u_3 - u_3^*u_2)$$

$$\langle\psi\Gamma_{23}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i\eta\zeta(-u_1^*u_4 + u_4^*u_1 - u_2^*u_3 + u_3^*u_2)$$

$$\langle\psi\Gamma_{12}\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i\eta\zeta(u_1^*u_3 - u_3^*u_1 - u_2^*u_4 + u_4^*u_2)$$

$$\langle\psi\psi\rangle = \frac{1}{4}\sum_{\eta\zeta}i\eta(u_1^*u_3 - u_3^*u_1 + u_2^*u_4 - u_4^*u_2)$$

Acknowledgement

The authors gratefully acknowledge the assistance of Peter Szekeres in the preparation of this article.

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