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A CRITERION FOR HYPERBOLICITY

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The usual definition of hyperbolicity of a group G demands that all geodesic triangles in the Cayley graph of G should be thin. Using the theorem that a subquadratic isoperimetric inequality implies a linear one, we show that it is in fact only necessary for all triangles from a given combing to be thin, thus giving a new criterion for hyperbolicity of finitely presented groups.

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1. Slim triangles

Given a group G, the *Cayley graph* $\Gamma_S(G)$ of G with respect to a generating set S of G is the graph whose vertex set is G and whose edge set is $\{(g, gs) | g \in G, s \in S\}$. Given a path p in $\Gamma_S(G)$, we write $l(p)$ for the number of edges in p. If p originates at the identity of G then we write $\bar{p}$ for the group element at the terminus of p (i.e. $\bar{p}$ is the group element represented by the word p in S).

Definition 1.1. A *triangle* in a group G is the data

$$(g_1, g_2, g_3, \theta_{12}, \theta_{23}, \theta_{31}),$$

where g_1, g_2, and g_3 are elements of G called the *vertices* of the triangle and θ_{ij} is a path in the Cayley graph of G from g_i to g_j (called a *side* of the triangle). If the sides are geodesic paths, the triangle is said to be *geodesic*.

For a triangle Δ as above, we denote by $\partial\Delta$ the loop $\theta_{12} * \theta_{23} * \theta_{31}$, called the *boundary* of Δ and we write $\pi(\Delta)$ for $l(\partial\Delta)$, the *perimeter* of Δ.

The following definition is based on the familiar geodesic case.

Definition 1.2. Let $\delta \geq 0$. A triangle Δ in G is *δ-slim* if for each of the three sides θ of Δ and for all $t \in \theta$, $d(t, \theta' \cup \theta'') \leq \delta$, where θ' and θ'' are the other two sides of Δ.

If there exists a finite generating set S of a group G and $\delta \geq 0$ such that all geodesic triangles in $\Gamma_S(G)$ are δ-slim then we say that G is *hyperbolic*. In fact hyperbolicity of groups does not depend on the finite generating set chosen. Also note that it is a

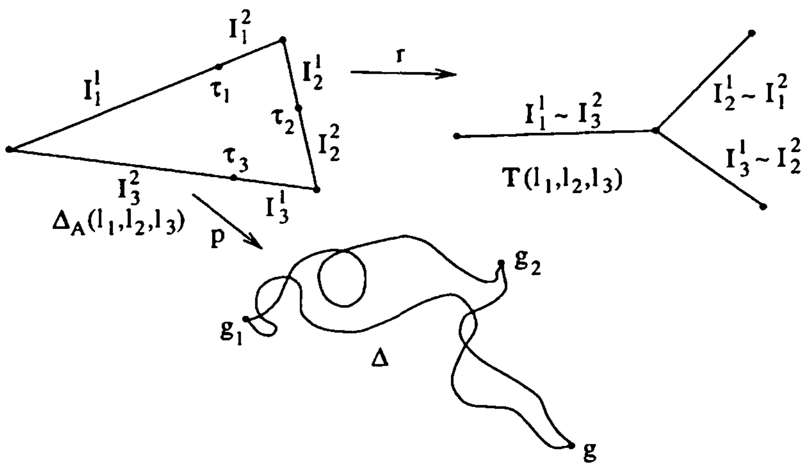

FIGURE 1: Collapsing a Triangle

consequence of the definitizon that all hyperbolic groups are finitely presented (these facts are proved in [**10**] and [**4**]).

2. Thin triangles

Let $l_1 \geq 0$, $l_2 \geq 0$ and $l_3 \geq 0$ be real numbers satisfying the triangle inequality (in all possible permutations) and let I_1, I_2 and I_3 be the real intervals $[0, l_1]$, $[0, l_2]$ and $[0, l_3]$ respectively with endpoints i_1, t_1, i_2, t_2, i_3 and t_3. Let $I(l_1, l_2, l_3)$ be the disjoint union of I_1, I_2 and I_3. Then we define the *abstract triangle* $\Delta_A(l_1, l_2, l_3)$ to be the quotient space of $I(l_1, l_2, l_3)$ by identifying t_1 with i_2, t_2 with i_3 and t_3 with i_1. Now there exist points $\tau_1 \in I_1$, $\tau_2 \in I_2$ and $\tau_3 \in I_3$ with the following property: Suppose that τ_1 divides I_1 into segments I_1^1 and I_1^2, I_2 into I_2^1 and I_2^2 and I_3 into I_3^1 and I_3^2 as in Figure 1. Then the length $l(I_1^1) = l(I_3^2)$, $l(I_2^1) = l(I_1^2)$ and $l(I_3^1) = l(I_2^2)$. So we may isometrically identify I_1^1 with I_3^2, I_2^1 with I_1^2 and I_3^1 with I_2^2 as in Figure 1 to obtain a quotient space of $\Delta_A(l_1, l_2, l_3)$ called the tripod $T(l_1, l_2, l_3)$. We denote the quotient map by r. Then $r(I_1) \cap r(I_2) \cap r(I_3)$ is a single point, which we call the *fork* of the tripod.

Definition 2.1. We say that a triangle $(g_1, g_2, g_3, \theta_{12}, \theta_{23}, \theta_{31})$ in a group is *proper* if the lengths of its sides satisfy the triangle inequality, i.e. for all $i \neq j \neq k$ with $1 \leq i, j, k \leq 3$,

$$l(\theta_{ij}) \leq l(\theta_{jk}) + l(\theta_{ki}).$$

For example, geodesic triangles are proper. Let $\Delta = (g_1, g_2, g_3, \theta_{12}, \theta_{23}, \theta_{31})$ be a proper triangle in a group G, with $l(\theta_{12}) = l_1$, $l(\theta_{23}) = l_2$ and $l(\theta_{31}) = l_3$. If x is a real number, denote by $[x]$ the greatest integer not exceeding x. There exists a map $p : I(l_1, l_2, l_3) \to G$ such that for all $x \in I_1$ with $0 \leq x \leq l_1$, $p(x) = \theta_{12}([x])$, for all $x \in I_2$ with $0 \leq x \leq l_2$, $p(x) = \theta_{23}([x])$, for all $x \in I_3$ with $0 \leq x \leq l_3$, $p(x) = \theta_{31}([x])$ and p induces a map $p : \Delta_A(l_1, l_2, l_3) \to G$ on passage to the quotient.

Definition 2.2. Let $\delta \geq 0$. We say that a proper triangle Δ is *δ-thin* if for all $t \in T(l_1, l_2, l_3)$, $diam(p(r^{-1}(t))) \leq \delta$.

We next give a lemma describing the area of thin triangles, where *area* is in the following sense. Let $G = \langle S|R \rangle$ be a finitely presented group. Then recall that a word in S is equal to the identity in G if and only if there exist words u_i in S for $1 \leq i \leq n$ such that, in the free group $F(S)$ generated by S,

$$w = \prod_{i=1}^{n} u_i r_i u_i^{-1}$$

where for all i with $1 \leq i \leq n$ either $r_i \in R$ or $r_i^{-1} \in R$.

Definition 2.3. With G as above, let w be a word in S which is equal to the identity in G. Then the *area* of w, $A(w)$, is defined to be

$$\min\left\{n \in \mathbb{N} | w = \prod_{i=1}^{n} u_i r_i u_i^{-1} \text{ in } F(S)\right\}.$$

We now describe an equivalent formulation of area which is more geometric and suited to our methods.

Definition 2.4. A *paired alphabet* is a finite set S together with an involution $f : S \to S$. We usually write $f(s) = s^{-1}$.

For example, an inverse closed set of generators of a group is a paired alphabet, where the involution is the group inverse.

Definition 2.5. A *map* is a finite, planar, oriented, connected and simply connected combinational 2-complex. We say that a map M is a *diagram* over a paired alphabet S if every edge e of M has a label $\phi(e) \in S$ such that $\phi(e^{-1}) = (\phi(e))^{-1}$.

The definition of a map ensures that it has a well defined boundary path. Note that every path in a diagram over S is labelled by a word in S.

Definition 2.6. A *van Kampen diagram* over a group $G = \langle S, R \rangle$ is a diagram M over S such that for all faces f of M the label of the boundary path of f is labelled by some $r^{\pm 1}$ with $r \in R$. The *area* of such a diagram is the number of its faces.

Proposition 2.7 (van Kampen's lemma). *Let $G = \langle S, R \rangle$ be a finitely presented group and let w be a word in S. Then $\overline{w} = 1_G$ if and only if there exists a van Kampen diagram over G with boundary labelled by w.*

See [10] for a proof of van Kampen's lemma. We define the *Dehn function* of G to be the function $D : \mathbb{N} \to \mathbb{N}$ given by $D(n) = \max\{A(w)\}$ where the maximum is taken over all words of length at most n in S such that $\overline{w} = 1_G$.

Proposition 2.8. *Let G be a group and let $\delta \geq 0$. Then there exists a linear function $y : \mathbb{N} \to \mathbb{N}$ such that if Δ is a δ-thin triangle in G then $A(\Delta) \leq y(\pi(\Delta))$.*

Proof. Suppose that Δ has side lengths l_1, l_2 and l_3. Let f be the fork of $T(l_1, l_2, l_3)$. Now $r^{-1}(f)$ consists of three points $\tau_1 \in I_1$, $\tau_2 \in I_2$ and $\tau_3 \in I_3$. Let $f_1 = p(\tau_1)$, $f_2 = p(\tau_2)$ and $f_3 = p(\tau_3)$. Join f_1, f_2 and f_3 pairwise by geodesics to bound a triangle, which we denote by $\alpha(\Delta)$, of perimeter no greater than $3\delta + 3$ as in Figure 2. If D is the Dehn function of G, we thus have $A(\alpha(\Delta)) \leq D(3\delta + 3)$. To finish the proof, therefore, it remains to show that the area of $\Delta - \alpha(\Delta)$ depends linearly on $\pi(\Delta)$.

Clearly it suffices to show that the area of each of these triangles depends linearly on $\pi(\Delta)$. So, consider λ_1, with sides $s_1 = [g_1, f_1]$, $s_2 = [g_1, f_3]$ and $s_3 = [f_1, f_3]$ (note that $|l(s_1) - l(s_2)| \leq 1$ and $l(s_3) \leq \delta$). We now subdivide λ_1 into further regions as follows. $I_1^1(l_1, l_2, l_3)$ and $I_3^2(l_1, l_2, l_3)$ are the segments of $\Delta_A(l_1, l_2, l_3)$ which map by p onto $[g_1, f_1]$ and $[g_1, f_3]$. Suppose that the maximum of their lengths is l_4. Define $p_i^1 = p(i)$ for $i \in I_1(l_1, l_2, l_3)$ if i is an integer with $0 \leq i \leq l_4 - 1$ and $p_i^2 = p(l_3 - i)$ for $i \in I_3(l_1, l_2, l_3)$ if i is an integer with $0 \leq i \leq l_4 - 1$. If $l(I_1(l_1, l_2, l_3)) = l(4)$ then define $p_{l_4}^1 = p(l_4)$ and otherwise define $p_{l_4}^1 = p(l_4 - 1)$, and make similar endpoint adjustments for the other segments. For each i pick a geodesic η_i between p_i^1 and p_i^2, and let e_j^i be the edge from p_i^j to p_{i+1}^j for $j = 1$ and 2. Define Q_i to be a least area quadrilateral bounded by $\eta_i * e_i^2 * \eta_{i+1}^{-1} * (e_i^1)^{-1}$ for $0 \leq i \leq m - 1$. Then

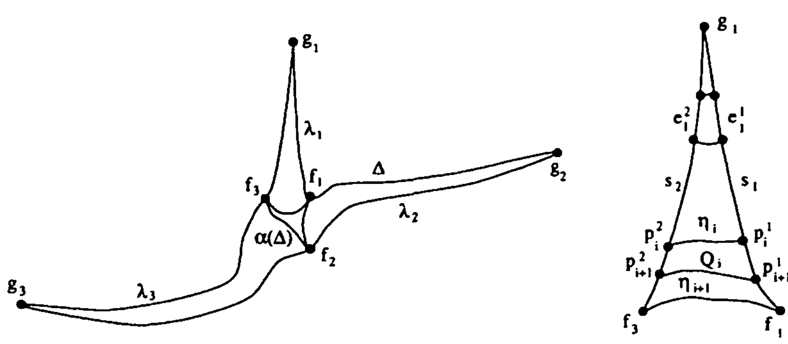

FIGURE 2: Dividing A Thin Triangle

$$A(\lambda_1) = \sum_{i=0}^{m-1} A(Q_i) + D(2\delta + 1)$$
$$\leq mD(2\delta + 2) + D(2\delta + 1)$$
$$\leq \pi(\Delta)D(2\delta + 2) + D(2\delta + 1),$$

since $m \leq \pi(\Delta)$. So we can take $y(n) = 3D(2\delta + 2)n + D(3\delta + 3) + D(2\delta + 1)$, which is linear in n as required. □

The following example shows that the same property does not hold for slim triangles.

Example 2.9. Let G be the free abelian group of rank 2 with the presentation $\langle a, b | [a, b] \rangle$. Define the triangle Δ_n to have vertices e, a^n and b^n, and sides described by the words $a^n(a^{-1}b)^n$ from e to b^n, $(a^{-1}b)^n b^{-n}$ from a^n to e and $b^{-n}a^n$ from b^n to a^n (see Figure 3). Then Δ_n is a $(3, 0)$-quasigeodesic 0-slim triangle for all n. However, the perimeter of Δ_n is a linear function of n but the area of Δ_n depends quadratically on n.

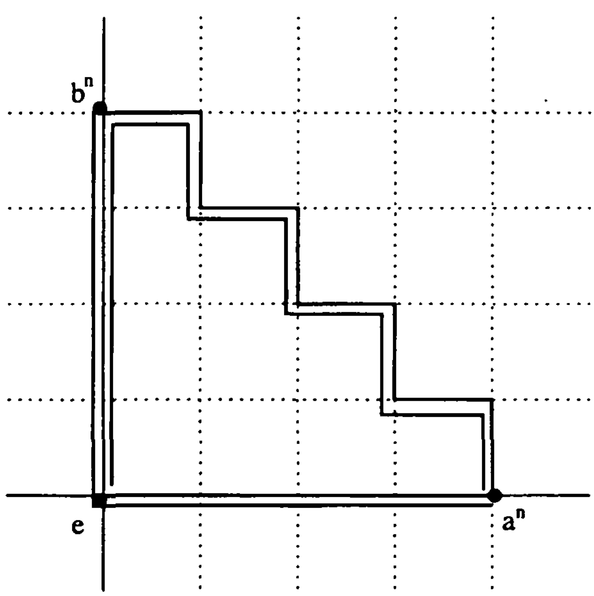

FIGURE 3: Slim Triangle with Quadratic Area

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3. Thin and slim combings

If X is a graph then let $P(X)$ denote the set of finite paths in X. Whenever $p \in P(X)$ we shall write $i(p)$ for the initial vertex of p and $t(p)$ for the terminal vertex of p.

Now let G be a finitely generated group and let S be a finite generating set for G.

Definition 3.1. A *combing* of a group G with respect to a generating set S is a map $\theta : G \to P(\Gamma_S(G))$ such that for all $g \in G$, $i(\theta(g)) = 1_G$ and $t(\theta(g)) = g$.

Note that we do not assume that the "fellow traveller property" of [3] holds for a combing.

We can equivariantly extend a combing θ of G to a map $\hat{\theta} : G \times G \to P(\Gamma_S(G))$ via the rule $\hat{\theta}(g_1, g_2) = g_1\theta(g_1^{-1}g_2)$. From now on, we shall also write θ for $\hat{\theta}$. Now if θ is a combing of G and g_1, g_2 and g_3 are three elements of G, then we use the notation $\Delta_\theta(g_1, g_2, g_3)$ for the triangle

$$(g_1, g_2, g_3, \theta(g_1, g_2), \theta(g_2, g_3), \theta(g_3, g_1)).$$

We now introduce two types of constraint on the triangles of a combing. Since the definition of a thin triangle only applies to proper triangles, we define a combing θ of a group G to be *triangular* if for all g_1, g_2 and g_3 in G, $\Delta_\theta(g_1, g_2, g_3)$ is a proper triangle.

Definition 3.2. Let $\delta \geq 0$. A combing (respectively triangular combing) θ of a group G is *δ-slim* (respectively *-thin*) if for all g_1, g_2 and g_3 in G, $\Delta_\theta(g_1, g_2, g_3)$ is a δ-slim (respectively -thin) triangle. We say that θ is *slim* (respectively thin) if there exists $\delta \geq 0$ such that θ is δ-slim (resp. -thin).

Proposition 3.3. *If a combing θ of a group G is triangular, then there exists a constant $K \geq 1$ such that for all g_1 and g_2, $l(\theta(g_1, g_2)) \leq Kd(g_1, g_2)$.*

Proof. Let G be a finitely generated group with a finite generating set $S = \{s_1, \ldots, s_p\}$ and let θ be a triangular combing of G with respect to S. Take K to be $\max_{s \in S}\{l(\theta(1_G, s))\}$. For g_1 and g_2 in G, let w be a geodesic word in S with $\overline{w} = g_1^{-1}g_2$. Suppose that $w = \prod_{j=1}^{m} s_{i_j}^{n_j}$ where $m = d(g_1, g_2)$ and for each j with $1 \leq j \leq m$, $n_j \in \mathbb{Z} - \{0\}$ and $1 \leq i_j \leq p$. Let $w_k = \prod_{j=1}^{k} s_{i_j}^{n_j}$ for $1 \leq k \leq m$ and let w_0 be the empty word. Then by repeated application of the triangle inequality we have

$$\begin{aligned} l(\theta(g_1, g_2)) &\leq \sum_{k=0}^{m-1} l(\theta(g_1\overline{w_j}, g_1\overline{w_{j+1}})) \\ &\leq Km \\ &= Kd(g_1, g_2). \end{aligned}$$

□

Let λ and μ be real numbers with $\lambda \geq 1$ and $\mu \geq 0$ and suppose that p is a path in a graph. If for all subpaths q of p we have

$$\frac{1}{\lambda} d(i(q), t(q)) - \mu \leq l(q) \leq \lambda d(i(q), t(q)) + \mu$$

then we call p a *(λ, μ)-quasigeodesic*. If for a path p there exist $\lambda \geq 1$ and $\mu \geq 0$ such that p is a (λ, μ)-quasigeodesic then we simply say that p is a *quasigeodesic*.

Definition 3.4. A combing θ of a group G is *quasigeodesic* if there exist $K \geq 1$ and $L \geq 0$ such that every combing line of θ is a (K, L)-quasigeodesic.

Note that the conclusion of Proposition 3.3 does not imply that θ is a quasigeodesic combing. Jean-Philippe Preaux has pointed out the following example due to Hamish Short.

Example 3.5. Consider the combing of $\mathbb{Z} = \langle x \rangle$ given by

$$\theta(x^n) = x^n(xx^{-1})^n$$

for all $n \in \mathbb{Z}$ and extending equivariantly to give combing lines between all pairs of integers. Then $l(\theta(x^n)) = 3n$. Let m_1, m_2 and m_3 be three integers. Now

$$\begin{aligned} l(\theta(m_1, m_3)) &= 3|m_1 - m_3| \\ &= 3|m_1 - m_2 + m_2 - m_3| \\ &\leq 3|m_1 - m_2| + 3|m_2 - m_3| \\ &= l(\theta(m_1, m_2)) + l(\theta(m_2, m_3)). \end{aligned}$$

So θ is a triangular combing. The conclusion of the previous proposition is clearly satisfied by θ, with $K = 3$. But consider the subword $(xx^{-1})^n$ of $\theta(x^n)$. The distance between the origin and the terminus of this path is 0, but $l((xx^{-1})^n) = 2n$. So for all $L \geq 0$ and for all $K \geq 1$ there exists an element of G, e.g. x^L, such that $\theta(x^L)$ is not a (K, L)-quasigeodesic. Hence θ is not a quasigeodesic combing.

It is known that if there exists $\delta_1 > 0$ such that every geodesic triangle in a group G is δ-slim then there exists $\delta_2 > 0$, depending on δ_1, such that every geodesic triangle in G is δ_2-thin (see e.g. **[10]**). If we no longer restrict to geodesic triangles, thin triangles are still slim, so if a group admits a thin combing then it admits a slim one.

Question. If a group admits a slim quasigeodesic combing then does it admit a thin combing?

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4. Hyperbolicity and thin combings

Reeves [9] has verified that $\mathbb{Z} \oplus \mathbb{Z}$ admits no thin combing, using work of Neumann and Shapiro which classifies biautomatic structures on free abelian groups [5]. Reeves has also asked the following question [8].

Question. Does admission of a thin combing characterise hyperbolicity of biautomatic groups?

We answer this question in the affirmative, and show that the hypothesis of biautomaticity is unnecessary. The usual concept of hyperbolicity requires *all* geodesic triangles to be δ-thin, but here we show that for hyperbolicity, it is only necessary for one triangle per triple of points to be thin. Also the sides of the triangle no longer need to be geodesics. In this sense our main theorem is an "efficient" criterion for hyperbolicity.

First, recall that a group is said to satisfy a *linear isoperimetric inequality* if its Dehn function D is bounded above by a linear function and that a finitely presented group satisfies a linear isoperimetric inequality if and only if it is hyperbolic (see [4] or [10]). We restrict to finitely presented groups because every finitely generated group G satisfies a linear isoperimetric inequality if we include all words equal to the identity in G as relators. A group is said to satisfy a *subquadratic isoperimetric inequality* if $\lim_{n\to\infty}(\frac{D(n)}{n^2}) = 0$. The following result, originally due to Gromov [4], will play an important part in our analysis. Proofs have been given by Ol'Shanskii [6], Papasoglu [7] and Bowditch [2].

Theorem 4.1. *If a group satisfies a subquadratic isoperimetric inequality then it satisfies a linear one (and so is hyperbolic).*

We can now state our main theorem.

Theorem 4.2. *A finitely presented group admits a thin combing if and only if it is hyperbolic.*

Proof. Clearly every hyperbolic group G admits a thin combing (in every Cayley graph with respect to a finite generating set, every geodesic combing is thin).

Conversely, let G be a finitely presented group admitting a δ-thin combing θ. We are going to show that G verifies a subquadratic isoperimetric inequality. Let $\phi : F(S) \to G$ be a choice of generators for G, where S is finite, and let w be a word of length n_w in S whose image is equal to the identity in G. In $\Gamma_S(G)$, $\overline{w}$ represents a loop of length n_w, originating and terminating at 1_G. We may assume that $\overline{w}$ is a simple loop since, if not, $\overline{w}$ may be divided into several loops $w_1, \dots, w_n$ with $A(\overline{w}) \leq \sum_{i=1}^{n} A(w_i)$.

We now define a finite sequence of elements of G. Let n be the smallest power of 2 such that $n \geq n_w$. Then $n \leq 2n_w$. For $0 \leq i \leq n_w$, let $h_i = \overline{w}(i)$ and for $n_w \leq i \leq n$, define $h_i = 1_G$. We now use the combing lines of θ to subdivide the loop $\overline{w}$. Let $g = h_{\frac{n}{2}}$, $g_1 = h_{\frac{n}{4}}$

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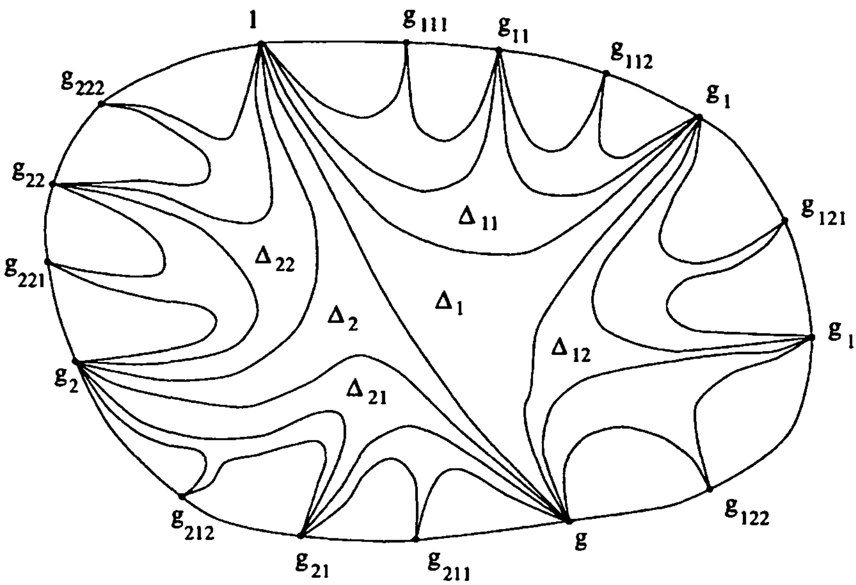

FIGURE 4: A Thin Combing Implies Hyperbolicity

and $g_2 = h_{\frac{3n}{4}}$ and define the triangles $\Delta_1 = \Delta_0(1_G, g_1, g_2)$ and $\Delta_2 = \Delta_0(g, g_2, 1_G)$. We next construct four more triangles on the sides of Δ_1 and Δ_2. Let $g_{11} = h_{\frac{n}{8}}$, $g_{12} = h_{\frac{3n}{8}}$, $g_{21} = h_{\frac{5n}{8}}$ and $g_{22} = h_{\frac{7n}{8}}$, and define $\Delta_{11} = \Delta_0(1_G, g_{11}, g_1)$, $\Delta_{12} = \Delta_0(g_1, g_{12}, g)$, $\Delta_{21} = \Delta_0(g, g_{21}, g_2)$ and $\Delta_{22} = \Delta_0(g_2, g_{22}, 1_G)$. The process of subdividing is continued for each $1 \leq p \leq \log_2(n) - 1$ to obtain group elements $g_{i_1 \cdots i_p}$ and triangles $\Delta_{i_1 \cdots i_p}$ for $i_j = 1$ and 2, as in Figure 4.

Let Q_p be the set of triangles introduced at the p^{th} stage of the subdivision and let Q be the union of all the triangles in the subdivision. We then have

$$A(Q) \leq \sum_{p=1}^{\log_2(n)-1} \sum_{T \in Q_p} A(T).$$

Suppose that K is as in Proposition 3.3 and that y is a linear function as in Proposition 2.8. If $T \in Q_p$ then we have $\pi(T) \leq \frac{Kn}{2^{p-1}}$ and

$$A(T) \leq \frac{Ky(n)}{2^{p-1}}.$$

Now Q_p contains 2^p triangles and if D is the Dehn function of $\Gamma_S(G)$ then we have

$$\begin{aligned} A(w) &\le A(Q) + nD(K+1) \\ &\le \sum_{p=1}^{\log_2(n)-1} \frac{2^p \cdot Ky(n)}{2^{p-1}} + nD(K+1) \\ &\le 2Ky(n)\log_2(n) + nD(K+1) \\ &\le 4Ky(n_w)\log_2(2n_w) + 2n_wD(K+1). \end{aligned}$$

Thus G satisfies an isoperimetric inequality which is $O(n \log_2 n)$. Since this is subquadratic, G is hyperbolic by Theorem 4.1. □

Note that the homogeneity (as metric spaces) of Cayley graphs, along with the equivariance of combings, does not play an important role in the arguments. In fact, it is possible to generalise the above theorem to path-metric spaces, using a formulation of area due to Bowditch [**2**]. This is described in the author's Ph.D. thesis [**1**].

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