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TAUBERIAN THEOREMS FOR BOREL-TYPE METHODS OF SUMMABILITY

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1. **Introduction.** Suppose throughout that s, a_n $(n=0, 1, 2, \ldots)$ are arbitrary complex numbers, that $\alpha>0$ and β is real and that N is a non-negative integer such that $\alpha N+\beta\geqq 1$. Let

$$s_n = \sum_{\nu=0}^{n} a_\nu, \quad S_{\alpha,\beta}(z) = \alpha e^{-z} \sum_{n=N}^{\infty} s_n \frac{z^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}, \quad A_{\alpha,\beta}(z) = \alpha e^{-z} \sum_{n=N}^{\infty} a_n \frac{z^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$$

where $z=x+iy$ is a complex variable and the power z^γ is assumed to have its principal value.

We shall be concerned with the Borel-type method of summability (B, α, β) defined as follows (see [1]): we write $s_n\to s(B, \alpha, \beta)$, or $\sum_0^\infty a_n=s(B, \alpha, \beta)$, if $S_{\alpha,\beta}(x)$ exists for all $x\geq 0$ and tends to s as $x\to\infty$. Further, we write $s_n=0(1)(B, \alpha, \beta)$ if $S_{\alpha,\beta}(x)$ exists and is bounded on $[0, \infty)$.

The actual choice of the integer N in the above definitions is clearly immaterial. We shall therefore tacitly assume whenever a finite number of methods (B, α, β_r) $(r=1, 2, \ldots, k)$ are under consideration that N is such that $\alpha N+\beta_r\geq 1$ $(r=1, 2, \ldots, k)$.

The following result is known (see [2]):

(A) *If $\beta>\mu$ and $\sum_0^\infty a_n=s(B, \alpha, \mu)$, then $\sum_0^\infty a_n=s$ (B, α, β).*

This result is "abelian" in character. Our object is to establish the "tauberian" results listed in the next section.

One of our tauberian conditions involves the notion of "slow decrease" defined as follows: a real-valued function $f(x)$, with domain $[0, \infty)$, is slowly decreasing if for every $\varepsilon>0$ there exist positive numbers X, δ such that $f(x)-f(y)>-\varepsilon$ whenever $x\geq y\geq X$ and $x-y\leq\delta$.

2. Statements of the main results

THEOREM 1. *If $\sum_0^\infty a_n=s(B, \alpha, \mu)$ and $a_n\to 0(B, \alpha, \beta)$, then $\sum_0^\infty a_n=s(B, \alpha, \beta)$.*

THEOREM 2. *If $s_n\to s(B, \alpha, \beta+\varepsilon)$ for some $\varepsilon>0$ and $s_n=0(1)(B, \alpha, \beta)$, then $s_n\to s(B, \alpha, \beta+\delta)$ for any $\delta>0$.*

THEOREM 2*. *If $\sum_0^\infty a_n=s(B, \alpha, \beta+\varepsilon)$ for some $\varepsilon>0$ and $a_n=0(1)(B, \alpha, \beta)$, then $\sum_0^\infty a_n=s(B, \alpha, \beta+\delta)$ for any $\delta>0$.*

THEOREM 3. *If $s_n\to s(B, \alpha, \beta+\varepsilon)$ for some $\varepsilon>0$ and $S_{\alpha,\beta}(x)$ is slowly decreasing, then $s_n\to s(B, \alpha, \beta)$.*

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THEOREM 3*. *If* $\sum_0^\infty a_n = s(B, \alpha, \beta+\varepsilon)$ *for some* $\varepsilon > 0$ *and* $A_{\alpha,\beta}(x)$ *is slowly decreasing, then* $\sum_0^\infty a_n = s(B, \alpha, \beta)$.

THEOREM 4. *If* $s_n = 0(1)(B, \alpha, \mu)$ *and* $s_n \geq -K$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $s_n = 0(1)(B, \alpha, \beta)$.

THEOREM 5. *If* $s_n \to s(B, \alpha, \mu)$ *and* $s_n \geq -K$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $s_n \to s(B, \alpha, \beta)$.

THEOREM 5*. *If* $\sum_0^\infty a_n = s(B, \alpha, \mu)$ *and* $a_n \geq -K$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $\sum_0^\infty a_n = s(B, \alpha, \beta)$.

The following theorems are extensions of a result due to Gaier [3].

THEOREM 6. *If* $s_n \to s(B, \alpha, \mu)$ *and if there are positive real numbers* A, a, δ *such that* $|S_{\alpha,\mu}(z)| \leq A \exp(a|z|)$ *whenever* $\operatorname{Re} z \geq \delta$, *then* $s_n \to s(B, \alpha, \beta)$.

THEOREM 6*. *If* $\sum_0^\infty a_n = s(B, \alpha, \mu)$ *and if there are positive real numbers* A, a, δ *such that* $|A_{\alpha,\mu}(z)| \leq A \exp(a|z|)$ *whenever* $\operatorname{Re} z \geq \delta$, *then* $\sum_0^\infty a_n = s(B, \alpha, \beta)$.

THEOREM 7. *If* $\sum_0^\infty a_n = s(B, \alpha, \mu)$ *and* $|a_n| \leq K^n$ *for all* $n \geq 0$ *where* K *is a positive constant, then* $\sum_0^\infty a_n = s(B, \alpha, \beta)$.

3. **Preliminary results.** It is known that the (B, α, β) method is regular (see [2]). Also, using the root test and a known result [1, Lemma 4], it can readily be shown that if either $S_{\alpha,\mu}(x)$ or $A_{\alpha,\mu}(x)$ exists for all $x \geq 0$ then both $S_{\alpha,\beta}(x)$ and $A_{\alpha,\beta}(x)$ exist for all $x \geq 0$.

LEMMA 1. *Let* $S_{\alpha,\beta}(x)$ *exist for* $x \geq 0$. *Then, for* $x \geq 0$,

(i) $S_{\alpha,\beta+\delta}(x) = \int_0^x h(x-t) S_{\alpha,\beta}(t)\, dt$ *where* $\delta > 0$ *and* $h(u) = u^{\delta-1} e^{-u} / \Gamma(\delta)$,
(ii) $A_{\alpha,\beta}(x) = S_{\alpha,\beta}(x) - S_{\alpha,\beta+\alpha}(x) + o(1)$ *as* $x \to \infty$.

LEMMA 2. *If* $\sum_0^\infty a_n = s(B, \alpha, \beta)$, *then* $a_n \to 0(B, \alpha, \beta)$.

The proof of Lemma 1 is straightforward, and Lemma 2 follows immediately from Lemma 1(ii) and result (A).

THEOREM 8. *Let* $f(t)$ *be Lebesgue integrable on every finite subinterval of* $[0, \infty)$ *and let* $F(x) = \int_0^x e^{-(x-t)} f(t)\, dt$. *If* $F(x) \to s$ *as* $x \to \infty$ *and* $f(t)$ *is slowly decreasing, then* $f(x) \to s$ *as* $x \to \infty$.

Proof. Since

$$F(x) = \frac{1}{w} \int_0^w g(u)\, du$$

where $w = e^x$ and

$$g(u) = \begin{cases} 0 & 0 \leq u < 1, \\ f(\ln u) & u \geq 1, \end{cases}$$

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Theorem 8 follows from a known result [4, p. 126]. (Observe that $g(u)$ is "slowly decreasing" in the sense given on p. 124 of [4]).

LEMMA 3. *If $f(t)$ is bounded on every finite subinterval of* $[0, \infty)$ *and is slowly decreasing, then there exist positive numbers M_1 and M_2 such that*

$$f(x)-f(y) \geq -M_1(x-y)-M_2 \quad \text{whenever} \quad x \geq y \geq 0.$$

Proof. Since $f(t)$ is slowly decreasing, there exist positive numbers X, δ such that $f(x)-f(y)>-1$ if $x \geq y \geq X$ and $x-y \leq \delta$. Hence, if $x \geq y \geq X$ and m is the smallest positive integer such that $(x-y)/m \leq \delta$, then

$$\begin{aligned} f(x)-f(y) &= \sum_{j=1}^{m}\left\{f\left(y+j\,\frac{x-y}{m}\right)-f\left(y+(j-1)\,\frac{x-y}{m}\right)\right\} \\ &> -m \\ &= -(m-1)-1 \\ &\geq -\frac{x-y}{\delta}-1. \end{aligned}$$

Thus, if $M=\sup_{0\leq x\leq X} |f(x)|$, then

$$f(x)-f(y) > -\frac{1}{\delta}(x-y)-2M-1 \quad \text{whenever} \quad x \geq y \geq 0.$$

THEOREM 9. *Let $h(u)$ be a real-valued, non-negative, Lebesgue measurable function such that*

$$0 < \int_0^\infty h(u)\,du < \infty \quad \textit{and} \quad \int_0^\infty uh(u)\,du < \infty.$$

Let $f(t)$ be a real-valued function such that, for some positive numbers M_1 and M_2,

$$f(x)-f(y) \geq -M_1(x-y)-M_2 \quad \textit{whenever} \quad x \geq y \geq 0,$$

and such that, for all $x \geq 0$,

$$F(x) = \int_0^x h(x-t)f(t)\,dt$$

exists as a Lebesgue integral. Then, $f(x)$ is bounded on $[0, \infty)$ *whenever $F(x)$ is bounded on* $[0, \infty)$.

Proof. Suppose that $M_3=\sup_{x\geq 0} |F(x)|<\infty$.

Choose X such that

$$L = \int_0^X h(u)\,du > 0.$$

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Now

$$\begin{aligned} f(x)\int_0^x h(x-t)\,dt &= \int_0^x h(x-t)\{f(x)-f(t)\}\,dt+F(x) \\ &\geq \int_0^x h(x-t)\{-M_1(x-t)-M_2\}\,dt+F(x) \\ &\geq -M_1\int_0^\infty uh(u)\,du-M_2\int_0^\infty h(u)\,du-M_3 \\ &= -M_4 \end{aligned}$$

say, and hence $f(x)\geq -M_4/L$ if $x\geq X$. But $f(x)\geq -M_1X-M_2+f(0)$ for $0\leq x\leq X$. Hence there exists a positive number M_5 such that $f(x)\geq -M_5$ for all $x\geq 0$.

If $x\geq X$, then

$$\begin{aligned} M_3 &\geq F(x) \\ &= \int_0^{x-X} h(x-t)f(t)\,dt+\int_{x-X}^x h(x-t)f(t)\,dt \\ &\geq -M_5\int_0^{x-X} h(x-t)\,dt+\int_{x-X}^x h(x-t)\{f(x-X)-M_1(t-x+X)-M_2\}\,dt \\ &\geq M_6+f(x-X)\int_0^X h(u)\,du \end{aligned}$$

where

$$M_6 = -M_5\int_X^\infty h(u)\,du+M_1\int_0^X uh(u)\,du-(M_1X+M_2)\int_0^X h(u)\,du.$$

It follows that $(M_3-M_6)/L\geq f(x)$ for $x\geq 0$.

Thus $f(x)$ is bounded on $[0, \infty)$.

THEOREM 10. *Let $f(t)$ be a real-valued non-decreasing fumction defined on* $[0, \infty)$ *and, for $\delta>0$, let*

$$F(x) = \frac{1}{\Gamma(\delta)}\, e^{-x}\int_0^x (x-t)^{\delta-1}f(t)\,dt, \qquad x\geq 0.$$

Then $e^{-x}f(x)$ is bounded on $[0, \infty)$ *whenever $F(x)$ is bounded on* $[0, \infty)$.

Proof. Suppose $M=\sup_{x\geq 0}|F(x)|<\infty$. Since $f(t)$ is non-decreasing, we have, for all $x\geq 0$,

$$\begin{aligned} \delta\Gamma(\delta)eM &\geq \delta\Gamma(\delta)eF(x+1) \\ &= \delta e^{-x}\int_0^x (x+1-t)^{\delta-1}f(t)\,dt+\delta e^{-x}\int_x^{x+1}(x+1-t)^{\delta-1}f(t)\,dt \\ &\geq \delta e^{-x}\int_0^x (x+1-t)^{\delta-1}f(0)\,dt+\delta e^{-x}\int_x^{x+1}(x+1-t)^{\delta-1}f(x)\,dt \\ &= f(0)\{(x+1)^\delta-1\}e^{-x}+e^{-x}f(x). \end{aligned}$$

It follows that $e^{-x}f(x)$ is bounded on $[0, \infty)$.

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In what follows, suppose that $H_b=\{z \mid \operatorname{Re} z\geq b\}$.

Gaier [3, Theorem 1] has proved the following result: *If* $f(z)$ *is analytic in* H_0 *and if there are positive numbers* A, a *such that* $|f(z)|\leq A \exp(a\,|z|)$ *for all* z *in* H_0, *then* $\lim_{x\to\infty} f'(x)=0$ *whenever* $\lim_{x\to\infty} f(x)=0$.

However, by first making the transformation $w=z-b$ and then using Cauchy's integral formula for $f^{(n)}(x)$ in Gaier's proof, we can easily prove:

THEOREM 11. *If* $f(z)$ *is analytic in* H_b *and if there are positive numbers* A, a *such that* $|f(z)|\leq A \exp(a\,|z|)$ *for all* z *in* H_b, *then* $\lim_{x\to\infty} f^{(n)}(x)=0$ $(n=1, 2, \ldots)$ *whenever* $\lim_{x\to\infty} f(x)=s$.

4. **Proof of the main results.** We shall first prove Theorems 1, 2, 3, 4, 5, 6.

Proof of Theorem 1. Let k be a positive integer. Then, by result (A), we have that $A_{\alpha,\beta+(k-1)\alpha}(x)\to 0$ as $x\to\infty$. Moreover, by Lemma 1(ii),

$$S_{\alpha,\beta+(k-1)\alpha}(x) = A_{\alpha,\beta+(k-1)\alpha}(x)+S_{\alpha,\beta+k\alpha}(x)+o(1) \quad \text{as} \quad x\to\infty.$$

Hence, we have that if $S_{\alpha,\beta+k\alpha}(x)\to s$ as $x\to\infty$, then $S_{\alpha,\beta+(k-1)\alpha}(x)\to s$ as $x\to\infty$. Since $S_{\alpha,\beta+k\alpha}(x)\to s$ as $x\to\infty$ when $\beta+k\alpha\geq\mu$ by result (A), it follows that $S_{\alpha,\beta}(x)\to s$ as $x\to\infty$.

Proof of Theorem 2. Suppose without loss of generality that the sequence $\{s_n\}$ is real. Let $\delta>0$ and let $M=\sup_{x\geq 0} |S_{\alpha,\beta}(x)|$. Let k be a positive integer and let $h(u)=u^{(k-2)+\delta}e^{-u}/\Gamma(k-1+\delta)$. Then, by Lemma 1(i), we have, for $x\geq y\geq 0$, that

$$\begin{aligned}
&|S_{\alpha,\beta+(k-1)+\delta}(x)-S_{\alpha,\beta+(k-1)+\delta}(y)| \\
&\qquad = \left| \int_y^x h(x-t)S_{\alpha,\beta}(t)\,dt+\int_0^y \{h(x-t)-h(y-t)\}S_{\alpha,\beta}(t)\,dt \right| \\
&\qquad \leq M\int_y^x h(x-t)\,dt+M\int_0^y |h(x-t)-h(y-t)|\,dt \\
&\qquad \leq M\int_0^{x-y} h(u)\,du+M\int_0^\infty |h(x-y+u)-h(u)|\,du \to 0 \quad \text{as} \quad x-y\to 0
\end{aligned}$$

since $h(u)$ is Lebesgue integrable on $[0, \infty)$. Further, by Lemma 1(i),

$$S_{\alpha,\beta+k+\delta}(x) = \int_0^x e^{-(x-t)}S_{\alpha,\beta+(k-1)+\delta}(t)\,dt.$$

Hence, by Theorem 8 (with $F(x)=S_{\alpha,\beta+k+\delta}(x), f(x)=S_{\alpha,\beta+(k-1)+\delta}(x)$), we have that if $S_{\alpha,\beta+k+\delta}(x)\to s$ as $x\to\infty$, then $S_{\alpha,\beta+(k-1)+\delta}(x)\to s$ as $x\to\infty$. Since $S_{\alpha,\beta+k+\delta}(x)\to s$ as $x\to\infty$ when $k+\delta\geq\varepsilon$ by result (A), it follows that $S_{\alpha,\beta+\delta}(x)\to s$ as $x\to\infty$.

Proof of Theorem 3. In view of Lemma 1(i) and Lemma 3, we have, by Theorem 9 (with $F(x)=S_{\alpha,\beta+\varepsilon}(x), f(x)=S_{\alpha,\beta}(x)$, $h(u)=u^{\varepsilon-1}e^{-u}/\Gamma(\varepsilon)$), that $S_{\alpha,\beta}(x)$ is bounded

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on $[0, \infty)$. Hence, by Theorem 2, $S_{\alpha,\beta+1}(x) \to s$ as $x \to \infty$. Thus, in view of Lemma 1(i), it follows, by Theorem 8 (with $F(x)=S_{\alpha,\beta+1}(x), f(x)=S_{\alpha,\beta}(x)$), that $S_{\alpha,\beta}(x) \to s$ as $x \to \infty$.

Proof of Theorem 4. In view of Lemma 1(i), we can assume without loss of generality that $\mu=\beta+\delta$ where $\delta>0$. The result then follows by Theorem 10 with

$$F(x) = S_{\alpha,\beta+\delta}(x)+\alpha e^{-x} \sum_{n=N}^{\infty} K \frac{x^{\alpha n+\beta+\delta-1}}{\Gamma(\alpha n+\beta+\delta)}$$

$$e^{-x} f(x) = S_{\alpha,\beta}(x)+\alpha e^{-x} \sum_{n=N}^{\infty} K \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}.$$

Proof of Theorem 5. Let k be a positive number such that $\mu-k<\beta$. Then, by Theorem 4, $s_n=0(1)(B, \alpha, \mu-k)$ and hence, by Theorem 2, $s_n \to s(B, \alpha, \beta)$.

Proof of Theorem 6. Let k be a positive integer such that $\mu-k \le \beta$. Since $S_{\alpha,\mu-1}(z)=S_{\alpha,\mu}(z)+dS_{\alpha,\mu}(z)/dz$, it is readily seen that

$$S_{\alpha,\mu-k}(z) = S_{\alpha,\mu}(z)+\sum_{j=1}^{k} c_j \frac{d^j}{dz^j} S_{\alpha,\mu}(z)$$

where $c_1, c_2, \ldots, c_k$ are integers. Since $d^j S_{\alpha,\mu}(x)/dx^j \to 0$ as $x \to \infty$ $(j=1, 2, \ldots, k)$ by Theorem 11, we have that $S_{\alpha,\mu-k}(x) \to s$ as $x \to \infty$. Hence $S_{\alpha,\beta}(x) \to s$ as $x \to \infty$ by result (A).

The proofs of Theorems 2*, 3*, 5*, 6* follow the same basic pattern which we illustrate by one example.

Proof of Theorem 2*. By Lemma 2, $a_n \to 0$ $(B, \alpha, \beta+\varepsilon)$ and hence, by Theorem 2, $a_n \to 0(B, \alpha, \beta+\delta)$ for any $\delta>0$. The desired conclusion follows by Theorem 1.

Proof of Theorem 7. Since $|a_n| \le K^n$ for all $n \ge 0$, we have that

$$|A_{\alpha,\mu}(z)| \le A \exp(K^{1/\alpha} |z|)$$

for some positive constant A. The desired result follows by Theorem 6*.

5. **Final remarks**

1. Theorem 2 is false for $\delta=0$. This is shown by the following example [cf. 4, p. 183]. Let $\{s_n\}$ be the sequence such that $S_{1,1}(x)=\sin e^x$. Then $S_{1,2}(x)=e^{-x}\{\cos 1-\cos e^x\}$. Hence $s_n \to 0(B, 1, 2)$ and $s_n=0(1)(B, 1, 1)$ but s_n does not tend to a limit $(B, 1, 1)$.

2. There exists a sequence $\{s_n\}$ which tends to a limit (B, α, β) but does not tend to a limit $(B, \alpha, \beta-1)$. Choose an integer m such that $\alpha m>1$. Let P be the smallest integer such that $mP \ge N$. Let $x^P \sin e^x=\sum_{n=0}^{\infty} b_n x^n$ and let

$$s_n = \begin{cases} \Gamma(\alpha n+\beta) b_k & \text{if } n = mk, \\ 0 & \text{otherwise.} \end{cases}$$

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Then

$$S_{\alpha,\beta}(x) = \alpha e^{-x}x^{\beta-1}x^{\alpha mP}\sin e^{x^{\alpha m}} \to 0 \quad \text{as} \quad x\to\infty$$

and

$$\begin{aligned} S_{\alpha,\beta-1}(x) &= S_{\alpha,\beta}(x)+S'_{\alpha,\beta}(x) \\ &= \alpha e^{-x}x^{\beta-1}x^{\alpha mP}(\alpha m)x^{\alpha m-1}e^{x^{\alpha m}}\cos e^{x^{\alpha m}}+o(1) \quad \text{as} \quad x\to\infty. \end{aligned}$$

Thus $s_n\to 0(B, \alpha, \beta)$ but $s_n\neq 0(1)(B, \alpha, \beta-1)$.

Hence the tauberian theorems proved in this paper are not "empty".

3. Corresponding to the Borel-type "exponential" method (B, α, β) is an "integral" method (B', α, β) defined as follows (see [1]): $\sum_0^\infty a_n=s(B', \alpha, \beta)$ if $A_{\alpha,\beta}(x)$ exists for all $x\geq 0$ and $\lim_{x\to\infty}\alpha^{-1}\int_0^x A_{\alpha,\beta}(t)\,dt=s-s_{N-1}$ (where $s_{-1}=0$).

The following result is due to Borwein [1, Theorem 2]: $\sum_0^\infty a_n=s(B, \alpha, \beta+1)$ *if and only if* $\sum_0^\infty a_n=s(B', \alpha, \beta)$.

The tauberian theorems proved in this paper suggest that analogous results hold for the method (B', α, β).

4. Let $p(z)=\sum_{n=0}^\infty p_nz^n$ be an integral function such that $p_n\geq 0$, $\sum_{r=n}^\infty p_r>0$ for all n. Associated with $p(z)$ is an integral function method of summability P defined as follows: $s_n\to s(P)$ if $\dfrac{1}{p(x)}\sum_{n=0}^\infty p_ns_nx^n\to s$ as $x\to\infty$.

The following result is due to Borwein (see [2]): *If $h(z)$ is analytic in H_b, $h(x)$ is real for $x\geq b$ and, when $x\geq b$ and $|z|$ is large, $h(z)=z^{\alpha z+\beta}e^{\gamma z}\{C+0(1/|z|)\}$ where C, α are positive and β, γ are real, then the method associated with the integral function $p(z)=\sum_{n=M}^\infty \dfrac{z^n}{h(n)}$ is equivalent to $(B, \alpha, \beta+1/2)$.*

There should therefore be tauberian theorems of the sort proved in this paper for a wide class of integral function methods.

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