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Metamathematics of modal logic

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The techniques employed in the semantic analysis of propositional languages fall roughly into two kinds. The *algebraic* method treats formulae as polynomial symbols by interpreting logical connectives as operators on certain kinds of lattices. In the *set-theoretic* approach the models, or *frames*, carry structural features other than finitary operations, such as neighbourhood systems and finitary relations. Formulae are interpreted as subsets of the frame in a manner constrained by its particular structure.

The two kinds of model are closely related. Algebras may be constructed as subset lattices of frames, and frames may be obtained from algebras through various lattice representations. The general concern of this thesis is to explore the relationships between these two semantical frameworks and to discuss their relative strengths and limitations. The vehicle chosen for this work is normal modal logic, although the concepts and results developed may be parallelled in other contexts, for example, intuitionist logic.

Sections 1 and 2 outline the syntax of modal logic, its algebraic semantics (modal algebras), and the "possible-worlds" model theory due to Kripke. A discussion of previous work on correspondences between frames and algebras leads in Section 3 to S.K. Thomason's notion of a *first-order frame*, that incorporates a restriction on the admissible interpretations of formulae. In Sections 4-7 we develop the basic structure theory of first-order frames, focusing on validity-preserving constructions such as homomorphisms, subframes, disjoint unions, and ultraproducts, each of which corresponds to an identity-preserving construction on modal-algebras.

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Ultraproducts are then used in Section 8 to discuss the compactness properties of two semantic consequence relations. In Section 9 we introduce what is probably the central concept of the thesis, the *descriptive frame*, designed to provide an exact set-theoretic analogue to the notion of a modal algebra. We show in Section 10 that the category of descriptive frames is dual to the category of modal algebras. Inverse limits of descriptive frames are defined in Section 11, and used in Section 12 in a characterisation, in terms of closure under various constructions, of those classes of descriptive frames that are axiomatic; that is, the class of all models of some set of formulae. After a discussion of characteristic models of particular logics (Section 13), we introduce in Section 14 the class of *d*-persistent formulae, those whose validity is preserved under full extensions of descriptive frames. Any logic axiomatised by such formulae has a modelling in the Kripke style. By a refinement of techniques due to Jónsson and Tarski we produce some wide-ranging syntactic criteria for *d*-persistence. In Section 15 descriptive frames are used to prove a conjecture of Lemmon and Scott concerning a completeness theorem for a general axiom schema. Section 16 employs ultraproducts to characterise those formulae whose models form an elementary class for the first-order language of a single dyadic predicate. Section 17 contains an example; the techniques thus far developed are used to analyse the celebrated logic *KM* , which is shown not to be determined by any class of Kripke models that is closed under ultraproducts. We conclude our work in Section 18 with a discussion of the relationships between some classes of formulae that are defined by special model theoretic properties, such as first-order definability, *d*-persistence, and validity on canonical frames.