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CR MAPPINGS OF CIRCULAR CR MANIFOLDS

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ABSTRACT. Let M be a circular CR manifold and let N be a rigid CR manifold in some complex vector spaces. The problem of the existence of local CR mappings from M into N is considered. Conditions are given which ensure that the space of such CR mappings depends on a finite number of parameters. The idea of the proof of the main result relies on a Bishop type equation for CR mappings. Roughly speaking, we look for CR mappings from M into N in the form $F = (f, g)$, we assume that g is given, then we find f in terms of g and some parameters, and finally we look for conditions on g. It works independently of assumptions on the Levi forms of M and N, and there is also some freedom on the codimension of the manifolds.

1. **Introduction.** A smooth embedded *submanifold* M of $\mathbb{C}^n$ is called *Cauchy-Riemann (CR)* if the complex dimension of the complex part of the tangent fibre $T_p(M)$ to M at p does not depend on p, *i.e.*, if $\dim_{\mathbb{C}}\big(T_p(M) \cap \sqrt{-1}\,T_p(M)\big) = l(p) \equiv$ const. The above constant is the CR dimension of M, which we denote by $\dim_{\mathrm{CR}} M$. By a CR *function* $f\colon U \longrightarrow \mathbb{C}$, U open in M, we mean a smooth function which satisfies the tangential Cauchy-Riemann equations on M. A mapping from M into $\mathbb{C}^k$ is called a CR *mapping* if each component is a CR function. For an introduction to CR theory, see [Bo].

The problem of local CR mappings between CR manifolds goes back to Poincaré [P], where he considered the case of real analytic hypersurfaces in $\mathbb{C}^2$. There is a vast literature devoted to the subject, including several review papers (see, for instance, [F2], [V]). Most of the papers about CR mappings deal with hypersurfaces with strong assumptions on the Levi form, or, in the higher codimensional case, with special classes of manifolds like standard or quadric CR manifolds (see, for instance, [ES], [F1], [S1], [S2], [Su], [T]).

In this paper we have chosen to study local CR mappings $F\colon M \longrightarrow N$ from circular CR manifolds M into rigid CR manifolds N. There exist some similarities between this problem and the problem of holomorphic mappings of circular or Reinhardt domains (*e.g.*, see [F2, Section 3] for a survey of the latter). A CR manifold $M \subset \mathbb{C}^{k+m}$ is called a *circular* CR *manifold* if it is given locally by an equation of the form

$$(1.1)\qquad M\colon\ y = h(|w_1|, \ldots, |w_m|),\ h = (h_1, \ldots, h_k), \quad h(0) = 0,\ dh(0) = 0,$$

where $(z, w) = (z_1, \ldots, z_k, w_1, \ldots, w_m) \in \mathbb{C}^{k+m}$, $z = x + iy$. A CR manifold $N \subset \mathbb{C}^{l+n}$ is called a *rigid* CR *manifold* if it is given locally by an equation

$$(1.2)\qquad N\colon\ \tau = \varphi(\xi_1, \ldots, \xi_n),\ \varphi = (\varphi_1, \ldots, \varphi_l), \quad \varphi(0) = 0,\ d\varphi(0) = 0,$$

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where $(\eta,\xi)=(\eta_1,\dots,\eta_l,\xi_1,\dots,\xi_n)\in\mathbb{C}^{l+n}$, $\eta=\sigma+i\tau$.

For CR mappings F between such types of manifolds it will be natural to assume that they are of the form $F=(f,g)$, where g depends only on w's. Under these assumptions we consider the following two questions: *How many* CR *mappings* $F\colon M\longrightarrow N$ *exist?* and *What is the form of these mappings?*

Our aim here is to apply an analytic disc approach (see Section 2) to study these questions. Recently, Baouendi and Rothschild [BR] also applied analytic discs to CR mappings but in a very different context than in our paper. Although it is our belief that our approach, with some modifications, can be carried out for more general CR manifolds (work in progress), we have chosen in this paper to concentrate on the above special cases because under these restrictions we were able to obtain *explicit* formulas (as in Theorems I and II), to allow some freedom on the codimension of the manifolds (as in Theorems I and 6.1) and to show that our method works independently of the type of points on the CR manifolds, in particular, independently of the behaviour of the Levi form.

Our idea of the proof of the main result relies on a Bishop type equation for CR mappings, which came from the Bishop equation for the lifting of analytic discs (see Sections 2 and 3). Roughly speaking, we look for CR mappings from M into N of the form $F=(f,g)$; we assume that g is given, then we find f in terms of g and some parameters, and finally we look for suitable conditions on g.

We now formulate some of the main results of the paper. For the proofs and other results and examples, see Section 5 and Section 6.

THEOREM I. *Let M be a circular* CR *manifold locally given by (1.1), and N be a rigid* CR *manifold locally given by (1.2).*

(a) Assume that a holomorphic mapping $g=g(w)$ is given such that $F=(f,g)$, $F(0)=0$, is a CR *mapping from M into N, then the mapping f is of the form*

$$(1.3)\qquad f(z,w)=a(\Re z)+i\sum_{\beta=1}^{k}a_\beta\Im z_\beta+p(w)\quad \textit{for } (z,w)\in M,$$

where $a=a(x)$, $a(0)=0$, is a real vector-valued function, $p=p(w)$, $p(0)=0$, is a holomorphic vector-valued function, and $a_\beta=\frac{\partial a(0)}{\partial x_\beta}$, $\beta=1,\dots,k$. Also we have

$$(1.4)\qquad \sum_{\beta=1}^{k}a_\beta h_\beta(r)+\Im\big(p(re^{i\theta})\big)=\varphi\big(g(re^{i\theta})\big)\quad \textit{where } re^{i\theta}=(r_1e^{i\theta_1},\dots,r_me^{i\theta_m}).$$

Moreover, after a holomorphic change of variables in a neighbourhood of the origin of $\mathbb{C}^{l+n}$, the function $p=p(w)$ can be made identically equal to zero,

(b) Conversely, if $g=g(w)$ is holomorphic and if f is of the form (1.3) with (1.4) being satisfied, then $F=(f,g)$ is a CR *mapping from M into N.*

THEOREM II. *Let M be a circular* CR *manifold as in (1.1), with* $\operatorname{rank}(\frac{\partial h_\beta}{\partial r_\alpha})_{\alpha\beta}\geq k$, *and let N be a rigid* CR *manifold as in (1.2) with* $\dim_{\mathrm{CR}}M=\dim_{\mathrm{CR}}N=m$. *Assume that*

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$g_0 = g_0(w)$ is a one-to-one holomorphic mapping between neighbourhoods of the origin in $\mathbb{C}^m$, and such that $F_0 = (f_0, g_0): M \longrightarrow N$, $F_0(0) = 0$, is a CR *embedding. Then (maybe after a holomorphic change of variables in a neighbourhood of N) it follows that*

(i) $\varphi \circ g_0 = a_1 h_1 + \cdots + a_k h_k$ for some constant vectors $a_1, \ldots, a_k$;

(ii) $f_0(z) = a_1 z_1 + \cdots + a_k z_k$;

(iii) for any holomorphic mapping $g = g(w)$ between neighbourhoods of the origin in $\mathbb{C}^m$ such that $F = (f, g)$, $F(0) = 0$, is CR *from M into N, we have that $g = g_0 \circ G$, with $G = (G_1, \ldots, G_m)$ satisfying ${}^t h\big(|G(w)|\big) = A\,{}^t h(|w|)$, where A is a constant $k \times k$ matrix, ${}^t h$ is the transpose of h, and $|G(w)| = \big(|G_1(w)|, \ldots, |G_m(w)|\big)$.*

2. **Definitions, notation, analytic discs and polydiscs.** If $a = (a_1, \ldots, a_p) \in \mathbb{C}^p$, $|a|$ will denote the vector $(|a_1|, \ldots, |a_p|)$ and $\|a\|$ will denote $(|a_1|^2 + \cdots + |a_p|^2)^{1/2}$. By D we shall denote the unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and by $\mathcal{H}(D)$ the space of holomorphic functions in D. Consider a map $\psi: \bar{D} \longrightarrow \mathbb{C}^p$, with each component belonging to $\mathcal{H}(D)$ and to some differentiability class on $\bar{D}$. The mapping ψ, or sometimes the image $\psi(\bar{D})$ will be called an *analytic disc* in $\mathbb{C}^p$. The restriction of ψ to $S^1 = \partial D$, or sometimes the image $\psi(S^1)$, will be called the *boundary of the disc*. The point $\psi(0)$ will be called the *center of the disc*.

For any compact $K \subset \mathbb{R}^n$, and $0 \le \alpha \le 1$, let $C^\alpha(K)$ be the Banach algebra of functions from K into $\mathbb{R}$ with the Lipschitz norm $\|u\|_\alpha \equiv \sup_{x \in K} |u(x)| + \sup_{x,y \in K} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty$. If $u = (u_1, \ldots, u_j)$ is vector-valued, then we use the same notation for the norm, namely $\|u\|_\alpha = (\|u_1\|_\alpha^2 + \cdots + \|u_j\|_\alpha^2)^{1/2}$, which should not lead to any confusion.

Fix $0 < \alpha < 1$ in this paper. It is well known (see, for instance, [HT, Section 3]) that for a function $x \in C^\alpha(S^1)$ there exists a unique function $y = Tx \in C^\alpha(S^1)$ such that $x + iy$ is the boundary value of a holomorphic function f in D with $\Im f(0) = 0$. The operator T is called the *conjugation operator on the circle*. The operator T is a bounded linear operator $T: C^\alpha(S^1) \longrightarrow C^\alpha(S^1)$.

For any function $x \in C(S^1)$, we denote its mean value by

$$J(x) = \frac{1}{2\pi} \int_0^{2\pi} x(e^{i\theta})\,d\theta. \tag{2.1}$$

If x is a vector-valued function, then J and T are defined to act component-wise.

If we want to find analytic discs (locally) with boundaries on a CR manifold $M \subset \mathbb{C}^{k+n}$ given by an equation $y = h(x, w)$, $h(0,0) = 0$, $dh(0,0) = 0$, then we have to solve the Bishop equation (see [Bi], [HT])

$$x = c - T[h(x, w)], \tag{2.2}$$

for the given data $(c, w) = (c_1, \ldots, c_k, w_1, \ldots, w_n)$, $w_j \in \mathcal{H}(D) \cap C^\alpha(\bar{D})$, $j = 1, \ldots, m$, $c \in \mathbb{R}^k$, where $\|w_j\|_\alpha$ and $\|c\|$ are sufficiently small. Such a solution exists, is unique, $x_i \in C^\alpha(S^1)$, $i = 1, \ldots, k$, and $J(x) = c$ (see [HT], [BP]). It corresponds in a one-to-one manner to the lifted analytic disc (z, w) with boundary on the manifold, where $z = (z_1, \ldots, z_k)$, $z_i \in \mathcal{H}(D) \cap C^\alpha(\bar{D})$, $i = 1, \ldots, k$, and $\|z\|_\alpha^{\bar{D}} \le \text{const}\{\|c\| + \|w\|_\alpha^{\bar{D}}\}$, (see [HT, p. 339]).

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It is natural to ask whether it is possible to lift polydiscs to a CR manifold. More precisely, if we have a holomorphic function

$$w(\zeta_1,\dots,\zeta_j) = \big(w_1(\zeta_1,\dots,\zeta_j),\dots,w_m(\zeta_1,\dots,\zeta_j)\big) \quad \text{for } |\zeta_1|<1,\dots,|\zeta_j|<1,$$

which is, say, C^1 up to the boundary of the polydisc, then is it possible to find a holomorphic mapping $\zeta = (\zeta_1,\dots,\zeta_j) \longrightarrow \big(z(\zeta), w(\zeta)\big)$ such that $\Im z(e^{i\theta}) = h\big(\Re z(e^{i\theta}), w(e^{i\theta})\big)$, where $e^{i\theta} = (e^{i\theta_1},\dots,e^{i\theta_j})$, for $0 \le \theta_p \le 2\pi, p = 1,\dots,j$?

We note that as in the 1-dimensional case, the real part $\Re z(\cdot)$ should satisfy a Bishop's type equation: $\Re z(\cdot) = J[\Re z(\cdot)] - T\big[h\big(\Re z(\cdot), w(\cdot)\big)\big]$, where this time J denotes the mean operator over the Šilov boundary of the unit polydisc, *i.e.*,

$$J[\Re z(\cdot)] = \frac{1}{(2\pi)^j}\int_0^{2\pi}\cdots\int_0^{2\pi} \Re z(e^{i\theta_1},\dots,e^{i\theta_j})\,d\theta_1\cdots d\theta_j, \tag{2.3}$$

and the operator T assigns to the boundary values of the real part of a holomorphic function restricted to the Šilov boundary of the unit polydisc the boundary values of its imaginary part with mean value zero.

Contrary to the 1-dimensional case, the above Bishop equation cannot always be solved when $j > 1$, even in the rigid case (simple example: $h\big(w(e^{i\theta_1}, e^{i\theta_2})\big) = \Im(e^{2i\theta_1}e^{-2i\theta_2})$). Some additional conditions are needed. However, if such a lifting of polydiscs exists, then it is unique up to a constant vector.

3. **Bishop's type equation for** CR **mappings.** As we have seen in the last section, analytic discs with boundaries on a CR manifold M are determined by their w's coordinates and real parameters. We use this idea for the construction of local CR mappings between CR manifolds; namely if $M \subset \mathbb{C}^{k+m}$ and $N \subset \mathbb{C}^{l+n}$ are given by

$$M\colon \quad y = h(x,w), \quad h(0,0)=0, \quad dh(0,0)=0,$$
$$N\colon \quad \tau = \varphi(\sigma,\xi), \quad \varphi(0,0)=0, \quad d\varphi(0,0)=0,$$

and if $F = (f,g)\colon M \longrightarrow N$ is CR, then we will show that the 'f' part is determined by the 'g' part and some parameters. Obviously since F is CR, g should satisfy additional restrictions. In this section we describe a general procedure, under the hypothesis on the existence of analytic polydiscs, on how to retrieve f from g.

Assume that a smooth family of analytic polydiscs $\{\mathcal{D}_s\}_{s\in\mathcal{S}}$ is given with the Šilov boundaries on M, and that the polydiscs are smooth up to the Šilov boundary (these smoothness conditions can be relaxed). Moreover we assume that the Šilov boundaries $\partial_{\check{S}}\mathcal{D}_s$ of the polydiscs fill up a neighbourhood of the origin in M, which, for simplicity, we denote again by M. So we have $\bigcup_{s\in\mathcal{S}} \partial_{\check{S}}\mathcal{D}_s = M$. More precisely, assume that we have the mapping $\Phi\colon \mathcal{S}\times\mathbb{R}^j \longrightarrow M$, given by

$$\Phi(s,\theta) = \Phi(s,\theta_1,\dots,\theta_j) = \big(z(s,e^{i\theta}), w(s,e^{i\theta})\big), \quad \text{where } e^{i\theta} = (e^{i\theta_1},\dots,e^{i\theta_j}),$$

so that for each $s \in \mathcal{S}$ we get an analytic polydisc $\mathcal{D}_s(\zeta) = \big(z(s,\zeta), w(s,\zeta)\big)$, $\zeta = (\zeta_1,\dots,\zeta_j) \in D^j = \{|\zeta_1|<1,\dots,|\zeta_j|<1\}$. We assume that the mapping Φ is smooth in both variables s and θ, and that the mapping $\mathcal{D}_s$ is holomorphic in ζ and smooth up to the Šilov boundary $\partial_{\check{S}}\mathcal{D}_s$. Since $\dim_{\mathbb{R}} M = k+2m$, we see that the minimal number of parameters is $k+2m-j$ in order that the Šilov boundaries $\{\partial_{\check{S}}\mathcal{D}_s\}_{s\in\mathcal{S}}$ fill up M.

LEMMA 3.1. *Let M and N be two generic* CR *manifolds embedded in* $\mathbb{C}^{k+m}$ *and* $\mathbb{C}^{l+n}$ *respectively. Let* $\{\mathcal{D}_s\}_{s\in\mathcal{S}}$ *be a smooth family of analytic polydiscs with their Šilov boundaries on M and such that* $\bigcup_{s\in\mathcal{S}}\partial_{\check{S}}\mathcal{D}_s = M$. *If a* CR *mapping* $F = (f,g)\colon M \longrightarrow N$ *exists, where* $f = (f_1,\ldots,f_l)$, *then f satisfies a Bishop's type equation on the Šilov boundary of the polydiscs; that is:*

(3.1)
$$\Re f\big(z(s,\cdot),w(s,\cdot)\big) = J\big[\Re f\big(z(s,\cdot),w(s,\cdot)\big)\big] - T\Big[\varphi\Big(\Re f\big(z(s,\cdot),w(s,\cdot)\big),g\big(z(s,\cdot),w(s,\cdot)\big)\Big)\Big].$$

PROOF. If $F=(f,g)\colon M\longrightarrow N$ is CR, then it can be holomorphically extended to the polydisc. This follows, for instance, from the Baouendi-Trèves approximation theorem [BT]. Therefore $F(\mathcal{D}_s)$ is an analytic polydisc with its Šilov boundary on N, and is of the form $F(\mathcal{D}_s)\colon\zeta\longrightarrow\Big(f\big(z(s,\zeta),w(s,\zeta)\big),g\big(z(s,\zeta),w(s,\zeta)\big)\Big)$. We have the following relations between $\Re f$ and $\Im f$ for $\zeta=e^{i\theta}$:

$$\Im f\big(z(s,e^{i\theta}),w(s,e^{i\theta})\big) = \varphi\Big(\Re f\big(z(s,e^{i\theta}),w(s,e^{i\theta})\big),g\big(z(s,e^{i\theta}),w(s,e^{i\theta})\big)\Big).$$

It follows from the discussion in Section 2 that $\Re f$ satisfies equation (3.1).

REMARK 3.2. Note that equation (3.1) is not sufficient for the existence of a CR mapping F in the above form. The function f should also satisfy a system of PDEs (see Section 5). Note also that equation (3.1) takes especially a simple form if N is rigid, and in this case, after suitably choosing the family of analytic discs, the function f can be determined by $J(\Re f)$.

4. **An application of Bishop's equation to** CR **mappings of circular** CR **manifolds.** In this section we apply the procedure described in Section 3 to "the most natural" CR manifolds for lifting polydiscs, namely for mappings from circular CR manifolds (1.1) to rigid manifolds (1.2). For the CR manifold M the most natural family of polydiscs to lift is

$$\begin{cases} w_1(\zeta_1,\ldots,\zeta_m) = r_1\zeta_1, \\ \qquad\vdots \\ w_m(\zeta_1,\ldots,\zeta_m) = r_m\zeta_m \end{cases} \quad \text{for } |\zeta_1|<1,\ldots,|\zeta_m|<1, \tag{4.1}$$

where $r_1,\ldots,r_m$ are positive real numbers that are sufficiently small. The lifted polydiscs are of the form $\big(x+ih(r),r_1\zeta_1,\ldots,r_m\zeta_m\big)$, $r=(r_1,\ldots,r_m)$, $x=(x_1,\ldots,x_k)\in\mathbb{R}^k$, with $\|r\|$ and $\|x\|$ sufficiently small.

LEMMA 4.1. *With the above assumptions and notation, let* $F\colon M\longrightarrow N$, $F=(f,g)$, *be a* CR *mapping, where* $g=(g_1,\ldots,g_n)$ *depends only on w, is holomorphic and is given. Then the mapping f is of the form*

$$f(z,re^{i\theta}) = a(x,r) + H(r,e^{i\theta}), \quad re^{i\theta} = (r_1e^{i\theta_1},\ldots,r_me^{i\theta_m}), \quad x=\Re z, \tag{4.2}$$

where $a = a(x,r)$ is a real vector-valued smooth function and $H = H(r, e^{i\theta})$, for fixed r, is the boundary values of a holomorphic function in the unit polydisc:

$$H(r, \zeta_1, \ldots, \zeta_m) = \sum_{\gamma_1, \ldots, \gamma_m \geq 0} h_{\gamma_1, \ldots, \gamma_m}(r) \zeta_1^{\gamma_1} \cdots \zeta_m^{\gamma_m}, \tag{4.3}$$

PROOF. First recall that the CR mapping F extends holomorphically to the polydiscs, and that the images of the polydiscs under F are (analytic) polydiscs too. Also since the manifold N is rigid, we have $\Im f(z, re^{i\theta}) = \varphi\big(g(re^{i\theta})\big)$ and we have that for fixed z and r, $e^{i\theta} \longrightarrow f(z, re^{i\theta})$ is the boundary values of a holomorphic function. By using (3.1) we obtain $\Re f(z, re^{i\theta}) = J[\Re f(z, re^{i\theta})] - T\big[\varphi\big(g(re^{i\theta})\big)\big]$. We note that the first term on the right-hand side of this equation depends smoothly on x and r, because $y = \Im z = h(r)$. If we denote this term by $a(x,r)$, and $H(r, e^{i\theta}) = -T\big[\varphi\big(g(re^{i\theta})\big)\big] + i\varphi\big(g(re^{i\theta})\big)$, then we immediately get the lemma.

REMARK. Note that if h is an embedding, then the mappings (4.2) and (4.3) can be stated using functions of y instead of r.

5. **Proof of Theorem I.** The main purpose of this section is to prove Theorem I. Also in this section and the next one, we will be able, by imposing further natural restrictions on the manifolds, to obtain explicit formulas for those CR mappings. This in turn will allow us to give conditions which will guarantee that the space of such CR mappings is finite dimensional. Also we give examples to show the necessity of most of these extra conditions.

PROOF OF THEOREM I. Let M be a circular CR manifold as in (1.1), and let N be a rigid CR manifold as in (1.2). The mapping $F: M \longrightarrow N$ should satisfy the system of tangential Cauchy-Riemann equations on M. Antiholomorphic tangent vector fields to M are of the form

$$\bar{X}_\alpha = -i\frac{\partial h_1}{\partial r_\alpha}\frac{w_\alpha}{|w_\alpha|}\frac{\partial}{\partial \bar{z}_1} - \cdots - i\frac{\partial h_k}{\partial r_\alpha}\frac{w_\alpha}{|w_\alpha|}\frac{\partial}{\partial \bar{z}_k} + \frac{\partial}{\partial \bar{w}_\alpha}, \quad \alpha = 1, \ldots, m. \tag{5.1}$$

Since the vector fields $\{\bar{X}_\alpha\}_{\alpha=1}^m$ given by (5.1) form a basis of the antiholomorphic tangent space to M, we have $\bar{X}_\alpha(F) = \bar{X}_\alpha(f,g) = 0$, $\alpha = 1, \ldots, m$. We only need to consider the 'f' part since the 'g' part automatically satisfies the equations by our assumptions. Moreover, we will assume that in (4.3), the real part of the free term of H (that is, when $\gamma_1 = \cdots = \gamma_m = 0$) is included in the function $a(x,r)$. Therefore, and without loss of generality, we suppose that the free term of H is purely imaginary.

To study the equations $\bar{X}_\alpha(f) = 0$, we first find $\bar{X}_\alpha(|w_\beta|)$ and $\bar{X}_\alpha(\theta_\beta)$. Using (5.1) and $\theta_\beta(z,w) = \frac{1}{i}\ln\frac{w_\beta}{|w_\beta|}$, we obtain

$$\bar{X}_\alpha(|w_\beta|) = \frac{1}{2}\delta_{\alpha\beta}\frac{w_\beta}{|w_\beta|} = \frac{1}{2}\delta_{\alpha\beta}e^{i\theta_\beta}, \quad \bar{X}_\alpha(\theta_\beta) = \frac{i}{2}\frac{\delta_{\alpha\beta}}{\bar{w}_\beta}, \quad \alpha, \beta = 1, \ldots, m, \tag{5.2}$$

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where $\delta_{\alpha\beta}$ is the Kronecker symbol. Now, using the form (4.2) of f, we get

$$\bar{X}_\alpha\Big(a(x,r)+H(r,e^{i\theta_1(z,w)},\dots,e^{i\theta_m(z,w)})\Big)=0,\quad \alpha=1,\dots,m.$$

Using (5.1), (5.2), and after simplification, the above equations become

$$-2i\sum_{\beta=1}^{k}\Big(\frac{\partial h_\beta}{\partial r_\alpha}\frac{\partial a}{\partial \bar{z}_\beta}\Big)+\frac{\partial a}{\partial r_\alpha}+\frac{\partial H}{\partial r_\alpha}-\frac{\partial H}{\partial \zeta_\alpha}\frac{1}{\bar{w}_\alpha}=0\quad\text{for }\alpha=1,\dots,m,$$

where $\zeta_\alpha=e^{i\theta_\alpha}=w_\alpha/|w_\alpha|$. Now using (4.3) with $\zeta_\alpha=e^{i\theta_\alpha}$, and comparing the coefficients of $e^{i\theta_1}\cdot\dots\cdot e^{i\theta_m}$ in the above equation, we obtain the following relations:

For $\gamma_1=\dots=\gamma_m=0$, we have

$$(5.3)\qquad -2i\sum_{\beta=1}^{k}\Big(\frac{\partial h_\beta(r)}{\partial r_\alpha}\frac{\partial a(x,r)}{\partial \bar{z}_\beta}\Big)+\frac{\partial a(x,r)}{\partial r_\alpha}+\frac{\partial h_{0,\dots,0}(r)}{\partial r_\alpha}=0.$$

For $\gamma_1+\dots+\gamma_m\geq 1$, we have

$$(5.4)\qquad \frac{\partial h_{\gamma_1\dots\gamma_m}(r)}{\partial r_\alpha}=\gamma_\alpha\frac{h_{\gamma_1,\dots,\gamma_m}(r)}{r_\alpha}.$$

First we consider (5.4). If $h_{\gamma_1,\dots,\gamma_m}(r)\neq 0$ at some point $r\in U$, we get $|h_{\gamma_1,\dots,\gamma_m}(r)|=\tilde{c}_{\gamma_1,\dots,\gamma_m}r_1^{\gamma_1}\cdots r_m^{\gamma_m}$ for some positive constants $\tilde{c}_{\gamma_1,\dots,\gamma_m}$. This holds also for the case $h_{\gamma_1,\dots,\gamma_m}(r)\equiv 0$ by simply allowing the constants $\tilde{c}_{\gamma_1,\dots,\gamma_m}$ to be nonnegative. So we have $h_{\gamma_1,\dots,\gamma_m}(r)=c_{\gamma_1,\dots,\gamma_m}r_1^{\gamma_1}\cdots r_m^{\gamma_m}$, for some complex constants $c_{\gamma_1,\dots,\gamma_m}$ such that $|c_{\gamma_1,\dots,\gamma_m}|=\tilde{c}_{\gamma_1,\dots,\gamma_m}$.

Now we take a closer look at equation (5.3). Since a is a real vector-valued function, and the real part of the free term of H is zero, taking the real and imaginary parts of (5.3), we obtain

$$(5.5)\qquad \frac{\partial a(x,r)}{\partial r_\alpha}=0,\quad \sum_{\beta=1}^{k}\frac{\partial h_\beta(r)}{\partial r_\alpha}\frac{\partial a(x,r)}{\partial x_\beta}=\frac{\partial \Im h_{0,\dots,0}(r)}{\partial r_\alpha}\quad\text{for }\alpha=1,\dots,m.$$

Combining the above form of $h_{\gamma_1,\dots,\gamma_m}$ with (4.2) and (4.3), we get that $f(z,w)=a(\Re z)+i\Im h_{0,\dots,0}(|w|)+p(w)$, where $p(w)=\sum_{\gamma_1+\dots+\gamma_m\geq 1}c_{\gamma_1,\dots,\gamma_m}w_1^{\gamma_1}\cdots w_m^{\gamma_m}$. Now integrating the second part of the system of (5.5) with respect to r_α's, we get $\Im h_{0,\dots,0}(r)=\sum_{\beta=1}^{k}h_\beta(r)\frac{\partial a(x)}{\partial x_\beta}+c(x)$ for some smooth real vector-valued function $c(x)$. Using the fact that F maps M into N, we get

$$\sum_{\beta=1}^{k}h_\beta(r)\frac{\partial a(x)}{\partial x_\beta}+c(x)+\Im\big(p(re^{i\theta})\big)=\varphi\big(g(re^{i\theta})\big),$$

and if we set $r=0$ here, we get $c(x)\equiv 0$. Moreover, if we set $x=0$, we obtain that $\Im h_{0,\dots,0}(r)=\sum_{\beta=1}^{k}h_\beta(r)\frac{\partial a(0)}{\partial x_\beta}$, and finally, denoting $\frac{\partial a(0)}{\partial x_\beta}$ by a_β we obtain (1.3) of Theorem I. It is obvious that after a holomorphic change of variables we can make $p\equiv 0$. This completes the proof of part (a).

The proof of (b) is obvious. The proposition is proved.

Let us note that it follows immediately from (5.5) that if $\operatorname{rank}\big(\frac{\partial h_\beta(r)}{\partial r_\alpha}\big)_{\alpha,\beta}<k$ for all r, then we cannot control in general all the coefficients a_β in (1.3) and it is very easy to see that the family of all mappings from M to N can depend on an infinite number of parameters. But we can show the following consequence of Theorem I:

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COROLLARY 5.1. *(a) With the assumptions of Theorem I and additionally that* $\operatorname{rank}\left(\frac{\partial h_\beta(r)}{\partial r_\alpha}\right) \geq k$ *for some r sufficiently small, it follows that the mapping f is of the form*

$$f(z,w) = a_1 z_1 + \cdots + a_k z_k + p(w) \quad \text{for } (z,w) \in M, \tag{5.6}$$

where $a_1, \ldots, a_m$ *are real constant vectors, and* $p = p(w)$ *is a holomorphic vector-valued function with* $p(0) = 0$. *Also we have*

$$a_1 h_1(r) + \cdots + a_k h_k(r) + \Im\big(p(re^{i\theta})\big) = \varphi\big(g(re^{i\theta})\big). \tag{5.7}$$

Moreover, after a holomorphic change of coordinates in a neighbourhood of the origin of $\mathbb{C}^{l+n}$, *the function p can be made identically equal to zero.*

(b) Conversely, if f is of the form (5.6) with (5.7) and $g = g(w)$ *is holomorphic, then* $F = (f,g)$ *is a* CR *mapping from M into N.*

PROOF OF COROLLARY 5.1. Using the assumption about $\operatorname{rank}\left(\frac{\partial h_\beta(r)}{\partial r_\alpha}\right)_{\alpha,\beta}$, we can solve the system of equations (5.5) with respect to $\frac{\partial a}{\partial x_\beta}$, and get $\frac{\partial a(x)}{\partial x_\beta} = \psi_\beta(r)$, $\beta = 1, \ldots, k$, for some functions ψ_β that depend on r only. Obviously both sides of these equations must be constant. So we obtain that $a(x) = a_1 x_1 + \cdots + a_k x_k + a_0$ for some constant vectors $a_0, \ldots, a_k$. If we look again at (5.5), and substitute these constant vectors for $\frac{\partial a}{\partial x_\beta}$, we obtain $\sum_{\beta=1}^k a_\beta \frac{\partial h_\beta(r)}{\partial r_\alpha} = \frac{\partial \Im h_{0,\ldots,0}(r)}{\partial r_\alpha}$, $\alpha = 1, \ldots, m$, which gives $\Im h_{0,\ldots,0}(r) = a_1 h_1 + \cdots + a_k h_k + \tilde{a}_0$, where $\tilde{a}_0$ is a constant vector. Since $f(0) = 0$ and because $y = h(r)$ on M, we must have $a_0 = 0 = \tilde{a}_0$, and we get (5.6) and (5.7). The proof of (b) is obvious.

6. **Counting the** CR **mappings.** In this section we will again assume that $M \subset \mathbb{C}^{k+m}$ is a circular CR manifold defined by (1.1), and that $N \subset \mathbb{C}^{l+n}$ is a rigid CR manifold given by (1.2).

In the previous section, for a given mapping g we constructed a CR mapping $F = (f,g)$ from M into N, under the assumption that F exists. Obviously, the construction is not possible for an arbitrary g. In this section we consider the following problem:

How many holomorphic mappings $g = g(w)$ *exist such that* $F = (f,g): M \longrightarrow N$ *is a* CR *mapping from a neighbourhood of* 0 *in M into a neighbourhood of* 0 *in N with* $F(0) = 0$, *and is there a relation between those g's?*

In Theorem I we obtained a general form for such mappings, and the condition on g is implicitly hidden in (1.4). In general, it is impossible to find a relation between different g's, if for instance we have that $g = g(w)$ and $\tilde{g} = \tilde{g}(w)$ both satisfy the hypothesis of Theorem I, and that the images of a neighbourhood of 0 in $\mathbb{C}^m$ under g and $\tilde{g}$ intersect only at $0 \in \mathbb{C}^n$. Therefore one of the natural assumptions is that there exists a mapping g which is a surjection onto a neighbourhood of 0 in $\mathbb{C}^n$ and which satisfies the assumptions of Theorem I. Obviously it is possible to consider intermediate cases but we do not want to go in this direction.

Before the formulation of the next theorem, we need some notation. Let $g_0 = g_0(w)$ be a surjective mapping from a neighbourhood of 0 in $\mathbb{C}^m$ onto a neighbourhood of 0 in $\mathbb{C}^n$. Without loss of generality we can assume that for a generic $w'' = (w_{n+1}, \dots, w_m)$ the mapping $(w_1, \dots, w_n) \longrightarrow g_0(w_1, \dots, w_n, w'')$ is not identically zero. For a generic point $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, which is sufficiently close to the origin there exists w with $g(w) = \xi$. We denote by $\mathfrak{g}_0$ the maximal branch of the inverse of the above mapping passing through $w = (w', w'')$, where $w' = (w_1, \dots, w_n)$, and w'' as above; *i.e.*, we have $g_0\big(\mathfrak{g}_0(\xi, w''), w''\big) = \xi$.

If we have any other mapping $g = g(w)$ from a neighbourhood of 0 in $\mathbb{C}^m$ into a neighbourhood of 0 in $\mathbb{C}^n$ (not necessarily surjective), then we use the notation

$$G(w) = \big(G_1(w), \dots, G_n(w), w''\big) = \Big(\mathfrak{g}_0\big(g(w), w''\big), w''\Big),$$
$$|G(w)| = \big(|G_1(w)|, \dots, |G_n(w)|, |w_{n+1}|, \dots, |w_m|\big).$$

We use the same notation if the sequence $1, \dots, n$ is replaced by an arbitrary strictly increasing sequence of n numbers between 1 and m, with the obvious change in the meaning of w' and w''. This should not lead to any confusion.

THEOREM 6.1. *Let $M \subset \mathbb{C}^{k+m}$ be a circular* CR *manifold as in (1.1), and let $N \subset \mathbb{C}^{l+n}$ be a rigid* CR *manifold as in (1.2). Assume that there exists $g_0 = g_0(w)$ which is holomorphic and is surjective from a neighbourhood of* 0 *in $\mathbb{C}^m$ onto a neighbourhood of* 0 *in $\mathbb{C}^n$, and such that $F_0 = (f_0, g_0): M \longrightarrow N$, $F_0(0) = 0$, is* CR*. Then for any other* CR *mapping $F = (f, g): M \longrightarrow N$ with $g = g(w)$ being holomorphic, we have the following relation between g and g_0:*

$$\sum_{\beta=1}^{k} a_\beta^0 h_\beta\big(|G(w)|\big) = \sum_{\beta=1}^{k} a_\beta h_\beta(|w|), \tag{6.1}$$

where a_β^0, a_β, $\beta = 1, \dots, k$, are the constants from Theorem I which correspond to g_0 and g respectively.

In particular, if $m = n$, i.e., $\dim_{\mathrm{CR}} M = \dim_{\mathrm{CR}} N$*, and if g_0 is a biholomorphic mapping between neighbourhoods of the origin in $\mathbb{C}^m$, then we get*

$$\sum_{\beta=1}^{k} a_\beta^0 h_\beta\Big(\big|g_0^{-1}\big(g(w)\big)\big|\Big) = \sum_{\beta=1}^{k} a_\beta h_\beta(|w|).$$

PROOF. From Theorem I, we can assume that f_0 and f have the form as in (1.3), *i.e.*, $f_0(z, w) = a_0(\Re z) + i\sum_{\beta=1}^{k} a_\beta^0 h_\beta(|w|)$ (here $p_0 \equiv 0$), and $f(z, w) = a(\Re z) + i\sum_{\beta=1}^{k} a_\beta h_\beta(|w|) + p(w)$, where the coefficients a_β^0, a_β are real constant vectors. Again using Theorem I, we have

$$\sum_{\beta=1}^{k} a_\beta^0 h_\beta(|w|) = \varphi\big(g_0(w)\big), \quad \sum_{\beta=1}^{k} a_\beta h_\beta(|w|) + \Im p(w) = \varphi\big(g(w)\big). \tag{6.2}$$

Using the assumption about g_0, and the notation introduced before the statement of the theorem, we get

$$\sum_{\beta=1}^{k} a_\beta^0 h_\beta\Big(\big|\big(\mathfrak{g}_0(\xi, w''), w''\big)\big|\Big) = \varphi(\xi).$$

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Combining this equality with the second equality of (6.2), we obtain

$$\sum_{\beta=1}^{k} a_\beta^0 h_\beta\big(|G(w)|\big) = \sum_{\beta=1}^{k} a_\beta h_\beta(|w|) + \Im p(w).$$

We note that the above equality can hold only if the function p is constant, which means in our case that $p \equiv 0$; so we get (6.1). The theorem is proved.

From the above theorem we see that without additional assumptions on the rank of the constant vectors a_β or on the independence of the functions h_β one cannot expect that the set of all CR mappings from M into N will depend only on a finite number of parameters (see examples below). Several different assumptions can be considered leading to quite similar results with quite similar proofs. In order to avoid repetition of the same arguments, we consider only one case given in Theorem II, which, we think, is the most important one, for instance, for classification purposes.

PROOF OF THEOREM II. It suffices to apply Corollary 5.1 to get immediately (i) and (ii). In order to prove (iii), let us take a closer look at (i):

$$\varphi\big(g_0(w)\big) = a_1 h_1(|w|) + \cdots + a_k h_k(|w|) \tag{6.3}$$

where $\varphi = (\varphi_1, \ldots, \varphi_l)$ and where $a_1, \ldots, a_k$ are some constant vectors in $\mathbb{C}^l$. Because F_0 is an embedding, we have that $k \leq l$ and $\operatorname{rank}(a_1, \ldots, a_k) = k$. Since g_0 is invertible, the function $\varphi = (\varphi_1, \ldots, \varphi_l)$ is uniquely determined by (6.3), and we obtain $\varphi(\xi) = a_1 h_1\big(|g_0^{-1}(\xi)|\big) + \cdots + a_k h_k\big(|g_0^{-1}(\xi)|\big)$.

Take any other holomorphic mapping g that satisfies the assumptions of Theorem 6.3. Then, by Corollary 5.1, the mapping f satisfies (5.6) for some set of constant vectors $b_1, \ldots, b_k$ and a holomorphic function $p(w)$. Combining (1.4) with the above formula for $\varphi(\xi)$, and denoting $G = (G_1, \ldots, G_m) = g_0^{-1} \circ g$, we obtain

$$b_1 h_1(|w|) + \cdots + b_k h_k(|w|) + \Im\big(p(w)\big) = a_1 h_1\big(|G(w)|\big) + \cdots + a_k h_k\big(|G(w)|\big). \tag{6.4}$$

From this equality we can express the imaginary part of the holomorphic function $p = p(w)$ in terms of $h_\alpha\big(|G(w)|\big)$ and $h_\alpha(|w|)$. But it is easy to see that such representation is possible only if the function $p = p(w)$ is constant. Because $p(0) = 0$, consequently $p \equiv 0$. Hence from these arguments, we get that the term $\Im\big(p(w)\big) \equiv 0$ in (6.4). Since $\operatorname{rank}(a_1, \ldots, a_k) = k$, we can express $h_\alpha\big(|G(w)|\big)$ in terms of $h_\beta(|w|)$, namely

$$h_\alpha\big(|G(w)|\big) = \sum_{\beta=1}^{k} a_{\alpha\beta} h_\beta(|w|), \quad \alpha = 1, \ldots, k,$$

for some real constants $a_{\alpha\beta}$. The above equality gives (iii) and completes the proof of the theorem.

COROLLARY 6.2. *With the assumptions of Theorem II and additionally assuming that the* CR *codimension of M is equal to its* CR *dimension, i.e. that $k = m$, then we have that the conclusion of Theorem II holds and moreover that G is of the form*

$$G(w) = \big(G_1(w), \ldots, G_m(w)\big), \; G_\alpha(w) = c_\alpha w_1^{j_{\alpha 1}} \cdot \cdots \cdot w_m^{j_{\alpha m}}, \quad \alpha = 1, \ldots, m,$$

where $c_1, \ldots, c_m \in \mathbb{C}$ are constants, and $j_{\alpha\beta}$ are nonnegative integers. Moreover, if g is biholomorphic, then

$$G(w_1, \ldots, w_m) = (c_1 w_{j_1}, \ldots, c_m w_{j_m}),$$

where $c_1, \ldots, c_m \in \mathbb{C}$ are constants, and $(j_1, \ldots, j_m)$ is a permutation of $(1, \ldots, m)$.

In particular, if $m = 1$, and if the function h vanishes to infinite order at $w = 0$, then either $G(w) \equiv 0$ or $G(w) = e^{it}w$, where $t \in \mathbb{R}$.

The proof of the corollary is easy and is left for the reader who should also compare this result with the theorem of Shimizu ([F2, Theorem 3.6], [Sh]).

EXAMPLE 6.3. Let $M \subset \mathbb{C}^2$, $k = 1$, $m = 1$, be given by $y = |w|^2$, and let $N \subset \mathbb{C}^3$, $l = 1, n = 2$, by $\tau = |\xi_1|^2$. So we have $\dim_{\mathrm{CR}} M < \dim_{\mathrm{CR}} N$, and it is very easy to see that the family of all embeddings of M into N depends on an infinite number of parameters.

EXAMPLE 6.4. In this example we want to show that if $\dim_{\mathrm{CR}} M = \dim_{\mathrm{CR}} N$ but $\operatorname{rank}(\frac{\partial \varphi_\beta}{\partial \xi_\alpha}) < \dim_{\mathrm{CR}} N$, then the family of embeddings of M into N also can be infinite dimensional. Let $h = h(r)$ be a real-valued smooth function defined for $r \geq 0$. Let $M \subset \mathbb{C}^4$ be given by $y_1 = h(|w_1|)$, $y_2 = h(|w_2|)$, and let $N \subset \mathbb{C}^4$ be given by $\tau_1 = h(|\xi_1|)$, $\tau_2 = h(|\xi_1|)$. It is easy to see that any holomorphic mapping F of the form $F(z_1, z_2, w_1, w_2) = \big(z_1, z_2, w_1, g(z_1, z_2, w_1, w_2)\big)$ maps M into N.

EXAMPLE 6.5. In this example we want to show a simple application of Corollary 6.2. Let $M \subset \mathbb{C}^4$ be given by $y_1 = \exp(-1/|w_1|)$, $y_2 = \exp(-1/|w_2|)$, and let $N \subset \mathbb{C}^4$ be given by $\tau_1 = \exp\big(-1/(4|\xi_1|)\big)$, $\tau_2 = \exp\big(-1/(9|\xi_2|)\big)$. Using Corollary 6.2 it is very easy to find all CR embeddings $F = (f, g)\colon M \longrightarrow N$, where $g = g(w)$ depends on w only. Namely the embeddings are of the form $F(z, w) = (e^{it_1}z_1, e^{it_2}z_2, \frac{1}{2}e^{it_1}w_1, \frac{1}{3}e^{it_2}w_2)$ or $(e^{it_2}z_2, e^{it_1}z_1, \frac{1}{2}e^{it_2}w_2, \frac{1}{3}e^{it_1}w_1)$, where t_1, t_2 are some real constants.

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