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MELLIN MULTIPLIERS AND RADIALLY SYMMETRIC RIESZ POTENTIALS

by W. LAMB and S. E. SCHIAVONE

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Riesz potentials with radially symmetric densities are examined from the standpoint of Mellin multipliers. Various results are deduced from the underlying multipliers, including a decomposition of the potential into a product of Erdélyi-Kober fractional integrals. Distributional versions of these results are also produced and shown to be valid under less severe restrictions on the parameters than those required in a weighted L^p setting.

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1. Introduction

A number of operators which arise in fractional calculus have now been treated as particular cases of more general theories of multiplier transforms. Prominent amongst these are the Erdélyi–Kober operators which have been extensively studied from the standpoint of Mellin multipliers, initially by Rooney **[9]**, **[10]**, and then later by McBride **[5]**, **[6]**, **[7]** who extended Rooney's L^p-based results to spaces of test functions and generalised functions. Other related operators have also received attention, including the Riemann–Liouville and Weyl fractional integrals which have been investigated using Mellin multipliers in **[5]**, and Fourier multipliers in **[3]**. More recently, a theory of bilateral Laplace multipliers has been developed in **[12]** and used to yield information on the two-dimensional Riesz fractional integral associated with the wave operator.

In the present paper, we turn our attention to the Riesz fractional integral which is linked with the n-dimensional Laplacian and demonstrate that the Mellin multiplier theories of Rooney and McBride are applicable to this operator when restricted to radially symmetric functions of the form $\phi(|\mathbf{x}|)$, $\mathbf{x}\in\mathbf{R}^n$. In this case, the corresponding Riesz potential is also radially symmetric and can be written as

$$(R_n^{2\alpha}\phi)(r)=C_{n,2\alpha}\int_{\mathbf{R}^n}|\mathbf{x}-\mathbf{y}|^{2\alpha-n}\phi(\rho)d\mathbf{y} \tag{1.1}$$

where $r=|\mathbf{x}|$, $\rho=|\mathbf{y}|$ and

$$C_{n,2\alpha}=2^{-2\alpha}\pi^{-n/2}\Gamma(n/2-\alpha)/\Gamma(\alpha). \tag{1.2}$$

Riesz potentials with radially symmetric densities have previously been studied by Rubin [11], who established that the operator $R_n^{2\alpha}$ defined by (1.1) is related to the Riemann–Liouville and Weyl fractional integrals via the formula

$$(SR_n^{2\alpha}\phi)(r)=2^{-2\alpha}r^{1-n/2}(I^{\alpha}r^{n/2-\alpha-1}K^{\alpha}S\phi)(r) \tag{1.3}$$

where

$$(S\phi)(r)=\phi(\sqrt{r}). \tag{1.4}$$

Formula (1.3) enabled Rubin to deduce properties of $R_n^{2\alpha}\phi$ for densities ϕ belonging to certain weighted L^p spaces.

Although not mentioned explicitly in [11], (1.3) can be expressed in the equivalent, but simpler form

$$R_n^{2\alpha}\phi=(r/2)^{2\alpha}I_2^{n/2-1,\alpha}K_2^{-\alpha,\alpha}\phi \tag{1.5}$$

where $I_2^{n/2-1,\alpha}$ and $K_2^{-\alpha,\alpha}$ are Erdélyi–Kober operators. This decomposition of $R_n^{2\alpha}$ into one-dimensional fractional integrals, as well as other more familiar results, will be seen to follow naturally from properties of the associated Mellin multipliers. In addition, we shall demonstrate that restrictions required in the L^p treatment of $R_n^{2\alpha}$ may be relaxed considerably when working within the framework of the spaces $F_{p,\mu}$ and $F'_{p,\mu}$ introduced by McBride.

2. The operator $\mathbf{R}_n^{2\alpha}$ on $\mathbf{L}_\mu^p$

As far as possible, we shall adhere to the notation and terminology of [5]. Thus, for $1\leqq p<\infty$ and $\mu\in\mathbf{C}$,

$$L_\mu^p=\{\phi:r^{-\mu}\phi\in L^p(0,\infty)\}$$

and

$$F_{p,\mu}=\{\phi\in C^\infty(0,\infty):r^k\phi^{(k)}\in L_\mu^p \quad \text{for} \quad k=0,1,2,\ldots\}.$$

Equipped with the norm

$$\|\phi\|_{p,\mu}=\left(\int_0^\infty |r^{-\mu}\phi(r)|^p\,dr\right)^{1/p},$$

L_μ^p is a Banach space, while $F_{p,\mu}$ is a Fréchet space with respect to the topology generated by the seminorms $\gamma_k^{p,\mu}$ given by

$$\gamma_k^{p,\mu}(\phi)=\|r^k\phi^{(k)}\|_{p,\mu}, \quad k=0,1,2,\ldots.$$

It is evident that the mapping r^λ, defined by

$$(r^\lambda\phi)(r)=r^\lambda\phi(r), \quad r>0,$$

is a homeomorphism from L^p_μ onto $L^p_{\mu+\lambda}$, and from $F_{p,\mu}$ onto $F_{p,\mu+\lambda}$, for each $\lambda\in\mathbf{C}$.

Properties of the radially symmetric potential $R^{2\alpha}_n\phi$, for $\phi\in L^p_\mu$, can be established by routine application of integral inequalities for Mellin convolutions. Concentrating initially on the case $n=1$, we rewrite (1.1) in the more suitable form

$$(R^{2\alpha}_1\phi)(r)=C_{1,2\alpha}r^{2\alpha}\int_0^\infty \phi(\rho)k(r/\rho)\rho^{-1}\,d\rho \tag{2.1}$$

where

$$k(r)=r^{-2\alpha}(|r+1|^{2\alpha-1}+|r-1|^{2\alpha-1}). \tag{2.2}$$

The behaviour of $R^{2\alpha}_1$ on L^p_μ now follows directly from [**9**, Lemma 3.1].

Theorem 2.1. *If* $0<\operatorname{Re}2\alpha<1/p-\operatorname{Re}\mu<1$ *then* $R^{2\alpha}_1$ *is a bounded linear mapping from* L^p_μ into $L^p_{\mu+2\alpha}$.

Proof. First we note that

$$R^{2\alpha}_1=r^{2\alpha}T^{2\alpha}_1 \tag{2.3}$$

where

$$(T^{2\alpha}_1\phi)(r)=C_{1,2\alpha}\int_0^\infty \phi(\rho)k(r/\rho)\rho^{-1}\,d\rho. \tag{2.4}$$

Now

$$\int_0^\infty r^{\nu-1}|k(r)|\,dr\,(\nu=1/p-\operatorname{Re}\mu)$$

$$=I_1+I_2+I_3,$$

where, for $0<\operatorname{Re}2\alpha<1/p-\operatorname{Re}\mu<1$,

$$I_1=\int_0^\infty r^{\nu-\operatorname{Re}2\alpha-1}(r+1)^{\operatorname{Re}2\alpha-1}\,dr=\beta(\nu-\operatorname{Re}2\alpha,1-1/p+\operatorname{Re}\mu), \tag{2.5}$$

$$I_2=\int_0^1 r^{\nu-\operatorname{Re}2\alpha-1}(1-r)^{\operatorname{Re}2\alpha-1}\,dr=\beta(\nu-\operatorname{Re}2\alpha,\operatorname{Re}2\alpha), \tag{2.6}$$

$$I_3=\int_1^\infty r^{\nu-\operatorname{Re}2\alpha-1}(r-1)^{\operatorname{Re}2\alpha-1}\,dr=\beta(1-1/p+\operatorname{Re}\mu,\operatorname{Re}2\alpha). \tag{2.7}$$

Therefore, on applying [9, Lemma 3.1] (with μ replaced by $1-p\mu$ to conform with the definition of L^p_μ used here; see [5 (I), Remark 2.2]) it follows that $T_1^{2\alpha}$ is a bounded linear mapping from L^p_μ into L^p_μ under the stated conditions.

For the case $n>1$, we use polar coordinates and a suitable rotation (see [11]) to write (1.1) as

$$(R_n^{2\alpha}\phi)(r)=r^{2\alpha}(T_n^{2\alpha}\phi)(r) \tag{2.8}$$

where

$$(T_n^{2\alpha}\phi)(r)=\int_0^\pi\int_0^\infty k_n(r/\rho,\theta)\phi(\rho)\rho^{-1}\,d\rho\,d\theta, \tag{2.9}$$

$$k_n(r,\theta)=C'_{n,2\alpha}(\sin\theta)^{n-2}r^{-2\alpha}(r^2+2r\cos\theta+1)^{\alpha-n/2}, \tag{2.10}$$

$$C'_{n,2\alpha}=2^{1-2\alpha}\pi^{-1/2}\Gamma(n/2-\alpha)/\Gamma(n/2-1/2)\Gamma(\alpha). \tag{2.11}$$

To determine the behaviour of $T_n^{2\alpha}$ on L^p_μ, we require the following generalisation of the inequality used in the previous theorem.

Lemma 2.2. *Let K be defined by*

$$(K\phi)(r)=\int_a^b\int_0^\infty k(r/\rho,\theta)\phi(\rho)\rho^{-1}\,d\rho\,d\theta,\quad r>0,$$

where k is measurable on $(0,\infty)\times(a,b)$ and

$$\int_a^b\int_0^\infty r^{1/p-\operatorname{Re}\mu-1}|k(r,\theta)|\,dr\,d\theta=C<\infty.$$

Then K is a bounded linear mapping from L^p_μ into L^p_μ, and

$$\|K\phi\|_{p,\mu}\leqq C\|\phi\|_{p,\mu},\quad \forall\phi\in L^p_\mu.$$

Proof. For $\phi\in L^p_\mu$, let

$$\psi(r,\theta)=\int_0^\infty k(r/\rho,\theta)\phi(\rho)\rho^{-1}\,d\rho,\quad r>0, a<\theta<b.$$

Then

$$(K\phi)(r)=\int_a^b \psi(r,\theta)\,d\theta, \quad r>0,$$

and

$$\|K\phi\|_{p,\mu}=\left\|\int_a^b r^{-\mu}\psi(r,\theta)\,d\theta\right\|_{p,0}$$

$$\leqq\int_a^b \|r^{-\mu}\psi(r,\theta)\|_{p,0}\,d\theta \quad \text{(by [8, pp. 158–159])}$$

$$=\int_a^b \left\|\int_0^\infty k(r/\rho,\theta)\phi(\rho)\rho^{-1}\,d\rho\right\|_{p,\mu} d\theta$$

$$\leqq\left(\int_a^b\int_0^\infty r^{1/p-\operatorname{Re}\mu-1}|k(r,\theta)|\,dr\,d\theta\right)\|\phi\|_{p,\mu} \quad \text{(by [9, Lemma 3.1])}$$

$$=C\|\phi\|_{p,\mu}.$$

Theorem 2.3. *If* $0<\operatorname{Re}2\alpha<1/p-\operatorname{Re}\mu<n$ *then* $R_n^{2\alpha}$ *is a bounded linear mapping from* L_μ^p *into* $L_{\mu+2\alpha}^p$.

Proof. The case $n=1$ has been dealt with in Theorem 2.1. To prove the result for $n>1$, we apply Lemma 2.2 to the operator $T_n^{2\alpha}$ given by (2.9). For $\operatorname{Re}2\alpha<1/p-\operatorname{Re}\mu<n$ and $v=1/p-\operatorname{Re}\mu$, we have

$$\int_0^\infty (r^2+2r\cos\theta+1)^{\operatorname{Re}\alpha-n/2}r^{v-\operatorname{Re}2\alpha-1}\,dr=A(\alpha,n,v)P_\delta^\gamma(\cos\theta)(\sin\theta)^\gamma$$

by [1, p. 310 (22)], but see Note 2.4 below, where $\gamma=\operatorname{Re}\alpha+1/2-n/2$, $\delta=v-\operatorname{Re}\alpha-n/2-1/2$, $A(\alpha,n,v)=2^{-\gamma}\Gamma(1/2+n/2-\operatorname{Re}\alpha)\beta(v-\operatorname{Re}2\alpha,n-v)$ and P_δ^γ is the Legendre function of the first kind [2, p. 143].

Moreover, for $\operatorname{Re}\alpha>0$,

$$\int_0^\pi (\sin\theta)^{\gamma+n-2}P_\delta^\gamma(\cos\theta)\,d\theta$$

$$=\frac{2^\gamma\pi\Gamma(n/2-1/2)\Gamma(\operatorname{Re}\alpha)}{\Gamma([n+\gamma+\delta]/2)\Gamma([n+\gamma-\delta-1]/2)\Gamma(1+[\delta-\gamma]/2)\Gamma([1-\gamma-\delta]/2)},$$

by [2, p. 172 (27)]. Combining these, we obtain

$$\int_0^\pi \int_0^\infty |k_n(r,\theta)| r^{\nu-1}\, dr\, d\theta < \infty$$

and hence $T_n^{2\alpha}$ is a bounded linear mapping from L_μ^p into L_μ^p for $0<\operatorname{Re} 2\alpha < 1/p - \operatorname{Re}\mu < n$. This completes the proof.

Note 2.4. The term $(\sin\theta)^{\nu-1/2}$ in [**1**, p. 310 (22)] is incorrect and should be replaced by $(\sin\theta)^{1/2-\nu}$.

Our last result in this section is an identity involving the gamma function which will be required later to simplify the Mellin multiplier associated with $R_1^{2\alpha}$.

Lemma 2.5. *Let* $\{a-b, b, 1-a\} \subset \mathbf{C} - \{0,-1,-2,\ldots\}$. *Then*

$$\beta(a-b,1-a)+\beta(1-a,b)+\beta(b,a-b) \tag{2.12}$$

$$=\frac{\sqrt{\pi}\,\Gamma(1/2-a/2)\Gamma(b/2)\Gamma(a/2-b/2)}{\Gamma(a/2)\Gamma(1/2-b/2)\Gamma(1/2-a/2+b/2)}. \tag{2.13}$$

Proof. We can express (2.12) in the form $(N_1+N_2+N_3)/D$, where

$$D=\Gamma(1-b)\Gamma(1-a+b)\Gamma(a), \quad N_1=\Gamma(a-b)\Gamma(1-a+b)\Gamma(a)\Gamma(1-a),$$

$$N_2=\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b), \quad N_3=\Gamma(b)\Gamma(1-b)\Gamma(a-b)\Gamma(1-a+b).$$

By repeated application of Legendre's duplication formula,

$$D=2^{-1}\pi^{-3/2}\Gamma(1/2-b/2)\Gamma(1-b/2)\Gamma(a/2)\Gamma(1/2+a/2)\Gamma(1-a/2+b/2)\Gamma(1/2-a/2+b/2).$$

Moreover, the identities

$$\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z), \quad \Gamma(1/2+z)\Gamma(1/2-z)=\pi/\cos(\pi z) \tag{2.14}$$

can be combined to produce

$$\Gamma(z)\Gamma(1-z)\Gamma(1/2+z)\Gamma(1/2-z)=2\pi\Gamma(2z)\Gamma(1-2z). \tag{2.15}$$

If we assume that $\{a-b, b, 1-a\} \subset \mathbf{C} - \mathbf{Z}$, then (2.14) and (2.15) lead to

$$\frac{N_1}{D}=\frac{\Gamma(1-a/2)\Gamma(1/2-a/2)\Gamma(a/2-b/2)\Gamma(1/2+a/2-b/2)}{2\sqrt{\pi}\,\Gamma(1/2-b/2)\Gamma(1-b/2)}$$

$$=\frac{H(a,b)\sin(\pi b/2)}{2\sin(\pi a/2)\cos(\pi(a-b)/2)}, \tag{2.16}$$

where $H(a,b)$ is given by (2.13), and similarly

$$N_2/D = H(a,b)\sin(\pi(a-b)/2)/2\sin(\pi a/2)\cos(\pi b/2), \tag{2.17}$$

$$N_3/D = H(a,b)\cos(\pi a/2)/2\cos(\pi(a-b)/2)\cos(\pi b/2). \tag{2.18}$$

The result now follows on adding (2.16), (2.17) and (2.18), and using the fact that the functions defined by (2.12) and (2.13) are analytic in $\mathbf{C}-\{0,-1,-2,\ldots\}$.

3. Mellin multipliers and $\mathbf{R}_n^{2\alpha}$ on $\mathbf{F}_{p,\mu}$

For completeness, we include one or two details of the theory of Mellin multipliers. The Mellin transform $\mathcal{M}\phi$ of a function ϕ is defined formally by

$$(\mathcal{M}\phi)(s) = \int_0^\infty r^{s-1}\phi(r)\,dr. \tag{3.1}$$

When $\phi \in L_\mu^p$ and $1 < p \leqq 2$, it is known that $\mathcal{M}\phi$ exists almost everywhere on the line $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$, the integral in (3.1) being interpreted in terms of mean convergence. As in [5], Ω will denote a domain in the complex plane which is the union of a finite or countably infinite collection of disjoint, open strips parallel to the imaginary axis. For any such domain Ω and $p \in [1,\infty)$, we let

$$\Omega_p = \{\mu\colon 1/p - \mu \in \Omega\}. \tag{3.2}$$

Since we assume throughout that $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$, it follows that

$$s \in \Omega \Leftrightarrow \mu \in \Omega_p. \tag{3.3}$$

Definition 3.1. The complex-valued function g is an L_μ^p multiplier if

(a) g is analytic on a domain Ω,

(b) there exists a (unique) linear operator R, depending on g, such that

  (i) for each $p \in (1,\infty)$ and $\mu \in \Omega_p$, R is a continuous linear mapping from L_μ^p into L_μ^p,

  (ii) for $1 < p \leqq 2$, $\mu \in \Omega_p$ and $\phi \in L_\mu^p$,

$$(\mathcal{M}(R\phi))(s)) = g(s)(\mathcal{M}\phi)(s). \tag{3.4}$$

Note 3.2. (a) To obtain the corresponding definition of an $F_{p,\mu}$ multiplier, we simply replace L_μ^p by $F_{p,\mu}$ throughout Definition 3.1.

(b) The operator R is referred to as the (Mellin) multiplier transform associated with the multiplier (or symbol) g.

(c) Sufficient conditions for a function g to be an L^p_μ multiplier are given in [7] and [**10**]. Results on $F_{p,\mu}$ multipliers can be found in [5] and [7]. For example, L^p_μ multipliers are $F_{p,\mu}$ multipliers [**5** (II), Theorem 3.3]. Moreover, if g is an L^p_μ multiplier with associated transform R, and P is a polynomial, then Pg is the symbol of $P(-\delta)R$, where $\delta \equiv r\,d/dr$; see [**5**, (II), Theorem 3.6].

To illustrate these ideas, and also introduce one or two results which are required later, let us review some of the facts established on the Erdélyi–Kober operators $I^{\eta;\alpha}_m$ and $K^{\eta;\alpha}_m$, where

$$(I^{\eta;\alpha}_m\phi)(r)=(m/\Gamma(\alpha))\int_0^r(r/\rho)^{-m\eta-m\alpha}((r/\rho)^m-1)^{\alpha-1}\rho^{-1}\phi(\rho)\,d\rho, \tag{3.5}$$

and

$$(K^{\eta;\alpha}_m\phi)(r)=(m/\Gamma(\alpha))\int_r^\infty(r/\rho)^{m\eta}(1-(r/\rho)^m)^{\alpha-1}\rho^{-1}\phi(\rho)\,d\rho. \tag{3.6}$$

Example 3.3. Rooney [9] has shown that the functions

$$g_I(\eta,\alpha,m;s)=\Gamma(\eta+1-s/m)/\Gamma(\eta+\alpha+1-s/m),\quad \operatorname{Re}s<m\operatorname{Re}\eta+m \tag{3.7}$$

and

$$g_K(\eta,\alpha,m;s)=\Gamma(\eta+s/m)/\Gamma(\eta+\alpha+s/m),\quad \operatorname{Re}s>-m\operatorname{Re}\eta \tag{3.8}$$

are L^p_μ multipliers with respective transforms $I^{\eta;\alpha}_m$ and $K^{\eta;\alpha}_m$, defined by (3.5) and (3.6), provided that $\operatorname{Re}\alpha>0$. The conditions on α and s (and hence on α, p and μ) can be relaxed considerably by working in $F_{p,\mu}$ rather than L^p_μ. For example, formulae such as

$$g_I(\eta,\alpha,m;s)=P_l(s)g_I(\eta,\alpha+l,m;s)\quad (l=1,2,\ldots),$$

where

$$P_l(s)=\prod_{j=1}^{l}(\eta+\alpha+j-s/m),$$

can be used to show that g_I is an $F_{p,\mu}$ multiplier for each $\alpha\in\mathbf{C}$ and has associated domain

$$\Omega_I=\{s:\operatorname{Re}s\neq m\operatorname{Re}\eta+km\quad\text{for}\quad k=1,2,\ldots\}. \tag{3.9}$$

Similarly, g_K is an $F_{p,\mu}$ multiplier for each $\alpha\in\mathbf{C}$, and has domain

$$\Omega_K=\{s:\operatorname{Re}s\neq -m\operatorname{Re}\eta-mk\quad\text{for}\quad k=0,1,\ldots\}. \tag{3.10}$$

Concrete representations for the corresponding multiplier operators are again given by the Erdélyi–Kober operators. However, definitions in terms of integrodifferential expressions have now to be used which guarantee the existence of $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ as continuous operators on $F_{p,\mu}$ whenever $p\in[1,\infty)$ and $\mu\in\mathbf{C}$ satisfy the respective conditions $1/p-\mu\in\Omega_I$, and $1/p-\mu\in\Omega_K$; see [4, Chapter 3] for details.

We now consider the operator $R_n^{2\alpha}$ given by (1.1).

Theorem 3.4. *If* $0<\operatorname{Re}2\alpha<n$ *then the function*

$$g(\alpha,n;s)=\frac{2^{-2\alpha}\Gamma(s/2-\alpha)\Gamma(n/2-s/2)}{\Gamma(s/2)\Gamma(n/2-s/2+\alpha)},\quad \operatorname{Re}2\alpha<\operatorname{Re}s<n, \tag{3.11}$$

is an L_μ^p *multiplier and has associated transform* $r^{-2\alpha}R_n^{2\alpha}$.

Proof. A straightforward application of the results given in [9, p. 1203] shows that $g(\alpha,n;\cdot)$ is an L_μ^p multiplier. Moreover, from Theorem 2.3, $r^{-2\alpha}R_n^{2\alpha}$ is a bounded linear mapping from L_μ^p into L_μ^p for $0<\operatorname{Re}2\alpha<1/p-\operatorname{Re}\mu<n$. To complete the proof, we examine the cases $n=1$ and $n>1$ separately and verify that

$$(\mathcal{M}(r^{-2\alpha}R_n^{2\alpha}\phi))(s)=g(\alpha,n;s)(\mathcal{M}\phi)(s)$$

for $1<p\leqq 2$, $0<\operatorname{Re}2\alpha<1/p-\operatorname{Re}\mu<n$ and $\phi\in C_0^\infty(0,\infty)$. The result will then follow immediately from standard continuity arguments and the fact that $C_0^\infty(0,\infty)$ is dense in L_μ^p.

n = 1: For $0<\operatorname{Re}2\alpha<\operatorname{Re}s<1$,

$$(\mathcal{M}(r^{-2\alpha}R_1^{2\alpha}\phi))(s)=(\mathcal{M}(T_1^{2\alpha}\phi))(s)=C_{1,2\alpha}(\mathcal{M}k)(s)(\mathcal{M}\phi)(s)$$

where k and $T_1^{2\alpha}$ are given by (2.2) and (2.4) respectively. If we now apply formulae (2.5)–(2.7) and Lemma 2.5, then we obtain

$$\begin{aligned}
&C_{1,2\alpha}(\mathcal{M}k)(s)\\
&=C_{1,2\alpha}(\beta(s-2\alpha,1-s)+\beta(s-2\alpha,2\alpha)+\beta(1-s,2\alpha))\\
&=g(\alpha,1;s).
\end{aligned}$$

n > 1: The calculations used in proving Theorem 2.3 show that

$$(\mathcal{M}(r^{-2\alpha}R_n^{2\alpha}\phi))(s)=h(\alpha,n;s)(\mathcal{M}\phi)(s),$$

where

$$h(\alpha, n; s) = \frac{2^{1-2\alpha}\sqrt{\pi}\,\Gamma(n/2-\alpha)\Gamma(n/2+1/2-\alpha)\Gamma(s-2\alpha)\Gamma(n-s)}{\Gamma(n-2\alpha)\Gamma(s/2)\Gamma(s/2+1/2-\alpha)\Gamma(n/2+1/2-s/2)\Gamma(n/2-s/2+\alpha)}.$$

On applying Legendre's duplication formula to $\Gamma(n-2\alpha)$, $\Gamma(s-2\alpha)$ and $\Gamma(n-s)$, we find that $h(\alpha, n; s) = g(\alpha, n; s)$.

Corollary 3.5. *If $1 \leqq p < \infty$ and $0 < \operatorname{Re} 2\alpha < 1/p - \operatorname{Re}\mu < n$, then*

$$R_n^{2\alpha}\phi = 2^{-2\alpha} r^{2\alpha} I_2^{n/2-1,\alpha} K_2^{-\alpha,\alpha}\phi \tag{3.12}$$

for each $\phi \in L_\mu^p$.

Proof. This follows immediately from the fact that

$$g(\alpha, n; s) = 2^{-2\alpha} g_I(n/2-1, \alpha, 2; s) g_K(-\alpha, \alpha, 2; s).$$

As in Example 3.3, the various restrictions on the parameters can be relaxed by working in $F_{p,\mu}$, and it is not difficult to show that the function $g(\alpha, n; s)$ is an $F_{p,\mu}$ multiplier with associated domain given by the larger set

$$\Omega(\alpha, n) = \{s\colon \operatorname{Re} s \neq n+2l, \operatorname{Re} 2\alpha - 2l \quad \text{for} \quad l = 0, 1, 2, \ldots\}. \tag{3.13}$$

If we continue to represent the corresponding multiplier transform by $r^{-2\alpha} R_n^{2\alpha} \equiv T_n^{2\alpha}$, then (3.12) provides a concrete expression for $R_n^{2\alpha}$ in terms of the more general versions of the Erdélyi–Kober operators. More explicitly, for $1 \leqq p < \infty$ and

$$\mu \in \Omega(\alpha, n, p) \equiv \{\mu\colon 1/p - \mu \in \Omega(\alpha, n)\}, \tag{3.14}$$

we define $R_n^{2\alpha}$ on $F_{p,\mu}$ by

$$R_n^{2\alpha}\phi = 2^{-2\alpha} r^{2\alpha} I_2^{n/2-1,\alpha} K_2^{-\alpha,\alpha}\phi, \quad \phi \in F_{p,\mu}, \tag{3.15}$$

where $I_2^{n/2-1,\alpha}$ and $K_2^{-\alpha,\alpha}$ are defined in accordance with the values of α, p, μ and n. In particular, when $0 < \operatorname{Re} 2\alpha < 1/p - \operatorname{Re}\mu < n$, $R_n^{2\alpha}$ can be represented by (1.1).

The following properties of $R_n^{2\alpha}$ on $F_{p,\mu}$ are now easily derived.

Theorem 3.6. *Let $\Omega(\alpha, n, p)$ be given by (3.14) for each $\alpha \in \mathbf{C}$, $p \in [1, \infty)$ and $n \in \mathbf{N}$.*

(a) *If $\mu \in \Omega(\alpha, n, p)$ then $R_n^{2\alpha}$ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+2\alpha}$.*

(b) *If $\mu \in \Omega(\alpha, n, p)$ and $\mu + 2\alpha \in \Omega(\beta, n, p)$ then*

$$R_n^{2\beta} R_n^{2\alpha} = R_n^{2\alpha+2\beta} \tag{3.16}$$

as operators on $F_{p,\mu}$.

(c) *If* $\mu\in\Omega(\alpha,n,p)$ *and* $\mu+2\alpha\in\Omega(-\alpha,n,p)$ *then* $R_n^{2\alpha}$ *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu+2\alpha}$ *and*

$$(R_n^{2\alpha})^{-1}\phi=R_n^{-2\alpha}\phi \quad \forall\phi\in F_{p,\mu+2\alpha}. \tag{3.17}$$

(d) *Let*

$$\Delta_n=\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}. \tag{3.18}$$

Then, as operators on $F_{p,\mu}$,

$$\Delta_n R_n^{2\alpha}=-R_n^{2\alpha-2} \quad \text{if} \quad \mu\in\Omega(\alpha,n,p) \tag{3.19}$$

and

$$R_n^{2\alpha}\Delta_n=-R_n^{2\alpha-2} \quad \text{if} \quad \mu-2\in\Omega(\alpha,n,p). \tag{3.20}$$

Proof. (a) This follows immediately from [**4**, Theorems 3.31, 3.34].

(b) Under the given conditions, each side of (3.16) defines a continuous mapping from $F_{p,\mu}$ into $F_{p,\mu+2\alpha+2\beta}$. Moreover, for $\phi\in C_0^\infty(0,\infty)$,

$$\begin{aligned}(\mathcal{M}(r^{-2\alpha-2\beta}R_n^{2\beta}R_n^{2\alpha}\phi))(s)&=(\mathcal{M}(r^{-2\beta}R_n^{2\beta}R_n^{2\alpha}\phi))(s-2\alpha)\\&=g(\beta,n;s-2\alpha)(\mathcal{M}(r^{-2\alpha}R_n^{2\alpha}\phi))(s) \quad (\text{for } s-2\alpha\in\Omega(\beta,n))\\&=g(\beta,n;s-2\alpha)g(\alpha,n;s)(\mathcal{M}\phi)(s) \quad (\text{for } s\in\Omega(\alpha,n))\\&=g(\alpha+\beta,n;s)(\mathcal{M}\phi)(s)\\&=(\mathcal{M}(r^{-2\alpha-2\beta}R_n^{2\alpha+2\beta}\phi))(s),\end{aligned}$$

and therefore, since $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$,

$$R_n^{2\beta}R_n^{2\alpha}\phi=R_n^{2\alpha+2\beta}\phi \quad \forall\phi\in F_{p,\mu}.$$

(c) This can be deduced from (b) on setting $\beta=-\alpha$ and noting that R_n^0 is the identity operator on $F_{p,\mu}$ for $\mu\in\Omega(0,n,p)$.

(d) The proof is similar to that given for (b). Under the stated conditions, each side of (3.19) defines a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+2\alpha-2}$. To establish that these operators are identical, we need only show that they agree on $C_0^\infty(0,\infty)$, and this follows since

$$\begin{aligned}(\mathcal{M}(r^{2-2\alpha}\Delta_n R_n^{2\alpha}\phi))(s) &= (\mathcal{M}(\Delta_n R_n^{2\alpha}\phi))(s+2-2\alpha)\\ &= (s-2\alpha)(s+2-2\alpha-n)(\mathcal{M}(R_n^{2\alpha}\phi))(s-2\alpha)\\ &= (s-2\alpha)(s+2-2\alpha-n)g(\alpha,n;s)(\mathcal{M}\phi)(s)\\ &= -g(\alpha-1,n;s)(\mathcal{M}\phi)(s)\\ &= -(\mathcal{M}(r^{2-2\alpha}R_n^{2\alpha-2}\phi))(s).\end{aligned}$$

Equation (3.20) can be established in the same way.

Note 3.7. On comparing the index law (3.16) and inverse formula (3.17) with [5 (II), Theorem 4.8], it would appear that the operator $R_n^{2\alpha}$ is a suitable candidate for analysis via the fractional power theory presented in [5]. Indeed, if we follow the terminology of [5] and examine the triple

$$h(s) = 2^s\Gamma(s/2)/\Gamma(n/2-s/2)$$

$$\Omega = \{s\colon \operatorname{Re} s \neq 2-2l, n+2l \quad \text{for} \quad l=0,1,2,\dots\}$$

$$\gamma = 2,$$

then the associated operator is given by

$$T(h,\Omega,\gamma) = R_n^2 = (r/2)^2 I_2^{n/2-1,1} K_2^{-1,1}.$$

Moreover,

$$h(s-2\alpha)/h(s) = g(\alpha,n;s)$$

which suggests that, for suitable $\alpha \in \mathbf{C}$,

$$R_n^{2\alpha} = (R_n^2)^\alpha \tag{3.21}$$

where the right-hand side of (3.21) represents the αth power of R_n^2. Unfortunately, a problem arises when we attempt to identify the admissible set A_F for this triple. As A_F consists of all $\alpha \in \mathbf{C}$ for which the function

$$h(s-2\alpha)/h(s) = g(\alpha,n;s)$$

is an $F_{p,\mu}$ multiplier **with associated domain Ω**, it follows that

$$A_F = \{\alpha \in \mathbf{C}\colon \Omega \subseteq \Omega(\alpha,n)\}.$$

Clearly there are very few admissible values of α and this places severe restrictions on the fractional power approach. Although it seems likely that the theory in [5] can be modified to allow the domain of the multiplier associated with T^α to vary with α, this will not be pursued here.

4. Distributional results

The advantages of studying the operator $R_n^{2\alpha}$ within the framework of $F_{p,\mu}$ rather than L_μ^p, have been highlighted in the previous section. Clearly these are achieved at the cost of restricting the size of the domain of $R_n^{2\alpha}$. Fortunately, we can easily remedy this by producing a distributional version of the $F_{p,\mu}$ theory.

First we note that $F_{q,-\mu}$ is a space of test functions for each $q\in[1,\infty)$ and $\mu\in\mathbf{C}$, and so its dual $F'_{q,-\mu}$ can be regarded as a space of generalised functions. Moreover, when p and q are related by $1/p+1/q=1$, each function $\eta\in L_\mu^p$ generates a functional $\tilde{\eta}\in F'_{q,-\mu}$ via the formula

$$(\tilde{\eta},\phi)=\int_0^\infty \eta(r)\phi(r)\,dr,\quad \phi\in F_{q,-\mu},$$

and therefore we obtain the chain of inclusions

$$F_{p,\mu}\subset L_\mu^p\subset F'_{q,-\mu}.$$

As a result, it is meaningful to discuss the extension of operators from L_μ^p to $F'_{q,-\mu}$ and also from $F_{p,\mu}$ to $F'_{q,-\mu}$. For example, appropriate definitions of the Erdélyi–Kober operators on $F'_{q,-\mu}$ are shown in [4, Definitions 3.45, 3.49] to be

$$(I_m^{\eta,\alpha}f,\phi)=(f,K_m^{\eta+1-1/m,\alpha}\phi), \tag{4.1}$$

$$(K_m^{\eta,\alpha}f,\phi)=(f,I_m^{\eta-1+1/m,\alpha}\phi), \tag{4.2}$$

where $f\in F'_{q,-\mu}$ and $\phi\in F_{q,-\mu}$.

Multiplier transforms $T\equiv T(h,\Omega,\gamma)$ of the type discussed in Note 3.7 can also be extended from L_μ^p to $F'_{q,-\mu}$ by means of the formulae

$$(Tf,\phi)=(f,T'\phi)$$

$$(T'\phi)(r)=r^\gamma(UTUr^\gamma\phi)(r)$$

$$(U\phi)(r)=r^{-1}\phi(r^{-1}),$$

where $f\in F'_{q,-\mu}$ and $\phi\in F_{q,-\mu-\gamma}$; see [5, (II), §6] for details. In particular, if we consider the triple

$$h(s)=2^{s}\Gamma(s/2)/\Gamma(n/2-s/2),\ \Omega=\Omega(\alpha,n),\ \gamma=2\alpha,$$

which has associated multiplier transform $R_n^{2\alpha}$, then it follows from [5 (III), Theorem 4.17] that the operator $(R_n^{2\alpha})'$ is also a multiplier transform with corresponding triple

$$h(s')=2^{s'-1}\Gamma(n/2-1/2+s'/2)/\Gamma(1/2-s'/2),$$

$$\Omega'\equiv\Omega'(\alpha,n)=\{s'\colon \operatorname{Re} s'\neq 1+2l,\ 1+\operatorname{Re} 2\alpha-n-2l \quad \text{for } l=0,1,\ldots\}$$

$$\gamma'=2\alpha.$$

Routine calculations show that

$$(R_n^{2\alpha})'\phi=2^{-2\alpha}I_2^{-\alpha-1/2,\alpha}K_2^{n/2-1/2,\alpha}r^{2\alpha}\phi,\quad \phi\in F_{q,-\mu-2\alpha},$$

and therefore, on using (4.1) and (4.2), we arrive at the expected result that

$$R_n^{2\alpha}=2^{-2\alpha}r^{2\alpha}I_2^{n/2-1,\alpha}K_2^{-\alpha,\alpha}$$

as extended operators on $F'_{q,-\mu}$.

It is now an easy matter to produce more general versions of the results stated in Theorem 3.6. For example, $R_n^{2\alpha}$ is a continuous linear mapping from $F'_{q,\mu}$ into $F'_{q,-\mu-2\alpha}$ whenever $\mu\in\Omega(\alpha,n,p)$. Moreover if $\mu\in\Omega(\alpha,n,p)$ and $\mu+2\alpha\in\Omega(\beta,n,p)$ then

$$(R_n^{2\alpha})'(R_n^{2\beta})'\phi=(R_n^{2\alpha+2\beta})'\phi \quad \forall\phi\in F_{q,-\mu-2\alpha-2\beta}$$

and therefore, from standard properties of adjoints,

$$R_n^{2\beta}R_n^{2\alpha}f=R_n^{2\alpha+2\beta}f \quad \forall f\in F'_{q,-\mu}$$

under the same conditions on α, β and μ. Finally, (3.19) and (3.20) continue to hold in $F'_{q,-\mu}$ when Δ_n is defined as a continuous linear mapping from $F'_{q,-\mu}$ into $F'_{q,2-\mu}$ by

$$(\Delta_n f,\phi)=(f,\Delta'_n\phi),\quad f\in F'_{q-\mu},\phi\in F_{q,2-\mu},$$

where

$$\Delta'_n=\frac{d^2}{dr^2}+\frac{1-n}{r}\frac{d}{dr}.$$

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Department of Mathematics
University of Strathclyde
Livingstone Tower
26 Richmond Street
Glasgow G1 1XH
Scotland

Department of Mathematics
University of Alberta
Edmonton
Alberta T6G 2G1
Canada