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OSCILLATIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Sufficient conditions are established for the oscillation of solutions of hyperbolic equations of neutral type of the form

$$\frac{\partial^2}{\partial t^2}\left[u(x,t)+p(t)u(x,\ t-\tau)\right]=a(t)\Delta u(x,t)+q(t)f\left(u(x,\ \sigma(t))\right)$$
$$(x,t)\in\Omega\times R_+\equiv G,$$

where $R_+ = \{0,\infty)$, Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$.

Recently there has been much interest in studying the oscillatory behavior of solutions of partial differential equations with deviating arguments. We refer the reader to the papers by Georgiou and Kreith [**1**], Mishev and Bainov [**3**, **4**], and Yoshida [**6**].

The purpose of this paper is to obtain the sufficient conditions for the oscillation of the solutions of hyperbolic equations of neutral type of the form

$$\frac{\partial^2}{\partial t^2}\left[u(x,t)+p(t)u(x,t-\tau)\right]=a(t)\Delta u(x,t)+q(t)f\Big(u\big(x,\ \sigma(t)\big)\Big) \tag{1}$$
$$(x,t)\in\Omega\times R_+\equiv G,$$

where $R_+ = [0,\infty)$, Ω is a bounded domain in R^n with a piecewise smooth bounded $\partial\Omega$.

Suppose that the following conditions (A) hold:

- (A_1) $\tau = \text{const} > 0$, $\sigma(t)$ is a continuous function in R_+ such that $\lim\limits_{t\to\infty}\sigma(t) = \infty$, $\sigma(t)\leqslant t$;
- (A_2) $a(t)$ is a nonnegative continuous function on R_+, $f(u)\in C(R,R)$ are convex in $(0,\infty)$ and $uf(u)>0$ for $u\neq 0$; and
- (A_3) $q(t)\in C\big([0,\infty)\big)$ and $p(t)\in C\big([0,\infty),[0,1]\big)$.

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Consider first the boundary condition

$$\frac{\partial u}{\partial n} + \mu u = 0, \quad (x,t) \in \partial\Omega \times R_+, \tag{2}$$

where μ is a continuous, nonnegative function on $\partial\Omega \times R_+$, and n denotes the unit exterior normal vector to $\partial\Omega$.

DEFINITION: The solution $u(u,t)$ of the problem (1), (2) is called oscillatory in G if $u(x,t)$ has zero in $\Omega \times [t_0, \infty)$ for each $t_0 > 0$.

THEOREM 1. *Let the conditions (A) hold and suppose there exists a positive constant α such that*

$$\frac{f(u)}{u} > \alpha \quad \text{for} \quad u \neq 0, \ \sigma'(t) \geqslant 0 \quad \text{for every} \quad t \geqslant 0. \tag{3}$$

If

$$\int^{\infty} q(s)\Big[1 - p\big(\sigma(s)\big)\Big] ds = \infty, \tag{4}$$

then every solution $u(x,t)$ of the problem (1), (2) is oscillatory.

PROOF: Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (2) which has no zero in $\Omega \times [t_0, \infty)$. Without loss of generality we may assume that $u(x,t) > 0$ in $\Omega \times [t_0, \infty)$. From condition ($A_1$) there exists a $t_1 > t_0$ such that $u(x,t) > 0$, $u\big(x, \sigma(t)\big) > 0$ and $u(x,\ t - \tau) > 0$ in $\Omega \times [t_1, \infty)$. We integrate (1) with respect to x over the domain Ω and obtain for $t \geqslant t_1$

$$\begin{aligned} &\frac{d^2}{dt^2}\left[\int_\Omega u(x,t)dx + p(t)\int_\Omega u(x,\ t-\tau)dx\right] \\ &\qquad = a(t)\int_\Omega \Delta u(x,t)dx - q(t)\int_\Omega f\Big(u\big(x,\sigma(t)\big)\Big)dx. \end{aligned} \tag{5}$$

Green's formula yields

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = -\int_{\partial\Omega} \mu u ds \leqslant 0. \tag{6}$$

Moreover, from (A_2) and using Jensen's inequality it follows that

$$\int_\Omega f\Big(u\big(x,\sigma(t)\big)\Big)dx \geqslant |\Omega| \left(\frac{\int_\Omega u\big(x,\sigma(t)\big)dx}{|\Omega|}\right), \quad \text{for} \quad t \geqslant t_1, \ |\Omega| = \int_\Omega dx. \tag{7}$$

Then from (5), (6) and (7) it follows that for $t \geqslant t_1$

$$\frac{d^2}{dt^2}\left[V(t) + p(t)V(t-\tau)\right] + q(t)f\Big(V\big(\sigma(t)\big)\Big) \leqslant 0, \tag{8}$$

where $V(t) = \int_\Omega u(x,t)dx, \quad t \geqslant t_0$. Thus $V(t)$ is a positive solution of the inequality (8). Set $Z(t) = V(t) + p(t)V(t-\tau)$. Obviously $Z(t) > 0$ for $t \geqslant t_1$ and

$$Z''(t) \leqslant 0 \quad \text{for} \quad t \geqslant t_1. \tag{9}$$

Hence $Z'(t)$ is a decreasing function. We claim that $Z'(t) > 0$ for $t \geqslant t_1$. If $Z'(t) < 0$ for $t \geqslant t_1$, then there exists a $t_2 \geqslant t_1$ such that $Z'(t_2) < 0$. Then

$$Z(t) - Z(t_2) \leqslant \int_{t_2}^{t} Z'(t_2)ds = Z'(t_2)(t-t_2) \quad \text{for} \quad t \geqslant t_2$$

and $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts the fact that $Z(t) > 0$. In view of (3), we have

$$Z''(t) + \alpha q(t)V\big(\sigma(t)\big) \leqslant 0, \quad \text{for} \quad t \geqslant t_1,$$

or

$$Z''(t) + \alpha q(t)\Big[Z\big(\sigma(t)\big) - p\big(\sigma(t)\big)V\big(\sigma(t)-\tau\big)\Big] \leqslant 0, \quad t \geqslant t_1.$$

Since $Z(t) \geqslant V(t)$ and $Z(t)$ is nondecreasing, it follows that

$$Z''(t) + \alpha q(t)\Big[1 - p\big(\sigma(t)\big)\Big]Z\big(\sigma(t)\big) \leqslant 0, \quad t \geqslant t_1. \tag{10}$$

Let

$$W(t) = t^\beta \frac{Z'(t)}{Z\big(\sigma(t)\big)}, \quad t \geqslant t_1, \quad \beta \geqslant 0.$$

We obtain for $t \geqslant t_1$

$$\begin{aligned} W'(t) &= \beta t^{\beta-1}\frac{Z'(t)}{Z\big(\sigma(t)\big)} + t^\beta \frac{Z''(t)Z\big(\sigma(t)\big) - Z'(t)Z'\big(\sigma(t)\big)\sigma'(t)}{\Big[Z\big(\sigma(t)\big)\Big]^2} \\ &= \beta t^{\beta-1}\frac{Z'(t)}{Z\big(\sigma(t)\big)} - t^\beta \alpha q(t)\Big[1 - p\big(\sigma(t)\big)\Big] - \frac{Z'(t)Z'\big(\sigma(t)\big)\sigma'(t)}{\Big[Z\big(\sigma(t)\big)\Big]^2} \\ &\leqslant \beta t^{-1}W(t) - t^\beta \alpha q(t)\Big[1 - p\big(\sigma(t)\big)\Big]. \end{aligned}$$

From this it follows that

$$\left(\frac{W(t)}{t^\beta}\right)' \leqslant -\alpha q(t)\Big[1 - p\big(\sigma(t)\big)\Big] \quad \text{for} \quad t \geqslant t_1,$$

or

$$0 \leqslant \frac{W(t)}{t^\beta} \leqslant \frac{W(t_1)}{t_1^\beta} - \int_{t_1}^{t} \alpha q(s)\Big[1 - p\big(\sigma(s)\big)\Big]ds, \quad t \geqslant t_1.$$

Thus we have

$$\lim_{t\to\infty} \frac{W(t)}{t^\beta} = -\infty,$$

which leads to a contradiction.

If $u(x,t) \leqslant 0$ for $(x,t) \in \Omega \times [t_0,\infty)$, then the proof follows from the fact that $-u(x,t)$ is a positive solution of the problem (1), (2). This completes the proof of the theorem. $\square$

REMARK. If $u(x,t) \equiv u(t)$, $f(u) \equiv u$ and $\sigma = t - \sigma$, then Theorem 1 and the Theorem in [**2**] are the same.

We consider now the boundary condition

$$u = 0, \quad \text{on} \quad \partial\Omega \times [0, \infty), \tag{11}$$

and we consider the following Dirichlet problem in the domain Ω

$$\begin{gathered} \Delta v + \alpha v = 0, \quad \text{in} \quad \Omega \\ v|_{\partial\Omega} = 0, \end{gathered} \tag{12}$$

where α is a constant. It is well-known that the smallest eigenvalue α_0 of the problem (12) is positive and the corresponding eigenfunction $\psi(x) \geqslant 0$ for $x \in \Omega$.

With a solution $u(x,t)$ of the problem (1), (11) we associate the function

$$H(t) = \frac{\int_\Omega u(x,t)\psi(x)dx}{\int_\Omega \psi(x)dx}, \quad t \geqslant 0. \tag{13}$$

THEOREM 2. *If all conditions of Theorem 1 hold, then every solution of the problem (1), (11) is oscillatory in G.*

PROOF: Let $u(x,t)$ be a positive solution of the problem (1), (11) in $\Omega \times [t_0, \infty)$ for some t_0. By condition (A_1) there exists a $t_1 \geqslant t_0$ such that $u\big(x, \sigma(t)\big) \geqslant 0$ and $u(x, t-\tau) > 0$ in $\Omega \times [t_1, \infty)$. Multiply both sides of equation (1) by the eigenfunction $\psi(x) \geqslant 0$, and integrate with respect to x over the domain Ω; then we have

$$\begin{aligned} &\frac{d^2}{dt^2}\left[\int_\Omega u(x,t)\psi(x)dx + p(t)\int_\Omega u(x,t-\tau)\psi(x)dx\right] \\ &\quad = a(t)\int_\Omega \Delta u\psi(x)dx - q(t)\int_\Omega f\Big(u\big(x,\sigma(t)\big)\Big)\psi(x)dx, \quad t > t_1. \end{aligned} \tag{14}$$

From the divergence theorem it follows that

$$\int_\Omega \Delta u\psi(x)dx = -\alpha_0\int_\Omega u\psi(x)dx, \quad t \geqslant t_1, \tag{15}$$

where α_0 is the smallest eigenvalue of the problem (12).

Using condition (A_2) and Jensen's inequality it follows that

$$\begin{aligned} &\int_\Omega f\Big(u\big(x,\sigma(t)\big)\Big)\psi(x)dx \\ &\quad \geqslant \int_\Omega \psi(x)dx f\left(\frac{1}{\int_\Omega \psi(x)dx}\int_\Omega u\big(x,\sigma(t)\big)\psi(x)dx\right), \quad t \geqslant t_1. \end{aligned} \tag{16}$$

Using (13), (15) and (16), we obtain

$$\frac{d^2}{dt^2}[H(t)+p(t)H(t-\tau)] \leqslant -\alpha_0 a(t)H(t) - q(t)f\Big(H\big(\sigma(t)\big)\Big), \quad t \geqslant t_1. \tag{17}$$

Since, for $t \geqslant t_1$ $H(t) \geqslant 0$ and $H\big(\sigma(t)\big) \geqslant 0$, then by (17)

$$\frac{d^2}{dt^2}[H(t)+p(t)H(t-\tau)] + q(t)f\Big(H\big(\sigma(t)\big)\Big) \leqslant 0, \quad t \geqslant t_1.$$

The remainder of the proof is similar to that of Theorem 1; we omit it. ∎

We need the following lemma [5].

Lemma. *Consider the differential inequality*

$$x'(t) + b(t)x\big(g(t)\big) \leqslant 0, \quad t \geqslant t_0, \tag{18}$$

where $b(t) \in C\big(R,\ [0,\infty)\big)$, $g(t) \in C(R,R)$, $g(t) \leqslant t$ and $g(t)$ is a nondecreasing function with $\lim_{t\to\infty} g(t) = \infty$. If

$$\liminf_{t\to\infty} \int_{g(t)}^{t} b(s)ds \geqslant \frac{1}{e},$$

then the inequality (18) has no ultimately positive solutions.

Theorem 3. *Let conditions (A) and (3) hold. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)ds > \frac{1}{e}, \tag{19}$$

where $\quad Q(t) = \alpha_0\varepsilon_0 q(t)\sigma(t)\Big[1-p\big(\sigma(t)\big)\Big]\exp\left[\int_T^t \alpha_0\varepsilon_0 a(s)s[1-p(s)]ds\right],$

for all large T, $\varepsilon_0 \in (0,1)$ is a constant, then every solution $u(x,t)$ of the problem (1), (11) is oscillatory in G.

Proof: Let $u(x,t)$ be a positive solution of the problem (1)–(11). As in the proof of Theorem 2 we get (17). From (17) and (3) it follows that

$$\frac{d^2}{dt^2}[H(t)+p(t)H(t-\tau)] + \alpha_0 a(t)H(t) + \alpha q(t)H\big(\sigma(t)\big) \leqslant 0, \quad t \geqslant t_1.$$

Moreover,

$$Z''(t) + \alpha_0 a(t)\Big[Z(t) - p(t)H(t-\tau)\Big] + \alpha q(t)\Big[Z\big(\sigma(t)\big) - p\big(\sigma(t)\big)H\big(\sigma(t)-\tau\big)\Big] \leqslant 0,$$
$$t \geqslant t_1.$$

Since $Z(t) \geqslant H(t)$ and $Z(t)$ is nondecreasing, we have

$$Z''(t) + \alpha_0 a(t)\left[1 - p(t)\right] Z(t) + \alpha q(t)\Big[1 - p\big(\sigma(t)\big)\Big] Z\big(\sigma(t)\big) \leqslant 0, \quad t \geqslant t_1. \tag{20}$$

Obviously, $Z(t) \geqslant 0$, $Z''(t) \leqslant 0$ and $Z' \geqslant 0$ for $t \geqslant t_1$ (as in the proof of Theorem 1).

By Lemma 1 in [5] there exists a $T \geqslant t_1$ such that

$$Z(t) \geqslant \varepsilon_0 t Z'(t) \quad \text{and} \quad Z\big(\sigma(t)\big) \geqslant \varepsilon \sigma(t) Z'\big(\sigma(t)\big) \quad \text{for} \quad t \geqslant T.$$

Hence, from (20) we have

(21)

$$Z''(t) + \alpha_0 \varepsilon_0 T a(t)\left[1 - p(t)\right] Z'(t) + \alpha \varepsilon_0 q(t) \sigma(t)\Big[1 - p\big(\sigma(t)\big)\Big] Z'\big(\sigma(t)\big) \leqslant 0, \quad t \geqslant T.$$

Set
$$Y(t) = Z'(t) \exp\left[\int_{t_2}^{t} \alpha_0 \varepsilon_0 s a(s)\left[1 - p(s)\right]\right] ds, \quad t \geqslant T$$

and using (21) we obtain

$$Y'(t) + Q(t) Y\big(\sigma(t)\big) \leqslant 0, \quad t \geqslant T, \tag{22}$$

where
$$Q(t) = \alpha_0 \varepsilon_0 q(t) \sigma(t)\Big[1 - p\big(\sigma(t)\big)\Big] \exp\left[\int_{T}^{t} \alpha_0 \varepsilon_0 a(s) s\left[1 - p(s)\right]\right] ds.$$

Thus (22) has a positive solution $Y(t)$, which contradicts the conclusion of the lemma. ∎

Theorem 4. *Let conditions (A) and (3) hold. If*

$$\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \alpha_0 q(t) \sigma(t)\Big[1 - p\big(\sigma(t)\big)\Big] dt \geqslant 1,$$

then every solution $u(x,t)$ of the problem (1), (11) is oscillatory.

Proof: As in the proof of Theorem 3, we obtain (21). Thus

$$Y'(t) + \alpha_0 \varepsilon_0 q(t) \sigma(t)\Big[1 - p\big(\sigma(t)\big)\Big] Y\big(\sigma(t)\big) \leqslant 0, \quad t \geqslant T, \tag{23}$$

where $Y(t) = Z'(t)$, $\varepsilon_0 \in (0,1)$, such that

$$\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \alpha_0 \varepsilon_0 q(s) \sigma(s)\Big[1 - p\big(\sigma(s)\big)\Big] ds \geqslant 1. \tag{24}$$

Integrating (23) from $\sigma(t)$ to t, we have

$$Y(t) - Y\big(\sigma(t)\big) + \alpha_0 \varepsilon_0 \int_{\sigma(t)}^{t} q(s) \sigma(s)\Big[1 - p\big(\sigma(s)\big)\Big] Y\big(\sigma(s)\big) ds \leqslant 0, \quad t \geqslant T.$$

Since $Y(t)$ is decreasing and $\sigma'(t) \geqslant 0$, it follows that

$$Y(t) - Y\big(\sigma(t)\big) + \alpha_0\varepsilon_0 Y\big(\sigma(t)\big)\int_{\sigma(t)}^{t} q(s)\sigma(s)\Big[1 - p\big(\sigma(s)\big)\Big]ds \leqslant 0, \quad t \geqslant T,$$

or
$$\alpha_0\varepsilon_0\int_{\sigma(t)}^{t} q(s)\sigma(s)\big[1 - p\big(\sigma(s)\big)\Big]ds \leqslant 1 - \frac{Y(t)}{Y\big(\sigma(t)\big)} \leqslant 1, \quad t \geqslant T.$$

Hence
$$\limsup_{t\to\infty}\int \alpha_0\varepsilon_0 q(s)\sigma(s)\Big[1 - p\big(\sigma(s)\big)\Big]ds \leqslant 1,$$

which contradicts (24). Thus the proof of Theorem 4 is complete. □

EXAMPLE 1. Consider the equation

$$\frac{\partial^2}{\partial t^2}\left[u(x,t) + \frac{1}{2}u\left(x,\ t - \frac{\pi}{2}\right)\right]$$
$$= u_{xx}(x,t) - \frac{1}{2}u\left(x,\ t - \frac{\pi}{2}\right), \quad (x,t) \in (0,\pi) \times [0,\infty), \tag{25}$$

with a boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t \geqslant 0. \tag{26}$$

It is easy to check the functions: $f(u) = u$, $q(t) = 1/2 = p(t)$, $a(t) = 1$, $\tau = \pi/2$ and $\alpha = 1$. Moreover, $u \equiv 0$ on $\partial\Omega \times [0,\infty)$ where $\Omega = (0,\pi)$, and

$$\int^{\infty} q(s)\Big[1 - p\big(\sigma(s)\big)\Big]ds = \int^{\infty}\frac{1}{2}\left[1 - \frac{1}{2}\right]ds = \infty.$$

Hence the conditions of Theorem 1 are satisfied. Thus all solutions of the problem (25), (26) oscillate in $G = (0,\pi) \times [0,\infty)$. For instance the function $u(x,t) = \sin t\cos x$ is such a solution.

EXAMPLE 2. Consider the equation

$$\frac{\partial^2}{\partial t^2}\left[u(x,t) + e^{-\pi}u(x,t-\pi)\right] = u_{xx}(x,t) - e^{-\pi}u(x,t-\pi),$$
$$(x,t) \in (0,\pi) \times [0,\infty), \tag{27}$$

with a boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0. \tag{28}$$

It is easily verified that the functions

$$p(t) = e^{-\pi},\ q(t) = e^{-\pi},\ a(t) = 1,\ \sigma(t) = t - \pi \quad \text{and} \quad f(u) = u$$

satisfy the conditions of Theorem 4. Hence all solutions of the problem (27), (28) oscillate in $G = (0,\pi) \times [0,\infty)$. For instance, the function $u(x,t) = \sin t\sin x$ is such a solution.

References

[1] D. Georgiou and K. Kreith, 'Functional characteristic initial value problems', *J. Math. Anal. Appl.* **107** (1985), 414-424.

[2] M.K. Grammatikopoulos, G. Ladas and A. Meimaridou, 'Oscillations of Second Order neutral delay differential equations', *Rad. Mat.* **1** (1985), 267-274.

[3] D.P. Mishev and D.D. Bainov, 'Oscillation properties of the solutions of a class of hyperbolic equations of neutral type', *Funkcial. Ekvac.* **29** (1986), 213-218.

[4] D.P. Mishev and D.D. Bainov, 'Oscillation of the solutions of parabolic differential equations of neutral type', *Appl. Math. Comput.* **28** (1988), 97-111.

[5] Wei Junjie, 'Oscillation of second order delay differential equations', *Ann. Differential Eqations* **4** (1988), 473-478.

[6] N. Yoshid, 'Forced oscillations of solutions of parabolic equations', *Bull. Austral. Math. Soc.* **36** (1987), 289-294.

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