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Lifting *G*-Valued Galois Representations when $\ell \neq p$

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Abstract
In this paper, we study the universal lifting spaces of local Galois representations valued in arbitrary reductive group schemes when $\ell \neq p$. In particular, under certain technical conditions applicable to any root datum, we construct a canonical smooth component in such spaces, generalizing the minimally ramified deformation condition previously studied for classical groups. Our methods involve extending the notion of isotypic decomposition for a GL_n-valued representation to general reductive group schemes. To deal with certain scheme-theoretic issues coming from this notion, we are led to a detailed study of certain families of disconnected reductive groups, which we call *weakly reductive* group schemes. Our work can be used to produce geometric lifts for global Galois representations, and we illustrate this for G_2-valued representations.

1. Introduction

1.1. Galois deformations

Fix a local field F with residue characteristic ℓ. Fix a prime $p \neq \ell$ and a reductive group scheme G (with connected fibers) over the ring of integers $\mathcal{O}$ in a p-adic field. Let k be the residue field of $\mathcal{O}$ and Γ_F the absolute Galois group of F.

Given a G-valued representation of Γ_F over k (i.e., a continuous homomorphism $\overline{\rho} : \Gamma_F \to G(k)$), Tillouine[Til96] introduced a Galois lifting ring $R_{\overline{\rho}}^{\square}$ (building on work of Mazur treating the GL_n case [Maz89, Maz97]). These lifting rings and the associated formal schemes are central in many modern developments in number theory, and it is important to

- understand whether $\overline{\rho}$ lifts to characteristic zero; and
- understand the geometry of $\mathrm{Spf}(R_{\overline{\rho}}^{\square})$, especially to find formally smooth components and understand how they intersect.

(There are similar questions in the complementary situation when $\ell = p$, but the analysis is quite different and is connected with p-adic Hodge theory.) Work on these questions originally focused on the case that $G = \mathrm{GL}_n$ (especially GL_2), but recent developments in the Langlands program have made it increasingly important to understand general reductive groups G. For example, progress on these questions can be used to

○ produce 'nice' lifts of global mod-p representations to characteristic zero [Ram02, HR08, Pat16, Boo19b, Tan19, FKP21, FKP22];

○ investigate $\ell \neq p$ versions of the Breuil-Mézard conjecture [Sho18, Sho22, Sho23];

○ establish potential automorphy theorems and automorphy lifting theorems over global function fields for general G [BHKT19, BFH+22].

Theorem 1.1. *(see Theorems 6.1 and 6.20) Let $\overline{\rho} : \Gamma_F \to G(k)$ be a continuous homomorphism. Suppose p is large enough for the root datum of G (see Remark 6.21). Then there exists a continuous homomorphism $\rho : \Gamma_F \to G(\mathcal{O})$ lifting $\overline{\rho}$ such that $C_G(\rho(I_F))$ is $\mathcal{O}$-smooth. Moreover, there is a canonical $\mathcal{O}$-formally smooth irreducible component $\mathrm{Spf}\, R_{\overline{\rho}}^{\mathrm{m.r.},\square}$ of $\mathrm{Spf}\, R_{\overline{\rho}}^{\square}$.*

We call $R_{\overline{\rho}}^{\mathrm{m.r.},\square}$ the *minimally ramified lifting ring* and the associated Galois representations *minimally ramified*. Previous work about the smoothness of (components of) $R_{\overline{\rho}}^{\square}$ has been for classical groups [CHT08, Boo19a] or about the generic fiber [BP19, BG19] or has had strong dependencies between ℓ and p [DHKM24, §5].

The existence of lifts of $\overline{\rho}$ without the smooth centralizer condition also follows from $\mathbf{Z}_p$-flatness of $R_{\overline{\rho}}^{\square}$, a consequence of results about moduli of Langlands parameters in [DHKM24], [FS24] and [Zhu21] which are established using completely different techniques. In [DHKM24], one ingredient in the proof of $\mathbf{Z}_p$-flatness is to find a finite extension $\mathcal{O}'$ of $\mathcal{O}$ such that $\overline{\rho}$ lifts to $\rho : \Gamma_F \to G(\mathcal{O}')$. The lifts produced in [DHKM24] are different than those produced using our method: they have finite image, and we do not expect the centralizers of the inertia to be $\mathcal{O}$-smooth.

Our initial motivation for this project was to produce characteristic zero lifts of global $\overline{\rho} : \Gamma_K \to G(k)$ that are geometric in the sense of the Fontaine-Mazur conjecture when K is a number field, using variations on a local-to-global lifting result going back to Ramakrishna [Ram02]. As a sample application of Theorem 1.1, we give a lifting result for the exceptional group of type G_2, making use of [FKP21, Theorem A] (a generalization of the local-to-global lifting result) and [Lin20b, Theorem C] (giving local lifts at p). For a representation $\overline{\rho} : \Gamma \to G(k)$, let $\overline{\rho}(\mathfrak{g})$ denote the Lie algebra of G_k with Γ acting via the composition of $\overline{\rho}$ and the adjoint action.

Corollary 1.2. *Let G be the exceptional split group G_2 over $\mathbf{Z}$. Let p be a sufficiently large prime, and let $\overline{\rho} : \Gamma_{\mathbf{Q}} \to G(\overline{\mathbf{F}}_p)$ be a continuous representation. Assume that*

○ *$\overline{\rho}$ is odd (i.e. $\dim \mathrm{H}^0(\Gamma_{\mathbf{R}}, \overline{\rho}(\mathfrak{g})) = \dim \mathrm{Flag}_G$).*

○ *$\overline{\rho}|_{\Gamma_{\mathbf{Q}(\zeta_p)}}$ is absolutely irreducible (i.e., its image in $G(\bar{k})$ is not contained in any proper parabolic subgroup of G).*

Then $\overline{\rho}$ lifts to a $\rho : \Gamma_{\mathbf{Q}} \to G(\overline{\mathbf{Z}}_p)$ which is geometric in the sense of the Fontaine–Mazur conjecture.

The proof is given in Appendix C. All that is needed from Theorem 1.1 is the existence of local lifts at places away from p, which follows as above from [DHKM24, FS24, Zhu21].

Remark 1.3. We emphasize that $\mathcal{O}$-smoothness of the centralizer of the inertia in Theorem 1.1 is crucial for establishing the existence of a formally smooth component of the universal lifting ring; see Theorem 6.16. Besides producing minimally ramified lifts, our method can also produce lifts with other inertial types and sometimes establish smoothness of the component of the deformation ring containing the lift. We do not systematically explore this, as this is not the focus of this work, but we do build flexibility into our results in Section 6 and give a simple illustration in Example 6.10. This is of interest when studying generalizations of the $\ell \neq p$ version of the Breuil-Mézard conjecture and the irreducible components of the moduli of Langlands parameters as in [Sho18, Sho22, Sho23].

Remark 1.4. The Galois lifting space for $\overline{\rho}$ is a formal completion of the moduli spaces of [DHKM24], [FS24, §VIII] and [Zhu21] at $\overline{\rho}$. To make this precise, one can apply [Zhu21, Lemma 2.4.10] to the various moduli spaces of Galois representations and Weil-Deligne representations studied and compared in [Zhu21, §3.1]. Then the main results in these papers show that the Galois lifting spaces are flat local complete intersections, and [DHKM24, §5] provides some results on generic smoothness.

In this language, Theorem 1.1 shows in particular that the underlying reduced subscheme of the mod-p fiber of the moduli space $\mathscr{H}_W$ of Weil-Deligne representations is smooth away from the intersections of components for large p, and even at a point of intersection, there is some smooth component passing through this point. Indeed, Theorem 1.1 asserts that every point lies in a formally smooth irreducible component of Spf $R_{\overline{\rho}}^{\square}$, whose mod-$p$ fiber is therefore a smooth localization of the underlying reduced subscheme of an irreducible component of the mod-p fiber of $\mathscr{H}_W$. As our bound on p is independent of the size of the residue field of F, it applies in situations where the mod-p fibers of the above moduli space are not reduced (see [DHKM24, Proposition 5.26], and note that the banality of p depends on the size of the residue field of F by [DHKM24, Lemma 5.28]). See Remark 6.23 for a slight reinterpretation of our main result in terms of this moduli space.

Remark 1.5. In [BFH+22, Section 4.3], an automorphy lifting theorem for G-valued Galois representations over global function fields is established, assuming that the mod-p residual automorphic Galois representation has suitably large image and that the local deformation problems are *balanced* in the sense of [BFH+22, Definition 3.4]. Section 5 of *loc. cit.* shows that the *unrestricted* local condition at a place of ramification is balanced (and hence formally smooth) if p is larger than an ineffective constant depending on the automorphic representation using global arguments. In contrast, Theorem 1.1 produces a natural balanced local deformation condition in the general case with an effective lower bound on p depending only on the root datum of G (see Remark 6.21 for this lower bound).

Remark 1.6. The restrictions on p in Theorem 1.1 are effective but not optimal: see Remark 6.21. We expect a similar result should hold as long as p is a pretty good prime for G and $p > 3$.

1.2. Weakly reductive group schemes

To prove Theorem 1.1, we will directly adapt the argument of [CHT08] (which dealt with $G = \mathrm{GL}_n$) to a general G. For clarity, we will outline (a reinterpretation of) the argument from [CHT08] which constructs a canonical lift of $\overline{\rho} : \Gamma_F \to \mathrm{GL}(V \otimes_{\mathcal{O}} k)$ up to conjugacy, where V is a finite free $\mathcal{O}$-module of rank n. Let Λ_F be the maximal prime-to-p closed subgroup of the inertia subgroup I_F of Γ_F.

1. Lift $\overline{\rho}|_{\Lambda_F}$ to a representation $\rho_0 : \Lambda_F \to \mathrm{GL}(V)$. Let $V = \bigoplus_i V_i \otimes_{\mathcal{O}} W_i$ be the isotypic decomposition of V.
2. Show that if $p > n$, then there is a unique extension of the Λ_F-representation $\bigoplus_i V_i$ to a representation $\tau = (\tau_i) : I_F \to \prod_i \mathrm{GL}(V_i)$ such that $\det(\tau_i(\sigma))$ is of finite prime-to-p order for all $\sigma \in I_F$ and all i.
3. Choosing an identification $I_F/\Lambda_F \cong \mathbf{Z}_p$ and a splitting $I_F/\Lambda_F \to I_F$, show that there is a unipotent element $u_0 \in \prod_i \mathrm{GL}(W_i \otimes_{\mathcal{O}} k)$ such that $\overline{\rho}(n) = u_0^n$ for all $n \in \mathbf{Z}_p$. Show that u_0 lifts uniquely up to conjugacy to a section $u \in \prod_i \mathrm{GL}(W_i)$ with the same Jordan block decomposition on both fibers, and define $\rho_1 : I_F = \Lambda_F \rtimes \mathbf{Z}_p \to \mathrm{GL}(V)$ by

$$\rho_1(\lambda n) = \tau(\lambda n) u^n.$$

4. Using the uniqueness assertions of (2) and (3), finally extend ρ_1 to $\rho : \Gamma_F \to \mathrm{GL}(V)$.

To adapt this argument for general G in place of GL_n, one first needs to interpret the objects appearing. For instance, in steps 2 and 3, we need analogs of $\prod_i \mathrm{GL}(V_i)$ and $\prod_i \mathrm{GL}(W_i)$. The main observation is that when $G = \mathrm{GL}(V)$, the centralizer $\mathfrak{C} = C_G(\Lambda_F)$ is equal to $\prod_i \mathrm{GL}(W_i)$, while the double centralizer $\Delta = C_G(C_G(\Lambda_F))$ is equal to $\prod_i \mathrm{GL}(V_i)$. From this perspective, it is natural to study $\mathfrak{C} = C_G(\Lambda_F)$ and $\Delta = C_G(C_G(\Lambda_F))$ for general G, which we formalize using the notion of a *decomposition type*. This provides a structure for extending a prime-to-p inertial type to a G-valued representation of Γ_F. It is also necessary to understand the abelianization morphism $\Delta \to \Delta^{\mathrm{ab}}$ (the analog of the determinant) and the center $Z(\Delta)$ (the analog of the group of scalar matrices). We must, therefore, understand representability and smoothness properties of the $\mathcal{O}$-group schemes $\mathfrak{C}$, Δ, Δ^{ab}, and $Z(\Delta)$.

Remark 1.7. In step 3, it is also necessary to find a suitable meaning of 'the same Jordan block decomposition on both fibers' for a unipotent section of a general G and to show a suitable conjugacy result for these. This has been handled in [Cot22a]; see also [Har18] for similar results.

Unlike the case $G = \mathrm{GL}_n$, it is not evident that Δ is smooth (or even representable), and it is usually *not* true that $\mathfrak{C}$ and Δ are reductive group schemes; they often have disconnected fibers. This causes serious difficulties when working integrally, and the theory developed in [GP11] is not sufficient to handle this situation. These difficulties are well known to experts, and we describe some pathologies in Examples 2.16, 2.20 and 2.28.

In order to handle the families of disconnected reductive groups that are relevant to us, we introduce the notion of *weak reductivity*. Recall [Con14, Proposition 3.1.3] that if S is a scheme and G is a smooth affine S-group scheme with (possibly disconnected) reductive fibers, then the relative identity component G^0 is a reductive group scheme, and G/G^0 is an étale separated S-group scheme of finite presentation.

Definition 1.8. Over a scheme S, a weakly reductive group scheme is a smooth affine S-group scheme G such that G^0 is reductive and the component group G/G^0 is finite étale over S with order invertible on S.

We emphasize the condition that G/G^0 is of order invertible on S; if this assumption is omitted, such a smooth affine G is often called *geometrically reductive* or *generalized reductive*. However, as we show in Examples 2.20, 2.28 and 2.16, general geometrically reductive smooth affine group schemes are more pathological than their reductive counterparts.

Theorem 1.9 (Corollary 3.5, Corollary 3.7, Proposition 2.19, Proposition 2.27). *Let S be a scheme, and let G be a weakly reductive S-group scheme.*

1. *If Λ is a finite étale group scheme acting on G whose order is invertible on S, then the fixed point scheme $\mathfrak{C} = C_G(\Lambda)$ is weakly reductive. If $\mathrm{char}\, k(s)$ is pretty good[1] for G_s for all $s \in S$, then the centralizer $\Delta = C_G(\mathfrak{C})$ is weakly reductive.*
2. *If H is a simple reductive group scheme acting on G and $(\dim G/\mathrm{rank}\, H)!$ is invertible on S, then the fixed point scheme $C_G(H)$ is weakly reductive.*
3. *The center $Z(G)$ is a group scheme of multiplicative type, and it is smooth if $Z(G^0)$ is smooth.*
4. *The derived group $\mathscr{D}(G)$ (in the sense of fppf group sheaves) is represented by a weakly reductive S-group scheme, and the abelianization $G^{\mathrm{ab}} = G/\mathscr{D}(G)$ of G is a smooth group scheme of multiplicative type.*

The main new input in the proof of Theorem 1.9 is an analysis of schemes of homomorphisms between weakly reductive group schemes. If S is a scheme and G and H are S-group schemes, we let $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ denote the functor which sends an S-scheme S' to the set of S'-homomorphisms $H_{S'} \to G_{S'}$. In [GP11, Exp. XXIV, Corollaire 7.2.3], it is proved that if H is a reductive S-group scheme and G is a smooth affine S-group scheme, then $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is representable by a separated S-scheme locally of finite presentation. Usually $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is usually not quasi-compact or flat over S (see Example 2.4). However, we will show in Theorem 2.2 that $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is always a disjoint union of finitely presented S-*affine* S-schemes. Proving this involves revisiting the proof of representability of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ in [GP11], using ind-quasi-affine descent and affineness results for schemes of tori in [Ray70]. Over a field, this affineness result was proved in [Bri21, Theorem 6.3], and in general, it strengthens affineness results from [Rom22, Theorem 3.1.4].

Let us give a sense of some key steps in the proof of Theorem 1.9, starting with part (2).

1. Use known cohomology vanishing results (Theorem 3.4) to see that $C_G(H)$ is smooth affine, and use classical arguments over a field to show that $C_G(H)/C_G(H)^0$ is of order invertible on S.
2. Reduce to the case $S = \mathrm{Spec}\, A$ for a DVR A, and use Matsushima's theorem (Theorem 2.1) to reduce to showing that the quotient $G/C_G(H)$ is affine.

---

[1]See Definition 1.12.

3. Show that the natural monomorphism $i : G/C_G(H) \to \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$, given as the orbit map through the inclusion $H \to G$, is a closed embedding, and conclude using the above geometric property of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$.

The proof of Theorem 1.9(1) is similar, but when $H = C_G(\Lambda)$, we cannot show that i is a closed embedding, so our argument is slightly longer.

To prove Theorem 1.9(3), we first show that the automorphism functor $\underline{\mathrm{Aut}}_{G/S}$ is representable by a smooth clopen subscheme $\mathrm{Aut}_{G/S}$ of $\underline{\mathrm{Hom}}_{S\text{-gp}}(G, G)$. Since $Z(G)$ is the kernel of the natural S-homomorphism $\varphi : G \to \mathrm{Aut}_{G/S}$, this reduces us to understanding $\mathrm{Aut}_{G/S}$ and φ. We prove Theorem 1.9(4) through a somewhat complicated reduction to the separate cases of reductive group schemes and finite étale group schemes, both of which are understood.

Remark 1.10. Appendix B provides a curious consequence of Theorem 1.9 to the sizes of component groups of centralizers over fields. It illustrates the power of working with group schemes over rings, even when interested in questions over fields.

1.3. Outline of the paper

Sections 2 and 3 develop the theory of weakly reductive group schemes and establish a variety of results about centralizers. Section 4 introduces the notion of a decomposition type, and Sections 5 and 6 use this and the results about weakly reductive group schemes to construct lifts and the minimally ramified deformation condition. Depending on the reader's interests, Sections 4–6 can be read first relying on the properties of weakly reductive group schemes summarized in Theorem 1.9.

1.4. Notation and terminology

Given a group scheme H defined over a ring R and an R-algebra A, we write H_A for the base change of H to A, and write $H(A)$ for the A-points of H.

If S is a scheme and H is an S-group scheme acting on another S-group scheme G, then we denote by $C_G(H)$ the functor of fixed points for the action of H on G. If H is an S-subgroup scheme of G, then we denote by $N_G(H)$ the functor of sections of G normalizing H. Note that if H is an S-subgroup scheme of G, then $C_G(H)$ is the centralizer of H in G. For representability results, see [GP11, Exp. XII, Proposition 9.2] and [Con14, Proposition 2.1.6]; when these functors are representable, we will use the same notation to denote their representing objects.

We follow the convention in [GP11] and require that reductive group schemes have connected fibers. In Section 2, we introduce the notion of *weakly reductive* group schemes which allows disconnected fibers under some hypotheses. However, when working over a field, we do allow reductive groups to be disconnected, following general practice. We will require all groups of multiplicative type to be finitely presented, unlike the definition in [GP11, Exp. IX].

For a local field F, we use Γ_F to denote the absolute Galois group of F, I_F to denote the inertia subgroup of Γ_F, and $\Lambda_F \subset I_F$ to be the kernel of a homomorphism $I_F \to \mathbf{Z}_p$ as in Section 5.1.

We also recall the definition of a good and pretty good primes for a root datum $(X, \Phi, Y, \Phi^\vee)$.

Definition 1.11. A prime p is *good* if for every closed subsystem $\Sigma \subset \Phi$, $\mathbf{Z}\Phi/\mathbf{Z}\Sigma$ is p-torsion free.

Definition 1.12. We say that p is *pretty good* if the groups $X/\mathbf{Z}\Phi'$ and $Y/\mathbf{Z}\Phi'^\vee$ have no p-torsion for all subsets $\Phi' \subset \Phi$.

A prime is good (resp. pretty good) for a weakly reductive group G if it is good (resp. pretty good) for the root datum associated to G^0. By convention, we also say that 0 is good (and pretty good).

Remark 1.13. By [Cot22a, Lemma 2.2], a prime p is pretty good for a connected reductive group G over a field of characteristic p if and only if all of the following conditions hold:

1. *p* is good for *G*,
2. *p* does not divide the order of $\pi_1(\mathscr{D}(G))$,
3. $Z(G)$ is smooth.

2. Weakly reductive group schemes

In this section, we study weakly reductive group schemes. Weak reductivity is slightly more stringent than the condition of *geometric reductivity*, as introduced in [Alp14, Definition 9.1.1]. We will not recall the definition in general, but we quote the following theorem, which will be used several times in the sequel, especially in Section 3.

Theorem 2.1 [Alp14, Theorems 9.4.1, 9.7.6]. *Let S be a scheme, and let $H \subset G$ be flat, finitely presented, and separated S-group schemes, with H closed in G.*

1. *If G is smooth and affine, then it is geometrically reductive if and only if G^0 is reductive and G/G^0 is finite.*
2. *If G is affine and geometrically reductive, then H is geometrically reductive if and only if G/H is affine.*

In light of Theorem 2.1, a weakly reductive group scheme is just a geometrically reductive smooth affine group scheme with tame component group in the sense of [AOV08]. We will extend some fundamental constructions for reductive group schemes to weakly reductive group schemes. These extensions do not generally work for more general geometrically reductive smooth affine group schemes (see Examples 2.16, 2.20 and 2.28). We note that, by [AHR23, Theorem 9.9], all smooth affine linearly reductive group schemes are weakly reductive; we omit the definition of linear reductivity (which may be found in [AHR23, Definition 2.1] because it is of a technical nature orthogonal to the goals of this paper.

We work throughout with arbitrary base schemes, but many proofs begin by reducing to simpler cases. For the most part, we do not spell out these reductions in detail, and we refer the reader to [DG67, IV$_3$, Sections 8, 9, 11] for the techniques involved in such reduction steps.

2.1. Schemes of homomorphism

Let *S* be a scheme. If *G* and *H* are *S*-group schemes, then we define the set-valued functor $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ on *S*-schemes by

$$\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)(S') := \operatorname{Hom}_{S'\text{-gp}}(H_{S'}, G_{S'}).$$

The goal of this section is to study this functor. The first aim is the following theorem.

Theorem 2.2. *Let S be a scheme, let G be a smooth affine S-group scheme, and let H be a geometrically reductive smooth affine S-group scheme. The functor $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is representable by an ind-quasi-affine S-scheme locally of finite presentation. Moreover, suppose S is normal, quasi-compact and quasi-separated, and H^0 admits a maximal S-torus. Then $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is representable by a disjoint union of finitely presented S-affine S-schemes.*

As a general rule, we like to use underlines to refer to functors, and we omit the underline when referring to a representing object. However, in this case, the notation without the underline has an independent meaning, so it would be confusing to omit it.

It is not clear a priori that $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is even representable; for this, we begin with the following fundamental result of Demazure.

Lemma 2.3. *Suppose that H is a reductive S-group scheme and that G is smooth and quasi-projective over S with affine fibers. Then $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is representable by a separated S-scheme locally of finite presentation.*

Proof. This is [GP11, Exp. XXIV, Corollaire 7.2.3]. □

Example 2.4 [GP11, Exp. XXIV, 7.4]. The scheme $\underline{\mathrm{Hom}}_{\mathbf{Z}\text{-gp}}(\mathrm{SL}_{2,\mathbf{Z}}, \mathrm{SL}_{2,\mathbf{Z}})$ is the disjoint union of the following **Z**-schemes:

1. a scheme isomorphic to Spec **Z** (corresponding to the trivial homomorphism $\mathrm{SL}_{2,\mathbf{Z}} \to \mathrm{SL}_{2,\mathbf{Z}}$),
2. a scheme isomorphic to $\mathrm{PGL}_{2,\mathbf{Z}}$ (corresponding to conjugates of the identity homomorphism $\mathrm{SL}_{2,\mathbf{Z}} \to \mathrm{SL}_{2,\mathbf{Z}}$),
3. for each prime number p and each positive integer n, a scheme isomorphic to $\mathrm{PGL}_{2,\mathbf{F}_p}$ (corresponding to conjugates of the p^n-Frobenius morphism $\mathrm{SL}_{2,\mathbf{F}_p} \to \mathrm{SL}_{2,\mathbf{F}_p}$).

In particular, $\underline{\mathrm{Hom}}_{\mathbf{Z}\text{-gp}}(\mathrm{SL}_{2,\mathbf{Z}}, \mathrm{SL}_{2,\mathbf{Z}})$ is neither flat nor quasi-compact.

To prove Theorem 2.2, we will use étale descent to pass further to the case that H/H^0 is constant and then realize $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ as a closed subscheme of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H^0, G) \times G^n$ for some n. However, in order to make the descent argument, one needs to know effectivity of étale descent for closed subschemes of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H^0, G) \times G^n$. Since $\underline{\mathrm{Hom}}_{S\text{-gp}}(H^0, G)$ is usually not quasi-compact over S, this descent argument is not trivial. Thus, before showing representability, we will show that $\underline{\mathrm{Hom}}_{S\text{-gp}}(H^0, G)$ is *ind-quasi-affine* over S [Sta21, Tag 0AP6] in order to apply effectivity of fpqc descent for ind-quasi-affine morphisms [Sta21, Tag 0APK]. We begin with a more detailed study of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ in the case that H is reductive.

Lemma 2.5. *Let X be a locally noetherian scheme, and let $\pi : Y \to X$ be a finite surjective morphism, where Y is a disjoint union of affine schemes. Then X is ind-quasi-affine. If π is open (e.g., flat), then X is a disjoint union of affine schemes.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme; to show that X is ind-quasi-affine, we must show that U is quasi-affine. Note that $\pi^{-1}(U) \subset Y$ is quasi-compact, so it is contained in an affine clopen subscheme $V \subset Y$ by assumption. By Chevalley's theorem, that affineness can be checked after passing to a finite cover [DG67, II, Théorème 6.7.1], the closed subset $\pi(V) \subset X$ is affine (when considered with its reduced subscheme structure). The schematic closure $\overline{U}$ of U in X is a closed subset of $\pi(V)$, so $\overline{U}_{\mathrm{red}}$ is affine. Thus, by Chevalley's theorem again, $\overline{U}$ is affine. Since U is open in $\overline{U}$, it follows that U is quasi-affine.

If π is open, then the closed subset $\pi(V) \subset X$ is also open, so it is affine when considered with the structure of an open subscheme of X. □

Lemma 2.6. *In the setting of Theorem 2.2, suppose that H is a torus. Then $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is representable by a smooth ind-quasi-affine S-scheme. If S is normal, quasi-compact and quasi-separated, then $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is representable by a disjoint union of smooth affine S-schemes.*

Proof. First, smoothness of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is proved in [GP11, Exp. XI, Corollaire 4.2]. If S is normal and locally noetherian, then the result follows from [Ray70, Théorème IX 2.6]. For the remainder, we therefore assume that S is quasi-compact and quasi-separated. By spreading out (using [TT90, Theorem C.9]), we may assume that S is of finite type over Spec **Z**. In particular, the normalization $S' \to S$ is finite. Now $\underline{\mathrm{Hom}}_{S'\text{-gp}}(H_{S'}, G_{S'})$ is a disjoint union of affine S-schemes, and the morphism $\underline{\mathrm{Hom}}_{S'\text{-gp}}(H_{S'}, G_{S'}) \to \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is finite and surjective, so ind-quasi-affineness of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ follows from Lemma 2.5. □

Lemma 2.7. *In the setting of Theorem 2.2, assume that H is reductive. Then $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is representable by an ind-quasi-affine S-scheme. If S is normal, quasi-compact, and quast-separated, and H admits a maximal S-torus, then $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is representable by a disjoint union of finitely presented affine S-schemes.*

Proof. To show ind-quasi-affineness, we may work locally on S and spread out to assume that S is affine, noetherian and excellent. (For the second claim, we may use [TT90, Theorem C.9] to make the same reduction.) Using Lemma 2.5, we may also pass from S to its normalization to assume that S is normal.

Passing to a further étale cover, we may and do assume that H admits a maximal S-torus. By [GP11, Exp. XXIV, Corollaire 7.1.9], the natural restriction map

$$\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G) \to \underline{\operatorname{Hom}}_{S\text{-gp}}(T, G)$$

is finitely presented and affine, so the result follows from Lemma 2.6. (Note that a smooth affine S-scheme is automatically finitely presented.)   □

Proof of Theorem 2.2. Recall that H is now assumed to be a geometrically reductive smooth affine S-group scheme. In particular, H^0 is a reductive group scheme and H/H^0 is finite by Theorem 2.1 (and similarly for G). There are two issues: first, we need to show that $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is representable, and then we need to show that if S is normal, quasi-compact and quasi-separated, and H admits a maximal S-torus, then $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is a disjoint union of finitely presented S-affine S-schemes (at which point ind-quasi-affineness in general follows from Lemma 2.5). For both points, by working locally and spreading out, we may assume that S is noetherian and connected.

To prove representability, first assume that H/H^0 is constant and that the natural map $H(S) \to (H/H^0)(S)$ is surjective. Let $h_1, \ldots, h_n \in H(S)$ be a system of representatives for $(H/H^0)(S)$. We may and do assume $h_1 = 1$. There is then a natural morphism of functors

$$\beta : \underline{\operatorname{Hom}}_{S\text{-gp}}(H, G) \to \underline{\operatorname{Hom}}_{S\text{-gp}}(H^0, G) \times G^n,$$

given by $f \mapsto (f|_{H^0}, f(h_1), \ldots, f(h_n))$. We claim that β is a closed embedding. To this end, we need to understand when a tuple $(f_0, g_1, \ldots, g_n)$ in $\underline{\operatorname{Hom}}_{S\text{-gp}}(H^0, G)(S') \times G(S')^n$ lies in the image of β.

For indices i, j, let $h_i h_j = h_{\delta(i,j)} h_{i,j}$, where $1 \le \delta(i, j) \le n$ and $h_{i,j} \in H^0(S)$. In any case, there is a unique morphism of S-schemes $f : H \to G$ with $f|_{H^0} = f_0$ and $f(h_i) = g_i$ for all i: for this, note that for any S-scheme S' and any $h \in H(S')$, there is a unique open decomposition $S' = \bigsqcup_{i=1}^{n} S'_i$ such that $h|_{S'_i} = h_i h'_i$ for some $h'_i \in H^0(S'_i)$. Thus, f is defined uniquely by requiring $f(h_i h') = g_i f_0(h')$ for every S-scheme S' and every $h' \in H^0(S')$. Now $(f_0, g_1, \ldots, g_n)$ lies in the image of β if and only if the above-defined f is a homomorphism.

Unraveling, we find that f is a homomorphism if and only if the following conditions are satisfied:

1. $g_1 = 1$,
2. $g_i^{-1} f_0(h) g_i = f_0(h_i^{-1} h h_i)$ for all i,
3. $g_i g_j = g_{\delta(i,j)} f_0(h_{i,j})$.

So indeed, β is a finitely presented closed embedding, whence $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is representable, and in fact, it is ind-quasi-affine over S by Lemma 2.7. Moreover, if S is normal, then this shows that $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is a disjoint union of finitely presented S-affine S-schemes.

Now pass to the general case (i.e., no longer assume that H/H^0 is constant and that $H(S) \to (H/H^0)(S)$ is surjective). In any case, there is a finite étale cover $S' \to S$ such that $H_{S'}/H^0_{S'}$ is constant and $H(S') \to (H/H^0)(S')$ is surjective (e.g., take S' to be a Galois closure of the finite étale H/H^0), and so $\underline{\operatorname{Hom}}_{S'\text{-gp}}(H_{S'}, G_{S'})$ is ind-quasi-affine over S' by the above. Thus, by effectivity of fpqc descent for ind-quasi-affine morphisms [Sta21, Tag 0APK], we see that $\underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is representable and locally of finite presentation. Now that we have representability, we may assume that S is normal, quasi-compact and quasi-separated. As we have already seen, $\underline{\operatorname{Hom}}_{S'\text{-gp}}(H_{S'}, G_{S'})$ is representable by a disjoint union of finitely presented S'-affine S'-schemes, so because the morphism $\underline{\operatorname{Hom}}_{S'\text{-gp}}(H_{S'}, G_{S'}) \to \underline{\operatorname{Hom}}_{S\text{-gp}}(H, G)$ is finite étale, the result follows from Lemma 2.5.   □

Example 2.8. The schemes in Theorem 2.2 are usually not quasi-compact or flat, and they can fail to have smooth fibers. We saw examples of the first two phenomena in Example 2.4. For an example in which smoothness fails, let k be an algebraically closed field of characteristic $p > 0$, and consider the component C of $\underline{\operatorname{Hom}}_{k\text{-gp}}(\operatorname{SL}_2, \operatorname{GL}_{p+1})$ containing the representation $\operatorname{Sym}^p V$, where V is the standard representation of SL_2. This representation is not semisimple: in the notation of [Jan03, II, Chapter 2], it

has Jordan–Hölder factors $L(p)$ and $L(p-2)$. In fact, one can check that C consists of three orbits for the GL_{p+1}-action: the orbit of $\mathrm{Sym}^p V$, the orbit of $L(p) \oplus L(p-2)$ and the orbit of $(\mathrm{Sym}^p V)^*$. The first and last of these orbits are smooth and open, and their closures intersect in the second orbit.

Even worse, components of $\underline{\mathrm{Hom}}_{k\text{-gp}}(\mathrm{SL}_n, \mathrm{GL}_N)$ can have infinitely many orbits for large n and N; one can deduce this using [SX10, Theorem 5.2]. Consequently, it can be difficult to predict the dimensions of components of $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$.

Question 2.9.

1. If k is a field of characteristic $p > 0$ and H and G are reductive, can $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$ be non-reduced? This cannot occur if $p > \dim G / \operatorname{rank} H$, essentially by Theorem 3.4.
2. More generally, if H and G are reductive group schemes over $\mathbf{Z}$, can $\underline{\mathrm{Hom}}_{\mathbf{Z}\text{-gp}}(H, G)$ be non-reduced?
3. If A is a DVR and H and G are reductive, can $\underline{\mathrm{Hom}}_{A\text{-gp}}(H, G)$ have a non-flat component with an integral point? Again, this cannot happen if p is 'large'.

The geometry of Hom-schemes can be related to the theory of complete reducibility as in [BMR05]. Recall that if k is a field, G is a reductive k-group, and $H \subseteq G$ is a closed k-subgroup scheme, then H is G-completely reducible (or G-cr) if, for any R-parabolic $\bar{k}$-subgroup $P \subset G_{\bar{k}}$ such that $H_{\bar{k}} \subset P$, there exists an R-Levi $L \subset P$ such that $H_{\bar{k}} \subset L$. (The notions of R-parabolic and R-Levi subgroup are defined in [BMR05, Section 6]; they coincide with the usual notions of parabolic and Levi if G is connected.)

By [BMR05, Proposition 2.16, Theorem 3.1, Section 6], if H is topologically generated by $x_1, \ldots, x_n \in H(k)$ (in the sense that the subgroup of $H(k)$ generated by $x_1, \ldots, x_n$ is Zariski-dense in H), then H is G-cr if and only if the G-orbit of the n-tuple $(x_1, \ldots, x_n)$ is closed in G^n. Moreover, [Ser03, Property 4] shows that if H is smooth and G-cr, then H^0 is reductive. With these two facts in mind, the following lemma relates G-complete reducibility to orbits in Hom-schemes.

Lemma 2.10. *Let k be a field, and let G and H be (possibly disconnected) reductive k-groups. If H is topologically generated by $x_1, \ldots, x_n \in H(k)$, then the k-morphism $\iota : \underline{\mathrm{Hom}}_{k\text{-gp}}(H, G) \to G^n$ sending f to $(f(x_1), \ldots, f(x_n))$ is monic and satisfies the valuative criterion of properness. In particular, ι is a closed embedding when restricted to any connected component of $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$.*

Proof. It is clear that ι is monic, and the final claim follows from the others by [DG67, IV$_3$, Proposition 8.11.5] and Theorem 2.2, which shows that every connected component of $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$ is of finite type over k. We now verify that ι satisfies the valuative criterion of properness. Let A be a k-algebra which is a DVR with fraction field K, and let $(g_1, \ldots, g_n) \in G(A)^n$ be such that there exists a K-homomorphism $f_1 : H_K \to G_K$ satisfying $f_1(x_i) = g_i$ for all i. Let now $\Gamma \subset H \times_{\mathrm{Spec}\, A} G$ be the schematic closure of the graph of f_1, so that Γ is a flat closed A-subgroup scheme of $H \times_{\mathrm{Spec}\, A} G$ whose projection map π_1 to H is an isomorphism on generic fibers over A. Moreover, since $(x_i, g_i) \in H(A) \times G(A)$ for all i, it follows that $(x_i, g_i) \in \Gamma(A)$, and thus, $\pi_{1,s} : \Gamma_s \to H_s$ is surjective. Since H_s is smooth, we see that $\pi_{1,s}$ is flat, and by fibral flatness, it follows that π_1 is flat. Since $(\ker \pi_1)_K = \{1\}$, it follows from flatness that $\ker \pi_1 = \{1\}$. Thus, $\pi_{1,s}$ is a closed embedding, and since $\pi_{1,s}$ is surjective and H_s is smooth, it follows that $\pi_{1,s}$ is an isomorphism. By the fibral isomorphism criterion, π_1 is therefore an isomorphism, and it is the graph of an A-homomorphism $f : H \to G$ whose generic fiber is f_1. This verifies the valuative criterion. $\square$

Lemma 2.11. *Let k be a field, and let G and H be reductive k-groups. If $f : H \to G$ is a k-homomorphism, then $f(H) \subset G$ is G-cr if and only if the G-orbit through f in $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$ is closed.*

Proof. We may and do pass to a (possibly transcendental) field extension of k to assume that there exist $x_1, \ldots, x_n \in H(k)$ which topologically generate H. If $f(H)$ is G-cr, then by [BMR05, Proposition 2.16, Theorem 3.1], the G-orbit of $(f(x_1), \ldots, f(x_n))$ in G^n is closed. Thus, the G-orbit of f is closed in $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$ since this orbit is simply the preimage of the G-orbit of $(f(x_1), \ldots, f(x_n))$ under the map ι of Lemma 2.10. Conversely, if the G-orbit of f is closed, then $\iota(G \cdot f)$ is closed in G^n by Lemma 2.10, and we conclude with the fact that $\iota(G \cdot f)$ is simply the G-orbit of $(f(x_1), \ldots, f(x_n))$. $\square$

Remark 2.12. Using [Mar03, Proposition 3.2, Theorem 10.3] and Lemma 2.11, one can show that every component of $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$ contains only finitely many closed G-orbits. Consequently, [Ser05, Théorème 4.4] shows that if G is simple and $p > 1 + \mathrm{rank}\, G$, then $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)_{\mathrm{red}}$ is a disjoint union of G-orbits. For classical groups G, $\underline{\mathrm{Hom}}_{k\text{-gp}}(H, G)$ is reduced under these conditions, but we do not know what happens if G is exceptional.

2.2. Automorphism schemes

Next, if G is an S-group scheme, we define the set-valued functor $\underline{\mathrm{Aut}}_{G/S}$ by

$$\underline{\mathrm{Aut}}_{G/S}(S') = \{f \in \mathrm{Hom}_{S'\text{-gp}}(G_{S'}, G_{S'}) : f \text{ is an isomorphism}\}.$$

The following two lemmas are the crucial inputs needed to analyze this functor.

Lemma 2.13. *Let G be a finitely presented S-group scheme. If $f : G \to G$ is an S-homomorphism, then f is an isomorphism if and only if $\ker f_s = \{1\}$ for all $s \in S$.*

Proof. If f is an isomorphism, then certainly $\ker f = \{1\}$. If $\ker f_s = \{1\}$ for all $s \in S$, then $\ker f = \{1\}$: indeed, the identity section $e : S \to \ker f$ is a morphism of S-schemes which is an isomorphism on fibers over S, so because S is S-flat, it follows from the fibral isomorphism criterion [DG67, IV$_4$, Corollaire 17.9.5] that e is an isomorphism (i.e., $\ker f = \{1\}$). Thus, f is monic, and it follows from the Ax–Grothendieck theorem [DG67, IV$_4$, Proposition 17.9.6] that f is an isomorphism.   □

Lemma 2.14. *Let A be a DVR, and let G and H be geometrically reductive smooth affine A-group schemes. If $f : G \to H$ is an A-homomorphism, then the following are equivalent.*

1. *f_s is an isomorphism,*
2. *f_η is an isomorphism,*
3. *f is an isomorphism.*

Proof. Clearly, (3) implies (1) and (2). Conversely, if f_s and f_η are both isomorphisms, then f is an isomorphism by the fibral isomorphism criterion [DG67, IV$_4$, Corollaire 17.9.5]. Thus, it suffices to show that (1) and (2) are equivalent. We may and do further assume that A is complete with algebraically closed residue field.

First, suppose that G (and hence also H under either (1) or (2), due to finiteness of H/H^0) has connected fibers. Assume f_s is an isomorphism. By the fibral isomorphism criterion, $f_{A/\mathfrak{m}^n}$ is an isomorphism for all $n \geq 1$. By the local flatness criterion [Mat89, Theorem 22.3], f is flat. In fact, f is étale near 1 because the étale locus is open, and since it is a homomorphism with G fppf over A, it follows that f is étale. In particular, $\ker f$ is étale. One can check that an étale normal subgroup scheme of a connected group scheme over a field is automatically central, so $\ker f_\eta$ is central, and thus, $\ker f$ is contained in $Z(G)$. But $Z(G)$ is of multiplicative type, and $\ker f$ is a flat closed A-subgroup scheme of $Z(G)$, so it is also of multiplicative type by [Con14, Corollary B.3.3]. Since $\ker f_s = \{1\}$, we conclude that $\ker f_\eta = \{1\}$, and hence, f_η is a closed embedding by [GP11, Exp. VI$_B$, Corollaire 1.4.2]. For dimension reasons, f_η is dominant, so it is surjective by [GP11, Exp. VI$_B$, Proposition 1.2], and hence, it is an isomorphism since H_η is smooth.

Conversely, suppose that f_η is an isomorphism. If $g \in G(k(s))$ is a nontrivial semisimple element, then there is a maximal $k(s)$-torus $T_0 \subset G_s$ such that $g \in T_0(k(s))$. By [GP11, Exp. IX, Théorème 3.6], since A is complete, we may find a maximal A-torus $T \subset G$ with special fiber T_0. Since $\ker f|_{T_\eta} = \{1\}$, it follows from [GP11, Exp. IX, Théorème 6.8] that $\ker f|_T = \{1\}$, so in particular, $g \notin (\ker f)(k(s))$. Note that $(\ker f_s)^0_{\mathrm{red}}$ is a smooth connected closed normal subgroup of G_s, so it is *reductive*, and thus, its semisimple locus is dense. But the above argument shows that its semisimple locus is $\{1\}$, so in fact, $(\ker f_s)^0_{\mathrm{red}} = \{1\}$, whence $\ker f_s$ is *finite*. Thus, for dimension reasons, f_s is dominant, and since G is smooth we find that f_s is flat: this follows from generic flatness and a simple translation argument. By the fibral flatness criterion [DG67, IV$_3$, Théorème 11.3.10], f is flat, and so $\ker f$ is flat. Since

$\ker f_\eta = \{1\}$, it follows that $\ker f = \{1\}$: as a flat closed subscheme of G, $\ker f$ is the closure of its generic fiber. So f_s is a surjective closed embedding, and hence an isomorphism.

Now consider the general case – namely, that G/G^0 is finite. Suppose that either f_s or f_η is an isomorphism. By the reductive case settled above, we find that $f|_{G^0} : G^0 \to H^0$ is an isomorphism. Moreover, the homomorphism $G/G^0 \to H/H^0$ between constant groups is an isomorphism, since this can be checked on either the special or generic fiber. Thus, a diagram chase shows that f is an isomorphism. □

Theorem 2.15. *Let S be a scheme, and let G be a geometrically reductive smooth affine S-group scheme. The functor $\underline{\mathrm{Aut}}_{G/S}$ is representable by an open and closed subscheme $\mathrm{Aut}_{G/S}$ of $\underline{\mathrm{Hom}}_{S\text{-gp}}(G, G)$.*

Proof. By spreading out, we may and do assume that S is noetherian, so that $\mathscr{H} := \underline{\mathrm{Hom}}_{S\text{-gp}}(G, G)$ is locally noetherian by Lemma 2.3. Note that there is a universal S-homomorphism

$$f : G \times_S \mathscr{H} \to G \times_S \mathscr{H}$$

whose fiber over a given section of $\mathscr{H}$ is the corresponding endomorphism of G. By Lemmas 2.13 and 2.14, since $\mathscr{H}$ is locally noetherian, the locus U of $u \in \mathscr{H}$ such that $\ker f_u = \{1\}$ is open and closed, and $f_U : G \times_S U \to G \times_S U$ is an isomorphism. It follows from this reasoning that U represents $\underline{\mathrm{Aut}}_{G/S}$. □

To use Theorem 2.15, it will be necessary to establish some more properties of the Aut-scheme. If G is a weakly reductive group scheme over a scheme S, then we will see in Lemma 2.18 that $\mathrm{Aut}_{G/S}$ is always smooth. However, the following example shows that $\mathrm{Aut}_{G/k}$ may fail to be smooth if k is a field of characteristic $p > 0$ and G is a reductive k-group with component group of order divisible by p.

Example 2.16. Let $G = \mathbf{G}_m \times \mathbf{Z}/p\mathbf{Z}$ over a field k of characteristic $p > 0$. If S is a k-scheme, then S-automorphisms of G_S correspond to pairs (ϕ_0, g), where $\phi_0 : \mathbf{G}_{m,S} \to \mathbf{G}_{m,S}$ is an S-automorphism and $g \in G(S)$ is a section of order p such that for all $s \in S$, g_s does not lie in $G^0(k(s))$; the correspondence is given by sending an S-automorphism ϕ of G_S to $(\phi|_{G_S^0}, \phi(x))$, where $x = (1, 1) \in \mathbf{G}_m(S) \times \mathbf{Z}/p\mathbf{Z}(S)$. Using this, one sees that $\mathrm{Aut}_{G/k}^0 \cong \mu_p$. In particular, $\mathrm{Aut}_{G/k}$ is not smooth.

Lemma 2.17. *Let G be a (possibly disconnected) reductive group over a field k. The natural map $G \to \mathrm{Aut}_{G/k}$ is open. If k is perfect, then $\phi : G \to (\mathrm{Aut}_{G/k})_{\mathrm{red}}$ is flat.*

Proof. Since formation of $(\mathrm{Aut}_{G/k})_{\mathrm{red}}$ commutes with separable field extensions on k and purely inseparable extensions leave topological spaces unchanged, we may and do assume that k is algebraically closed. To show that ϕ is flat, it suffices to show that the map $G^0(k) \to (\mathrm{Aut}_{G/k})^0(k)$ is surjective. Indeed, then ϕ is a dominant map from a smooth finite type k-scheme to a smooth k-scheme, so it is generically flat, and being a group homomorphism translation, arguments show that it is flat.

Let $r : \mathrm{Aut}_{G/k} \to \mathrm{Aut}_{G^0/k} \times \mathrm{Aut}_{(G/G^0)/k}$ denote the natural restriction homomorphism. The map $G^0 \to \mathrm{Aut}_{G/k}$ induces a map $Z(G^0) \to \ker r$. We claim that the image of $Z(G^0)$ in $\ker r$ is of finite index. Once this is done, the lemma will follow: indeed, then

$$\dim \ker r|_{\phi(G^0)} = \dim \ker r|_{\phi(Z(G^0))} = \dim \ker r.$$

Since the group scheme of outer automorphisms of G^0 is etale, $\mathrm{pr}_1 \circ r : G^0 \to \mathrm{Aut}_{G^0/k}^0$ is surjective. Since $\mathrm{Aut}_{(G/G^0)/k}$ is finite, we find in particular that $\dim r(\phi(G)) = \dim r(\mathrm{Aut}_{G/k})$, so

$$\dim \phi(G^0) = \dim \ker r + \dim r(\mathrm{Aut}_{G/k}) = \dim \mathrm{Aut}_{G/k}.$$

Therefore, $\phi(G^0)$ is an open subgroup scheme of the smooth k-group scheme $(\mathrm{Aut}_{G/k})_{\mathrm{red}}$, so it contains $(\mathrm{Aut}_{G/k})^0$ and hence is equal to it.

So it remains to show that the image of $Z(G^0)$ of finite index in $(\ker r)(k)$. First, we must understand some properties of $\mathrm{Aut}_{G/k}$. Suppose that $f : G \to G$ is a k-homomorphism inducing the identity on

G^0 and G/G^0 (i.e., $f \in (\ker r)(k)$). We define a morphism $\lambda : G \to G$ by

$$\lambda(g) = f(g)g^{-1}.$$

One checks that λ is a 1-cocycle. Since $f|_{G^0}$ is the identity, we have

$$\lambda(gh) = f(gh)(gh)^{-1} = f(g)f(h)h^{-1}g^{-1} = \lambda(g)$$

for all functorial points g and h of G and G^0, respectively. Thus, λ factors through a morphism $G/G^0 \to G$. Moreover, for such g and h, we have

$$\lambda(g) = \lambda(hg) = f(hg)(hg)^{-1} = h\lambda(g)h^{-1},$$

so in fact, λ has image lying in $Z(G^0)$. Thus, λ is a 1-cocycle $G/G^0 \to Z(G^0)$. Note that any such λ determines f uniquely.

As in ordinary group cohomology, there is a short exact sequence

$$0 \to \mathrm{B}^1(G/G^0, Z(G^0)) \to \mathrm{Z}^1(G/G^0, Z(G^0)) \to \mathrm{H}^1(G/G^0, Z(G^0)) \to 0.$$

Notice that $\mathrm{B}^1(G/G^0, Z(G^0)) = \mathrm{B}^1((G/G^0)(k), Z(G^0)(k))$ (and so on) because G/G^0 is constant. Under the correspondence between f and λ as in the previous paragraph, $\mathrm{B}^1(G/G^0, Z(G^0))$ consists of those k-homomorphisms f induced by conjugation by an element of $Z(G^0)(k)$. Moreover, if G/G^0 is of order n, then $\mathrm{H}^1(G/G^0, Z(G^0))$ is n-torsion, so $\mathrm{H}^1(G/G^0, Z(G^0))$ admits a surjection from $\mathrm{H}^1(G/G^0, Z(G^0)[n])$ and hence is finite. So indeed, the map $Z(G^0)(k) \to (\ker r)(k)$ has finite index image, and we are done.   □

Lemma 2.18. *If G is a weakly reductive group scheme over S, then $\mathrm{Aut}_{G/S}$ is smooth and the natural map $\phi : G \to \mathrm{Aut}_{G/S}$ is flat. If $Z(G^0)$ is smooth, then ϕ is smooth.*

Proof. Let $\mathfrak{g} = \mathrm{Lie}\, G$. We claim that $\mathrm{H}^2(G_s, \mathfrak{g}_s) = 0$ for all $s \in S$. There is a Hochschild–Serre spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(G_s/G_s^0, \mathrm{H}^q(G_s^0, \mathfrak{g}_s)) \Rightarrow \mathrm{H}^{p+q}(G_s, \mathfrak{g}_s),$$

and since G_s/G_s^0 has order prime to char $k(s)$ for all $s \in S$, we find

$$\mathrm{H}^n(G_s, \mathfrak{g}_s) = \mathrm{H}^0(G_s/G_s^0, \mathrm{H}^n(G_s^0, \mathfrak{g}_s))$$

for all n. By [GP11, Exp. XXIV, Corollaire 1.13(ii)] (in which reference reductive groups over *fields* are also required to be connected), we have $\mathrm{H}^2(G_s^0, \mathfrak{g}_s) = 0$, so indeed, $\mathrm{H}^2(G_s, \mathfrak{g}_s) = 0$ for all $s \in S$. The same argument, using [GP11, Exp. XXIV, Corollaire 1.15.1], shows that $\mathrm{H}^1(G_s, \mathfrak{g}_s) = 0$ for all $s \in S$ if $Z(G^0)$ is smooth.

To show that $\underline{\mathrm{Aut}}_{G/S}$ is smooth, it suffices to verify the infinitesimal criterion of smoothness, and for this [GP11, Exp. III, Corollaire 2.9(ii)] shows that it suffices to show $\mathrm{H}^2(G_s, \mathfrak{g}_s) = 0$ for all $s \in S$, which we showed above. Moreover, [GP11, Exp. III, Corollaire 2.9(i)] shows that if $\mathrm{H}^1(G_s, \mathfrak{g}_s) = 0$ for all $s \in S$, then the morphism ϕ satisfies the infinitesimal criterion of smoothness, so it is smooth. In particular, the previous paragraph shows that if $Z(G^0)$ is smooth, then ϕ is smooth.

Finally, to show that ϕ is flat in general, we may use the fibral flatness criterion [DG67, IV_3, Théorème 11.3.10] to assume that $S = \mathrm{Spec}\, k$ for a field k. Thus, since $\mathrm{Aut}_{G/k}$ is smooth, we may conclude using Lemma 2.17.   □

Proposition 2.19. *If G is a weakly reductive S-group scheme, the functorial center $Z(G)$ is an S-group scheme of multiplicative type. If $Z(G^0)$ is smooth, then $Z(G)$ is smooth.*

Example 2.20. Before proving the proposition, we offer the following example to show that geometric reductivity for a smooth affine G is not enough for flatness of $Z(G)$. Let $A = \mathbf{Z}_p[\zeta_p]$. There is a non-split central extension of constant A-group schemes

$$1 \to \mathbf{Z}/p \to U \to (\mathbf{Z}/p)^2 \to 1,$$

where U is isomorphic to the group of $\mathbf{F}_p$-points of the unipotent radical of a Borel in SL_3 (i.e., U is the 'Heisenberg group' over $\mathbf{F}_p$). There is a homomorphism of A-group schemes $\mathbf{Z}/p \to \mu_p$ which is trivial on the special fiber and an isomorphism on the generic fiber, given by the choice of a primitive pth root of unity ζ_p. Pushing the above extension forward by the map $\mathbf{Z}/p \to \mu_p \to \mathbf{G}_m$ gives an extension

$$1 \to \mathbf{G}_m \to G \to (\mathbf{Z}/p)^2 \to 1,$$

where G has commutative special fiber and non-commutative generic fiber. In fact, $Z(G)$ has special fiber G and generic fiber G^0, so it is not flat.

Proof. Working locally and spreading out, we may and do assume that S is locally noetherian. Consider the morphism $\phi : G \to \mathrm{Aut}_{G/S}$. By Lemma 2.18, ϕ is flat, so $Z(G) = \ker\phi$ is a flat closed S-subgroup scheme of G, and it is smooth provided that $Z(G^0)$ is smooth. So we need only show that $Z(G)$ is of multiplicative type.

We define $C_{G^0}(G) := \ker\phi|_{G^0} = \ker\phi|_{Z(G^0)}$; note that $Z(G^0)$ is of multiplicative type. By [GP11, Exp. IX, Théorème 6.8], it follows that $C_{G^0}(G)$ is of multiplicative type. Moreover, there is a short exact sequence

$$1 \to C_{G^0}(G) \to Z(G) \to Z(G)/C_{G^0}(G) \to 1.$$

Since $Z(G)/C_{G^0}(G)$ is a closed commutative flat and finitely presented S-subgroup scheme of G/G^0, it is finite étale commutative of order invertible on S, and in particular, it is of multiplicative type. Thus, $Z(G)$ is a commutative extension of multiplicative type group schemes, so it is of multiplicative type by [GP11, Exp. XVII, Prop. 7.1.1]. □

Remark 2.21. With more work, the kind of arguments used in the proof of Lemma 2.17 (expanded upon in the case of 'abstract' groups in [Wel71]) can be used to proved that if G is weakly reductive over S, then there is a short exact sequence

$$1 \to G/Z(G) \to \mathrm{Aut}_{G/S} \to \mathrm{Out}_{G/S} \to 1$$

in which $\mathrm{Out}_{G/S}$ is an étale-locally constant S-group scheme, just as in the theory of reductive group schemes. Proving this would take us somewhat far afield, so we omit it.

We note that the above discussion gives a slight strengthening of [Mar03, Lemma 6.8].

Corollary 2.22. *Let k be a field, let G be a finite type k-group scheme, and let $H \subset G$ be a closed (not necessarily connected) reductive k-subgroup. The quotient $N_G(H)/HC_G(H)$ is finite. If H/H^0 has order prime to* char k*, then $N_G(H)/HC_G(H)$ is étale.*

Proof. We may and do assume that k is algebraically closed. Let $\phi : H \to \mathrm{Aut}_{H/k}$ be the natural map. By Lemma 2.17, $\phi(H)$ is an open subset of $\mathrm{Aut}_{H/k}$, so $\phi(H)$ is an open subgroup scheme of $(\mathrm{Aut}_{H/k})_{\mathrm{red}}$ and the quotient $A := \mathrm{Aut}_{H/k}/\phi(H)$ (which exists as a scheme by [GP11, Exp. VI$_A$, Théorème 3.2]) is locally of finite type. Moreover, A_{red} is étale: since $\phi(H)$ is (topologically) open in $\mathrm{Aut}_{H/k}$, A is discrete, and any reduced discrete scheme locally of finite type over an algebraically closed field is étale. The natural map $N_G(H) \to A$ has kernel $HC_G(H)$, so $N_G(H)/HC_G(H)$ is a finite type closed subscheme of A. Since A_{red} is étale, it follows that $N_G(H)/HC_G(H)$ is finite. If H/H^0 has order prime to char k, so H is weakly reductive, then already A is étale by Lemma 2.18, and thus, $N_G(H)/HC_G(H)$ is étale. □

Definition 2.23. If G is a finite type k-group scheme and H is a reductive k-subgroup scheme of G, then we set $W_H := N_G(H)/HC_G(H)$, which we call the *Weyl group* of the pair (G, H).

If H is a maximal torus of a smooth affine G, then this is the usual Weyl group W of G.

Proposition 2.24. *Let G be a reductive group over a field k, and let H be a reductive k-subgroup scheme of G. For a prime p, if $p \nmid |W|$ and $p \geq |H/H^0|$, then $p \nmid |W_H|$.*

Proof. We may and do assume that k is algebraically closed. Let $g \in N_G(H)(k)$; we need to show that under our hypotheses, g^n acts by an inner automorphism on H for some integer n prime to p. Let T be a maximal torus of H, and note that conjugacy of maximal tori in H^0 implies that after translation by $H^0(k)$, we may assume $g \in N_G(T)(k)$. Since $N_G(T)/C_G(T)$ is a subquotient of W, it follows that $g^{|W|} \in C_G(T)(k)$. Thus, after replacing g by $g^{|W|}$, we may and do assume that g centralizes T. In particular, g acts trivially on the Dynkin diagram of (H^0, T), so g acts on H^0 by an inner automorphism. Thus, after further translation by $H^0(k)$, we may and do assume that g centralizes H^0. Since $p \geq |H/H^0|$, we may pass to a further prime-to-p power of g to assume that g acts trivially on H/H^0. (It is a general fact, easily checked, that if $p \mid |\mathrm{Aut}(A)|$ for a finite group A, then $p < |A|$.)

Now that g acts trivially on H^0 and H/H^0, the argument of Lemma 2.17 shows that $\mathrm{Ad}(g)$ corresponds to a 1-cocycle $\eta : H/H^0 \to Z(H^0)$ (which lands in $Z(H^0)(k)$ since H/H^0 is constant). The corresponding class in $\mathrm{H}^1(H/H^0, Z(H^0)(k))$ is killed by $|H/H^0|$, so further passing from g to $g^{|H/H^0|}$, we may and do assume that $\mathrm{Ad}(g)$ is cohomologically trivial, from which is follows that g acts on H by conjugation by an element of $h \in Z(H^0)(k)$. Thus, g now acts on H by inner automorphisms, and we are done. $\square$

2.3. Abelianization

Our final goal in this section is to show the existence of the 'abelianization' of a weakly reductive group scheme G. We begin with the following folkloric lemma.

Lemma 2.25. *Let S be a scheme, and let $f : G \to H$ be a homomorphism of finitely presented S-group schemes. The following are equivalent.*

1. *f is faithfully flat,*
2. *f is an epimorphism of fppf sheaves and $\ker f$ is flat.*

Proof. Omitted. $\square$

Lemma 2.26. *Let $1 \to M \to E \to H \to 1$ be a central extension, where M is an S-group scheme of multiplicative type and H is a finite étale group scheme of constant order n invertible on S. Letting $N = n^2$, for every integer $d \geq 1$, the S-morphism $[Nd] : E \to E$ is a homomorphism. If M is moreover a torus, then $E[N] \to H$ is faithfully flat, where $E[N] := [N]^{-1}(1)$.*

Proof. By spreading out and étale-localizing around a point of S, we may and do assume that $S = \mathrm{Spec}\, A$ for a strictly henselian noetherian local ring A. In particular, since H is finite étale, it is a constant group and there exists a scheme-theoretic section $H \to E$. We may moreover pushforward by an inclusion of M into a torus to assume that M is a torus. The existence of a section implies that for every S-scheme S', the sequence

$$1 \to M(S') \to E(S') \to H(S') \to 1$$

is exact.

First, note that as an extension of H by M which admits a section, E corresponds to a cohomology class in $\mathrm{H}^2(H, M)$, the Hochschild cohomology group (see, for example, [DG70, II, §3, Prop. 2.3] or [Dem15, Prop. 2.3.6]). Concretely, this is the space of 2-cocycles $H \times H \to M$ modulo coboundaries. Since H is a constant group scheme, we have $\mathrm{H}^2(H, M) = \mathrm{H}^2(H(S), M(S))$. Since $H(S)$ is finite of

order n, $\mathrm{H}^2(H(S), M(S))$ is killed by n by the classical theory. Since n is invertible on the strictly henselian S, via the correspondence between extensions and classes in H^2, the image of the extension corresponding to multiplication by n on M is

$$1 \to M(S) \xleftarrow[n]{\cong} M(S)/M[n](S) \to E(S)/M[n](S) \to H(S) \to 1.$$

Thus, this extension is split (i.e., there is a section $\alpha_0 : H(S) \to E(S)/M[n](S)$ which is a homomorphism). This is equivalent to a section $H \to E/M[n]$, which we also denote by α_0.

Since S is strictly henselian, α_0 lifts to a section $\alpha : H \to E$. Now $\alpha_0(x)^n = \alpha_0(x^n) = 1$ for all $x \in H(S)$, so we have $\alpha(x)^n \in M[n](S)$, and thus,

$$\alpha(x)^N = \alpha(x)^{n^2} = 1.$$

Moreover, since α_0 is a homomorphism, we have $\alpha(xx')^{-1}\alpha(x)\alpha(x') \in M[n](S)$ for all $x, x' \in H(S)$. In other words,

$$\alpha(x)\alpha(x') = \alpha(xx')g_{x,x'}$$

for some $g_{x,x'} \in M[n](S)$.

Now we conclude the proof that $[Nd] : E \to E$ is a homomorphism. We shall check this on S'-points for every S-scheme S'. Any element of $G(S')$ is Zariski-locally of the form $\alpha(x)g$ for some $x \in H(S')$ and $g \in M(S')$, and using centrality of M, we compute, for $x, x' \in H(S')$ and $g, g' \in M(S')$,

$$(\alpha(x)g\alpha(x')g')^{Nd} = \alpha(xx')^{Nd}g_{x,x'}^{Nd}g^{Nd}g'Nd = g^{Nd}g'Nd = (\alpha(x)g)^{Nd}(\alpha(x')g')^{Nd},$$

so indeed $[Nd] : E \to E$ is a homomorphism.

Finally, we show that $E[N] \to H$ is faithfully flat when M is a torus. For this, let h be a local section of H. After fppf localization, we may lift h to a section e of E. Note that e^n is a local section of M, so because M is a divisible group scheme, we may pass to a further fppf localization to assume $e^n = m^n$ for some local section m of M. Because M is central in E, we see that $(em^{-1})^n = 1$, and thus, em^{-1} is a local section of $E[N]$ mapping to h. So faithful flatness follows from Lemma 2.25. $\qquad\square$

Proposition 2.27. *Suppose G is a weakly reductive S-group scheme. There exists a smooth S-group scheme G^{ab} of multiplicative type and a faithfully flat homomorphism $\pi : G \to G^{\mathrm{ab}}$ with the universal property that for any fppf abelian sheaf H on the category of S-schemes and homomorphism of sheaves $f : G \to H$, there is a unique homomorphism $G^{\mathrm{ab}} \to H$ through which f factors.*

The kernel $\mathscr{D}(G)$ of π represents the fppf-sheafification of the functor $S' \mapsto [G(S'), G(S')]$ on S-schemes S', and in particular, the formation of G^{ab} commutes with any base change on S.

Example 2.28. Before giving the proof, we illustrate again the relevance of weak reductivity (as opposed to geometric reductivity). For the G in Example 2.20, we claim that G^{ab} cannot exist as a scheme. Indeed, note that G_s is commutative, so on the special fiber, we have $\mathscr{D}(G_s) = 1$. However, since ζ_p does not lie in pA, it follows that $G_{A/p}$ is not commutative, so $\mathscr{D}(G_{A/p}) \neq 1$. Since every scheme over A/p with trivial special fiber is trivial (by 'nilpotent Nakayama'), this is a contradiction.

Proof of Proposition 2.27. The claims in the second paragraph of the proposition follow directly from the universal property of the first paragraph. First, note that if G has connected fibers, then [Con14, Thm. 5.3.1] shows that G^{ab} exists and is a torus. In general, any S-homomorphism $f : G \to H$ as in the statement of the proposition induces an S-homomorphism $G^0 \to H$, so f kills $\mathscr{D}(G^0)$ and hence factors through an S-homomorphism $G/\mathscr{D}(G^0) \to H$, where $G/\mathscr{D}(G^0)$ is a smooth affine S-group scheme with torus identity component. Thus, we may and do assume that G^0 is a torus.

Working étale-locally on S (using effectivity of étale descent for affine S-schemes), we may choose representatives $h_1, \ldots, h_n \in G(S)$ for G/G^0. Define the map $\phi : (G^0)^n \to G^0$ by $\phi(g_1, \ldots, g_n) =$

$\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}$. Note that ϕ is an S-homomorphism, so by [GP11, Exp. IX, Thm. 6.8], there is some S-subgroup scheme M of G^0 of multiplicative type through which ϕ factors and such that the factored map $\phi : (G^0)^n \to M$ is faithfully flat. Since f vanishes on all commutators, it annihilates M, and thus, f must factor through G/M. By definition, if $g \in G^0(S')$ is a section, then $h_i g h_i^{-1} \in gM$, so since G^0 is commutative, it follows that G^0/M is central in G/M. Thus, replacing G by G/M, we may and do assume that G^0 is central in G.

Working Zariski-locally on S, we may and do assume that the index of G^0 in G is constant, say equal to n. By Lemma 2.26, $G[N] := [N]^{-1}(1)$ is an S-subgroup scheme of G where $N = n^2$. By the same lemma, the map $G[N] \to G/G^0$ is faithfully flat, so there is a short exact sequence

$$1 \to G^0[N] \to G[N] \to G/G^0 \to 1.$$

Since $G^0[N]$ and G/G^0 are both finite étale (the former because N is invertible on the base), it follows that $G[N]$ is also finite étale.

By working étale locally on S, we may assume that $G[N]$ is a constant group scheme. In this case, the functor on S-schemes $S' \mapsto [G[N](S'), G[N](S')]$ is represented by the constant S-group scheme $\mathscr{D}(G[N])$. Moreover, if $\mathscr{D}(G)$ denotes the sheafification of the functor $S' \mapsto [G(S'), G(S')]$, then we have $\mathscr{D}(G) = \mathscr{D}(G[N])$: indeed, for an S-scheme S' and $g, h \in G(S')$, centrality of G^0 in G shows that the commutator $ghg^{-1}h^{-1}$ depends only on the images of g and h in $(G/G^0)(S')$. In particular, since the map $G[N] \to G/G^0$ is an epimorphism of fppf sheaves, we may pass to an fppf cover of S' to assume $g, h \in G[N](S')$. So indeed, $\mathscr{D}(G) = \mathscr{D}(G[N])$, which is a finite étale group scheme killed by N, so its order is invertible on S.

Finally, note that $G^{ab} := G/\mathscr{D}(G[N])$ is a smooth S-group scheme by [GP11, Exp. V, Thm. 4.1]. It is commutative by the previous paragraph. Moreover, since G^0 is a torus the image of G^0 in G^{ab} (which makes sense by [GP11, Exp. IX, Thm. 6.8]) is also a torus, which must be equal to the relative identity component $(G^{ab})^0$ for dimension reasons. Moreover, $G/G^0 \to G^{ab}/(G^{ab})^0$ is an epimorphism of sheaves, so the component group of G^{ab} is of order invertible on S. Thus, it follows from [GP11, Exp. XVII, Prop. 7.1.1] that G^{ab} is a group scheme of multiplicative type, as desired. $\square$

The center and abelianization of G are related. We begin with a useful lemma.

Lemma 2.29. *Let T_1 and T_2 be S-group schemes of multiplicative type such that T_2 is a torus, and let $f : T_1 \to T_2$ be an isogeny. Let Γ be a finite group acting on T_1 and T_2 compatibly with f, and define maps*

$$\phi : T_1 \to \prod_{\gamma \in \Gamma} T_1, \qquad \phi(x) = ((\gamma \cdot x)x^{-1})_{\gamma \in \Gamma}$$

$$\psi : \prod_{\gamma \in \Gamma} T_2 \to T_2, \qquad \psi((y_\gamma)_{\gamma \in \Gamma}) = \prod_{\gamma \in \Gamma} (\gamma \cdot y_\gamma)y_\gamma^{-1}.$$

The natural map $\overline{f} : \ker \phi \to \operatorname{coker} \psi$ is an isogeny, and if Γ is of order n, then $\ker \overline{f}$ is contained in $[n]^{-1}(\ker f)$.

Note that $\ker \phi$ and $\operatorname{coker} \psi$ are S-group schemes of multiplicative type and that $\ker \overline{f}$ is of multiplicative type, and in particular flat, by [GP11, Exp. IX, Thm. 6.8].

Proof. By Lemma 2.25, to show that $\overline{f}$ is faithfully flat, it suffices to show that $\overline{f}$ is an epimorphism of fppf sheaves. To this end, let t_2 be a local section of T_2. Since T_2 is a torus, after localizing, we may assume there is some section t_2' such that $t_2'n = t_2$. Further localizing, there exists a section t_1 of T_1 such that $f(t_1) = t_2'$. We then have

$$f\left(\prod_{\gamma \in \Gamma} (\gamma \cdot t_1)\right) = \prod_{\gamma \in \Gamma} (\gamma \cdot t_2') = t_2 \cdot \prod_{\gamma \in \Gamma} (\gamma \cdot t_2')t_2'-1,$$

which has the same image in coker ψ as t_2. Since $\prod_{\gamma \in \Gamma}(\gamma \cdot t_1)$ lies in ker ϕ, we see that indeed $\overline{f}$ is an epimorphism of fppf sheaves.

Now suppose that t_1 is a local section of ker ϕ such that $\overline{f}(t_1) = 1$. We claim that then $f(t_1)^n = 1$. We may write $f(t_1) = \prod_{\gamma \in \Gamma}(\gamma \cdot t_{2,\gamma})t_{2,\gamma}^{-1}$ locally. Using Γ-equivariance of f and the fact that t_1 is fixed by Γ, we have

$$f(t_1)^n = \prod_{\gamma' \in \Gamma}(\gamma' \cdot f(t_1)) = \prod_{\gamma,\gamma' \in \Gamma}(\gamma'\gamma \cdot t_{2,\gamma})(\gamma' \cdot t_{2,\gamma}^{-1}) = \prod_{\gamma \in \Gamma}\left(\prod_{\gamma' \in \Gamma}(\gamma'\gamma \cdot t_{2,\gamma})(\gamma' \cdot t_{2,\gamma})^{-1}\right) = 1,$$

where the final equality follows by reindexing. So indeed, ker $\overline{f}$ lies in $[n]^{-1}(\ker f)$, and in particular, it is quasi-finite. Since it is of multiplicative type, it is finite, and hence, $\overline{f}$ is an isogeny. $\qquad\square$

Proposition 2.30. *Let G be a weakly reductive S-group scheme. If $Z(G)$ is smooth, then the natural map $f : Z(G)^0 \to G^{\mathrm{ab},0}$ is an isogeny of S-tori. If, moreover, $Z(\mathcal{D}(G^0))$ is smooth, then f is smooth (equivalently, étale).*

Proof. Propositions 2.19 and 2.27 show that $Z(G)^0$ and $G^{\mathrm{ab},0}$ are S-tori under our assumptions, so it suffices to check the result on geometric fibers. In other words, we may and do assume that $S = \mathrm{Spec}\, k$ for an algebraically closed field k. Note that the natural $Z(G^0) \to G^{0,\mathrm{ab}}$ (whose target is *not* $G^{\mathrm{ab},0}$) is an isogeny of multiplicative type groups with kernel $Z(\mathcal{D}(G^0))$ (irrespective of whether $Z(G^0)$ is smooth): a maximal central torus T_0 of G^0 is of finite index in $Z(G^0)$, and there is the standard central isogeny $T_0 \times \mathcal{D}(G^0) \to G^0$.

Now G/G^0 acts on $Z(G^0)$ and $G^0/\mathcal{D}(G^0)$ by conjugation, so we can apply Lemma 2.29 with $T_1 = Z(G^0)$, $T_2 = G^{0,\mathrm{ab}}$, $\Gamma = G/G^0$, and $f : T_1 \to T_2$ the natural map. We will also use the notation ϕ, ψ, $\overline{f}$ of Lemma 2.29. In this case, ker $\phi = C_{G^0}(G)$, which admits $Z(G)^0$ as an open and closed S-subgroup scheme: indeed, $C_{G^0}(G) = G^0 \cap Z(G)$, and $Z(G)^0$ (resp. G^0) is open and closed in $Z(G)$ (resp. G). Moreover, coker $\psi = G^{\mathrm{ab},0}$. Thus, to establish the proposition, it suffices to show that $\overline{f}$ is an isogeny, which is smooth provided $Z(\mathcal{D}(G^0))$ is smooth. The fact that $\overline{f}$ is an isogeny follows directly from Lemma 2.29. The last statement of Lemma 2.29 shows that ker $\overline{f}$ lies in $[n]^{-1}(Z(\mathcal{D}(G^0)))$, where n is the order of G/G^0. Since n is invertible on S by hypothesis, smoothness of $Z(\mathcal{D}(G^0))$ implies smoothness of $[n]^{-1}(Z(\mathcal{D}(G^0)))$. $\qquad\square$

3. Centralizers

In this section, we study various kinds of centralizers in weakly reductive group schemes. Before diving into the results, we would like to summarize what is available in the literature for centralizers of weakly reductive subgroups of weakly reductive groups.

1. Over a field of pretty good characteristic (see Definition 1.12), [Her13] shows that the centralizer of any subgroup scheme of any connected reductive group is smooth. In fact, this property characterizes pretty good characteristic.
2. Over a field of good characteristic $p > 0$, the centralizer of any subgroup scheme of a connected reductive group has no p-torsion in its component group; this follows via a short argument with the Springer isomorphism. This general statement is false in every bad characteristic, as Springer showed in [Spr66, Theorem 4.12] by considering centralizers of regular unipotent elements.
3. If G is a connected reductive group over a field k of characteristic $p \geq 0$ and H is a finite subgroup of $G(k)$ of order prime to p, then $C_G(H)$ has reductive identity component by [PY02, Theorem 2.1], and its component group is of order prime to p (even in bad characteristic) by [FS24, Proposition VIII.5.11].
4. If H is a connected reductive subgroup of a connected reductive group G over a field of characteristic $p > 0$, then we are not aware of any general results concerning reductivity of $C_G(H)$ apart from the 'classical' case that H is of multiplicative type [Con14, Lemma 2.2.4] or the upcoming Corollary 3.5.

5. If G is a weakly reductive group scheme over a base scheme S and H is a finite subgroup of $G(S)$ of order invertible on S, then $C_G(H)$ is smooth and affine with reductive identity component; this follows from simple deformation theory and (3). If H is moreover solvable, then [DHKM24, Theorem A.12] shows that $C_G(H)$ has finite component group. Apart from this, we are not aware of any results in the literature concerning smoothness, reductivity or finiteness of component groups for centralizers of weakly reductive subgroup schemes of G prior to Corollaries 3.5 and 3.7 below.

3.1. Centralizers of weakly reductive subgroup schemes

We will now input the general results of Section 2 into concrete results on centralizers. We begin with the following lemma.

Lemma 3.1. *Let A be a DVR, and let X be a locally noetherian A-scheme. If X_0 is an open and closed subscheme of the special fiber X_s, then the natural map $X - X_0 \to X$ is affine. In particular, if X is affine, then $X - X_0$ is also affine.*

Proof. Affineness of a morphism can be checked Zariski-locally on the target, so we may freely shrink X to assume that the special fiber of X is connected. In this case, X_0 is either empty or all of X_s. If X_0 is empty, then the lemma is obvious; otherwise, $X_0 = V(\pi)$, where π is a uniformizer of A, and the lemma is again clear. □

The proof of the following theorem is similar in spirit to one of the proofs of [PY02, Theorem 2.1], which shows reductivity of the centralizer $C_G(\Lambda)$ of a finite group Λ over a field k by realizing it as the stabilizer of the conjugation action of G on $\underline{\mathrm{Hom}}_{k\text{-gp}}(\Lambda, G)$.

Theorem 3.2. *Let S be a scheme, and let G and H be geometrically reductive smooth affine S-group schemes. Suppose $f : H \to G$ is an S-homomorphism such that*

1. $\mathrm{H}^1(H_s, \mathfrak{g}_s) = 0$ *for all $s \in S$,*
2. $f_s(H_s)$ *is G_s-cr for all $s \in S$.*

Then $C_G(H)$ is geometrically reductive smooth affine S-scheme. If G is weakly reductive and $\mathrm{char}\, k(s)$ is good for G_s for all $s \in S$, then $C_G(H)$ is also weakly reductive.

Proof. By (1) and deformation theory [GP11, Exp. III, Corollaire 2.8], the orbit map $\phi : G \to \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is smooth, and thus, $C_G(H)$ is smooth and affine. To show that $C_G(H)$ is geometrically reductive, it suffices by Theorem 2.1 to assume that $S = \mathrm{Spec}\, A$ for a complete DVR A with algebraically closed residue field. In this case, Theorem 2.2 shows that $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is a disjoint union of finite type S-affine S-schemes.

Let C be the schematic closure of the G-orbit of f_η in $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$, so C is a G-stable closed subscheme of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ through which the orbit map of f factors. Let C_1 be the G-stable open subscheme of C obtained by deleting all of the components of C_s not containing a G-translate of f_s, so C_1 is affine by Lemma 3.1. Since ϕ is smooth, ϕ has open image; moreover, each fiber of ϕ has closed image by (2) and Lemma 2.10. Consequently, each fiber of C_1 is the (open) orbit of f in that fiber, and the map $i : C_1 \to \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is an open embedding on both fibers. Since C_1 is flat, it follows that i is étale and radicial, and thus, [DG67, IV$_4$, Théorème 17.9.1] shows that i is an open embedding.

Now we show that $C_G(H)$ is geometrically reductive; for this, it is equivalent to show that $G/C_G(H)$ is affine by Theorem 2.1. We will show, in fact, that the natural map $G/C_G(H) \to C_1$ is an isomorphism. By definition, ϕ factors through C_1, and the previous paragraph shows that this factored map is surjective. Note that the quotient $G/C_G(H)$ exists as a smooth separated algebraic space of finite type by work of Artin [Art74, Corollary 6.3]. Moreover, the induced morphism $G/C_G(H) \to C_1$ is a monomorphism, so by [Knu71, II, 6.15], it follows that $G/C_G(H)$ is a scheme. We claim that the morphism $G/C_G(H) \to C_1$ is an isomorphism. Indeed, from the above, we see that it is a smooth surjective monomorphism. Thus, by [DG67, IV$_4$, Théorème 17.9.1], it is a surjective open embedding and thus an isomorphism. As remarked above, this shows that $C_G(H)$ is geometrically reductive.

For the final claim, we may and do assume that $S = \operatorname{Spec} k$ for an algebraically closed field k of characteristic $p > 0$. If $\pi_0 C_G(H)$ has any p-torsion, then a simple argument with the Jordan decomposition shows that $C_G(H)(k)$ admits unipotent elements not lying in $C_G(H)^0(k)$. This does not happen in good characteristic by the argument of [SS70, III, 3.15].   □

In order to apply Theorem 3.2, we need the following simple lemma.

Lemma 3.3. *Let k be an algebraically closed field, and let G and H be (possibly disconnected) reductive groups over k. Let T be a maximal k-torus of H. Suppose $f : H \to G$ is a k-homomorphism such that $\mathrm{H}^1(H, V) = 0$ for all representations V isomorphic to $\operatorname{Lie} G$ under some k-homomorphism $H \to G$ such that the multiset of weights for T on V is the same as the multiset of weights for T on $\mathfrak{g}$. Then $f(H)$ is G-cr.*

Proof. Note that $\underline{\operatorname{Hom}}_{k\text{-gp}}(H, G)$ admits a disjoint union decomposition into pieces on which the multiset of weights for T on $\mathfrak{g}$ is constant; let U be the piece containing f. By hypothesis and [GP11, Exp. III, Corollaire 2.8], every orbit map $G \to U$ is smooth, and thus, every orbit is an open subscheme of U. Thus, every orbit is also closed. By Lemma 2.11, it follows that $f(H)$ is G-cr.   □

Now we recall the following fundamental result of McNinch [McN98], which builds on work of Jantzen [Jan97]. If H is a (possibly disconnected) reductive group and $H_1, \ldots, H_n$ are the simple factors of $\mathscr{D}(H^0)$, we let $\ell_H = \inf_i(\operatorname{rank} H_i)$. Note that if H^0 is a torus, then $\ell_H = \infty$.

Theorem 3.4 [McN98, Corollary 1.1.2]. *Let k be a field of characteristic $p > 0$, and let H be a connected reductive k-group. If V is an algebraic k-representation of H and $\dim V \leq p\ell_H$, then V is semisimple. In particular, if $\dim V < p\ell_H$, then $\mathrm{H}^1(H, V) = \operatorname{Ext}_H^1(k, V) = 0$.*

Corollary 3.5. *Let S be a scheme, and let G and H be geometrically reductive smooth affine S-group schemes. Suppose that H is weakly reductive and that for every $s \in S$, either $\operatorname{char} k(s) = 0$ or $\operatorname{char} k(s) > \dim \mathscr{D}(G_s^0)/\ell_{H_s}$. If H acts faithfully on G, then $C_G(H)$ is smooth, affine and geometrically reductive. If G is reductive, then $C_G(H)$ is also weakly reductive.*

Note that if H^0 is a torus (in particular if H is finite étale), then Corollary 3.5 involves no hypothesis on the residue characteristics of S.

Proof. By passing separately from H to $\mathscr{D}(H^0)$, $H^0/\mathscr{D}(H^0)$, and H/H^0 (and from G to centralizers of these), we can assume that H is either semisimple, a torus or finite étale of order invertible on S. Assume first that H is either a torus or finite étale. In this case, let $G_1 = G \rtimes H$, a geometrically reductive smooth affine S-group scheme which is weakly reductive whenever G is. By Theorem 3.2 (whose hypotheses always hold in the current setting), the centralizer $C_{G_1}(H)$ is smooth, affine and geometrically reductive, and it is weakly reductive whenever G is. There is a natural projection map $f : C_{G_1}(H) \to H$, and $C_G(H) = f^{-1}(1)$. If H is a torus, then in particular, it is commutative, and so f is split by the natural inclusion $H \to G_1$. Thus, $C_{G_1}(H) = C_G(H) \times H$ and $C_G(H)$ is geometrically reductive, smooth and affine. Moreover, $C_{G^0}(H)$ has connected fibers by the classical theory, so $C_G(H)$ is weakly reductive if G is weakly reductive. Now assume that H is finite étale of order invertible on S. Note f factors through a constant map $C_{G_1}(H)/C_{G_1}(H)^0 \to H$, so $C_G(H)$ is an open and closed S-subgroup scheme of $C_{G_1}(H)$, from which the result follows in this case. If G is weakly reductive, then to show that $C_G(H)$ is weakly reductive, it suffices to show that $C_{G^0}(H)$ is weakly reductive, which follows from the above and [FS24, Proposition VIII 5.11] applied on S-fibers.

Now assume that H is semisimple. By Lemma 3.3, Theorem 3.4, [BMR05, Corollary 3.42], and our bounds on the residue characteristics, H_s is G_s-cr for all $s \in S$. The action of H on G^0 is given by a map $f : H \to G^0/Z(G^0)$, and Theorem 3.2 combines with Theorem 3.4 to show that $C_{G^0/Z(G^0)}(H)$ is geometrically reductive, smooth and affine. The assumption on residue characteristics implies (by considering the simple types) that $Z(G^0)$ is smooth, so as H acts faithfully, we conclude that $C_{G^0}(H)$ is geometrically reductive, smooth and affine. If X is the stabilizer of f in G, then $C_{G^0}(H) = X \cap G^0$ is an open S-subgroup scheme of X, so X is smooth affine, and we claim that X is geometrically reductive. For this, we may and do assume $S = \operatorname{Spec} A$ for a DVR A. Now the quotient G/X is affine: first,

$G^0/C_{G^0}(H)$ is isomorphic to an affine open subscheme U of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G^0/Z(G^0))$ which is also a closed subscheme on S-fibers, as follows from the proof of Theorem 3.2. The quotient G/X is a finite union of G/G^0-translates of U in $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G^0/Z(G^0))$, so it is an open subscheme which is closed on fibers. By Theorem 2.2 and Lemma 3.1, it follows that G/X is affine and thus X is geometrically reductive by Theorem 2.1. We may now pass from G to X to assume that H acts trivially on G^0.

Since H has connected fibers, it also acts trivially on the finite étale S-group scheme G/G^0. It would now be enough to show that H acts trivially on G. Thus, there is a natural map $\varphi : H \times G \to G^0$ given by $\varphi(h, g) = (hgh^{-1})g^{-1}$. Since H acts trivially on G^0, it follows that φ factors through a map $H \times G/G^0 \to Z(G^0)$. For fixed $g \in (G/G^0)(S)$, the map $\varphi(-, g) : H \to Z(G^0)$ is an S-homomorphism. Since $Z(G^0)$ is of multiplicative type and H is semisimple, $\varphi(-, g)$ is trivial, as desired. Finally, if G is weakly reductive, then to show that $C_G(H)$ is weakly reductive, it is easy to pass from G to $G^0/Z(G^0)$ and thus reduce to Theorem 3.2. □

Next, we obtain a similar result under better bounds when H is the centralizer of a finite group of order invertible on the base. For this, we will use recent fundamental results of Fargues–Scholze. First, we must recall the notion of good filtration. The following theorem is a basic consequence of [FS24, Proposition VIII.5.12].

Theorem 3.6 [FS24, Proposition VIII.5.12]. *Let k be an algebraically closed field of characteristic $p > 0$, and let G be a connected reductive k-group such that p is pretty good for G. If Λ is a finite group of order prime to p acting on G, then $\mathrm{H}^i(C_G(\Lambda), \mathfrak{g}) = 0$ for all $i > 0$.*

Proof. By [AJ84, 4.4], every symmetric power $\mathrm{Sym}^n \mathfrak{g}^*$ admits a good filtration: in other words, there is a filtration of $\mathrm{Sym}^n \mathfrak{g}^*$ such that every subquotient is isomorphic to $\mathrm{H}^0(\lambda)$ for some dominant weight λ. Since p is pretty good for G, [Her13, Theorem 5.2] shows that $\mathfrak{g} \cong \mathfrak{g}^*$ as G-representations and thus, in particular, $\mathfrak{g}$ admits a good filtration. Now if $H = C_G(\Lambda)$, then the above discussion and [FS24, Proposition VIII.5.12] show that $\mathcal{O}(G/H)$ also admits a good filtration. Using the equality $\mathrm{H}^i(H, \mathfrak{g}) = \mathrm{H}^i(G, \mathfrak{g} \otimes_k \mathcal{O}(G/H))$ (which relies on the fact that G/H is affine), it is enough to show that $\mathrm{H}^i(G, \mathrm{H}^0(\lambda) \otimes_k \mathrm{H}^0(\mu)) = 0$ for all $i > 0$ and all dominant weights λ and μ. This is proved in [Jan03, II, 4.13]. □

Corollary 3.7. *Let S be a scheme, and let G be a weakly reductive S-group scheme. Let Λ be a finite étale S-group scheme of order invertible on S which acts on G. Then $C_G(\Lambda)$ is a weakly reductive S-group scheme. Moreover, suppose that for every $s \in S$, $\mathrm{char}\, k(s)$ is pretty good for G_s. Then $C_G(C_G(\Lambda))$ is a weakly reductive S-group scheme.*

Proof. The first claim follows immediately from Corollary 3.5. Let $H = C_G(\Lambda)$. By Theorem 3.2, it is enough to show that $\mathrm{H}^1(H_s, \mathfrak{g}_s) = 0$ and H_s is G_s-cr for all $s \in S$. The first condition holds by Theorem 3.6, and the second condition holds by [BMR05, Corollary 3.17]. □

Example 3.8. In general, some bound on the residue characteristics as in Corollary 3.5 is necessary, even in pretty good characteristic. For example, let p be any prime number, let $S = \mathrm{Spec}\, \mathbf{F}_p[[t]]$, let $H = \mathrm{SL}_2$, and let $G = \mathrm{GL}_{p+1}$. With notation as in [Jan03, II, Chapter 2], there is an indecomposable $p + 1$-dimensional representation $\mathrm{H}^0(p)$ of SL_2 equipped with a filtration

$$0 \to \mathrm{L}(p) \to \mathrm{H}^0(p) \to \mathrm{L}(p - 2) \to 0.$$

This gives an extension class in $\mathrm{Ext}^1_H(\mathrm{L}(p - 2), \mathrm{L}(p))$. Multiplying this class by t gives a $p + 1$-dimensional representation V of SL_2 over S with special fiber $\mathrm{L}(p) \oplus \mathrm{L}(p - 2)$ and generic fiber $\mathrm{H}^0(p)$. Relative to the corresponding map $H \to G$, we have $C_G(H)_s \cong \mathbf{G}_m^2$, while $C_G(H)_\eta \cong \mathbf{G}_m$ [Jan03, Proposition II.2.8], so $C_G(H)$ is not flat.

Moreover, the centralizer of a connected reductive subgroup of a reductive group need not be reductive if the characteristic is too small; this is related to the fact that representations of connected reductive

groups can fail to be semisimple in positive characteristic. For explicit examples, see the MathOverflow answer [hm], as well as the comments on that answer.

We do not know the optimal bound on the residue characteristics in Corollary 3.5; if $G = \mathrm{GL}_n$, then one can improve the bound from n^2/ℓ_H to n/ℓ_H. If $G = \mathrm{Sp}_n$ (resp. SO_n), then one can improve the bound from $n(2n+1)/\ell_H$ (resp. $n(n-1)/2\ell_H$) to n/ℓ_H. In general, it seems reasonable to expect that something like h_G/ℓ_H is the correct bound, where h_G is the Coxeter number of G.

3.2. Centralizers of pure unipotent elements

Later arguments rely heavily on the smoothness of centralizers of pure fiberwise unipotent elements. The notion of purity will capture the idea that an element of G 'looks similar' across all fibers. For example, an element like $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)$ will not be pure.

Definition 3.9. Let G a reductive group over a scheme S. A section $g \in G(S)$ is *pure* if $s \mapsto C_{G_s}(g_s)$ is locally constant on s. There is an analogous definition for elements of the Lie algebra.

If S is the spectrum of a discrete valuation ring, this means that the special and generic fibers of the centralizer of g have the same dimension.

Using a Springer isomorphism, the smoothness for centralizers of unipotent elements is closely related to the smoothness for centralizers of nilpotent elements in the Lie algebra. This smoothness was claimed in [McN08], but the argument there has a gap. This has been fixed by Hardesty (for pure nilpotents) and by the second author (for pure unipotents and nilpotents) [Har18, Cot22a]. Since our later arguments are naturally phrased in terms of unipotent elements, we will build on the latter.

Theorem 3.10. *Let A be a DVR with residue characteristic p, and let G be a weakly reductive group scheme over* $\mathrm{Spec}\, A$. *If p is pretty good for G and $u \in G(A)$ is a pure fiberwise unipotent element, then $C_G(u)$ is A-smooth.*

Using Remark 1.13, it would be equivalent to suppose that p is good for G^0, that $\#\pi_1(\mathscr{D}(G^0))$ is prime-to-p, and that $Z(G^0)$ is A-smooth.

Proof. If G is connected, flatness comes from [Cot22a, Theorem 1.1]. The smoothness of the fibers follows, for example, from [Her13, Theorem 1.1].

In general, note that $\overline{u}$ has order a power of p, and hence, u lies in the identity component of G. It suffices to show that every component of G which contains a point in the special fiber centralizing u has an A-point centralizing u. For then, the centralizer $C_G(u)$ is a union of copies of the smooth $C_{G^0}(u)$.

Given $g \in G(A)$ such that $\overline{g}$ centralizes u_s in the special fiber, as u is pure, [Cot22a, Theorem 5.11] shows that gug^{-1} and u are $G^0(A)$-conjugate. Thus, there is $h \in G^0(A)$ such that $huh^{-1} = gug^{-1}$, and hence, $h^{-1}g \in G(A)$ centralizes u and lies in the desired component of G. $\square$

3.3. Complements on pure unipotents

We conclude with some additional results about pure unipotents which elaborate the sense in which a pure unipotent 'looks similar' in the special and generic fibers.

We begin by reviewing the Bala-Carter method which classifies nilpotent orbits for a connected reductive group over an algebraically closed field k of good characteristic $p \geq 0$. (More information can be found in [Jan04, §4], and a uniform proof without case-checking in small characteristic is due to Premet [Pre03].) Using a Springer isomorphism, this equivalently gives a classification of conjugacy classes of unipotent elements. To state it, we need to define some terminology.

Let H be a connected reductive k-group with p good for H, and $\mathfrak{h} = \mathrm{Lie}\, H$.

○ A nilpotent $N \in \mathfrak{h}$ is a *distinguished nilpotent* if each torus contained in $C_H(N)$ is contained in the center of H.

○ For a parabolic $P \subset H$ with unipotent radical U, the *Richardson orbit* associated to P is the unique nilpotent orbit of H with dense intersection with Lie U. Its intersection with Lie P is a single orbit under P.

○ A parabolic subgroup $P \subset H$ with unipotent radical U is a *distinguished parabolic* if $\dim P/U = \dim U/\mathscr{D}(U)$.

Bala and Carter classified nilpotent orbits when the characteristic is good. One can check that if p is good for H, it will also be good for any Levi factor of a parabolic subgroup of H. The following fact can be found in [Jan04, §4].

Fact 3.11. *If p is a good prime for H, the nilpotent orbits for H are in bijection with $H(k)$-conjugacy classes of pairs (L, P) where L is a Levi factor of a parabolic subgroup of H and P is a distinguished parabolic of L. The nilpotent orbit for H associated to (L, P) is the unique one meeting $\mathrm{Lie}(P)$ in its Richardson orbit for L.*

The *Bala-Carter data* for H is the set of $H(k)$-conjugacy classes of pairs (L, P) as above. It turns out it is independent of k in the sense that it can be described completely in terms of the root datum of H as follows. All Levi subgroups L of a parabolic k-subgroup Q of H are a single $\mathcal{R}_{u,k}(Q)$-orbit, so in Fact 3.11, we may restrict to one Q per $H(k)$-conjugacy class and one L per Q. We may pick L so that it contains a (split) maximal torus T. After conjugation by $L(k)$, the distinguished parabolic subgroup $P \subset L$ may be assumed to contain T as well. But we know that parabolic subgroups Q of H containing T are in bijection with parabolic subsets of $\Phi(H, T)$ via $Q \mapsto \Phi(Q, T)$, so the possible Levi factors L of Q containing T are described just in terms of the root datum. Likewise, parabolic subgroups P of L containing T are in bijection with parabolic subsets of $\Phi(L, T)$. If we can characterize the condition that P is distinguished just in terms of the root data, this would mean that the Bala-Carter data can be described solely in terms of the root data and so is completely combinatorial.

We do so by constructing a grading on the Lie algebra of a parabolic P. Pick a Borel subgroup $B \subset H$ satisfying $T \subset B \subset P$. Let $\mathfrak{t} = \mathrm{Lie}(T)$ and $\Delta \subset \Phi = \Phi(L, T)$ be the set of positive simple roots determined by B. There is a unique subset $I \subset \Delta$ such that $P = BW_I B$ where W_I is the subset of the Weyl group generated by reflections with respect to roots in I. Define a group homomorphism $f : \mathbf{Z}\Phi \subset \mathbf{Z}^\Delta \to \mathbf{Z}$ by specifying that on the basis Δ, we have

$$f(\alpha) = \begin{cases} 2 & \alpha \in \Delta - I, \\ 0 & \alpha \in I. \end{cases}$$

This function gives a grading on $\mathfrak{l} = \mathrm{Lie}(L)$:

$$\mathfrak{l}(i) = \bigoplus_{f(\alpha)=i} \mathfrak{l}_\alpha \quad \text{and} \quad \mathfrak{l}(0) = \left(\bigoplus_{f(\alpha)=0} \mathfrak{l}_\alpha \right) \oplus \mathfrak{t}$$

(sums indexed by $\alpha \in \Phi$). With respect to this grading,

$$\mathrm{Lie}\, P = \bigoplus_{i \geq 0} \mathfrak{l}(i) \quad \text{and} \quad \mathrm{Lie}\, U = \bigoplus_{i > 0} \mathfrak{l}(i).$$

The condition that P is distinguished is equivalent to the condition that

$$\dim \mathfrak{l}(0) = \dim \mathfrak{l}(2) + \dim Z_L$$

by [Car85, Corollary 5.8.3] as p is good for L. But this condition depends only on the root datum. Thus, the Bala-Carter data for H can be described in a manner independent of the choice of algebraically closed field. We call this combinatorial description the *Bala-Carter label* for the nilpotent (or unipotent) orbit.

Definition 3.12. If K is a field (not necessarily algebraically closed), then the Bala-Carter label label for a nilpotent (resp. unipotent) element of Lie H (resp. $H(K)$) is the Bala-Carter label for the corresponding nilpotent (resp. unipotent) orbit over the algebraic closure.

Remark 3.13. From the classification of nilpotent orbits over algebraically closed fields, it is known that the corresponding nilpotent orbits in characteristic zero and characteristic p have the same dimension. Thus, the dimension of centralizers of elements in an orbit is also independent of the characteristic and depends only on the Bala-Carter label. There is a similar statement for conjugacy classes of unipotent elements and their centralizers.

We now return to the relative setting.

Lemma 3.14. *Let G be a connected reductive group scheme over a DVR A with residue field k whose characteristic is good for G. Given a unipotent element $\overline{u} \in G(k)$ with Bala-Carter label σ, there exists a pure fiberwise unipotent element $u \in G(A)$ lifting $\overline{u}$ with Bala-Carter label σ in the generic fiber, and a similar statement for nilpotent elements of the Lie algebra.*

Proof. We will prove this for a nilpotent element $\overline{X}$; the unipotent case follows using an integral Springer isomorphism [Cot22b, Theorem 1.1]. Let $\tau_s : (\mathbf{G}_m)_s \to G_s$ be a cocharacter associated to $\overline{X}$, which exists since char(k) is good for G. This lifts to a cocharacter $\tau : \mathbf{G}_m \to G$ by smoothness of the scheme of maximal tori [Con14, Theorem 3.2.6]. This cocharacter defines

$$\bigoplus_{n \geq 2} \mathfrak{g}(\tau, n) \subset \mathfrak{g}.$$

Over the special fiber, $\overline{X}$ is in $\mathfrak{g}_s(\tau, 2)$, and the $\mathrm{Ad}_{P(\tau_s)}$-orbit of $\overline{X}$ is open and dense. Pick a fiberwise nilpotent $X \in \mathfrak{g}(\tau, 2)$ lifting $\overline{X}$ and consider the parabolic $P_G(\tau) \subset G$. It naturally acts on $\bigoplus_{n \geq 2} \mathfrak{g}(\tau, n)$ and the stabilizer of X is $C_G(X)$. As we know, $\dim C_G(X)_\eta \leq \dim C_G(X)_s$, and since the orbit is open and dense in the special fiber, it follows that X is pure. Then argue as in [Cot22a, Lemma 5.2] to show that $P_G(\tau)$ gives the instability parabolic for X and τ is an associated cocharacter for X in the special and generic fibers. By [Cot22a, Proposition 2.13(6)], this information determines the Bala-Carter data in the special and generic fibers. □

Definition 3.15. Let K be a p-adic field and G be a connected reductive group over the ring of integers $\mathcal{O}_K$. We say that $g_1, g_2 \in G(\mathcal{O}_K)$ are geometrically conjugate if there exists a finite extension L of K such that g_1 and g_2 are $G(\mathcal{O}_L)$-conjugate.

Proposition 3.16. *Let K be a p-adic field with residue field k, and let G be a connected reductive group over $\mathcal{O}_K$ such that p is pretty good for G. Fix a Bala-Carter label σ and a pure unipotent u_σ with Bala-Carter label σ in $G(K)$ and $G(k)$ using Lemma 3.14. The following are equivalent for a fiberwise unipotent $u \in G(\mathcal{O}_K)$:*

1. *u is geometrically conjugate to u_σ;*
2. *the images of u in $G(K)$ and $G(k)$ have Bala-Carter label σ;*
3. *u is pure, and it has Bala-Carter label σ in $G(K)$;*
4. *u is pure, and it has Bala-Carter label σ in $G(k)$;*

Proof. Note that (1) implies (2) as the images of u_σ in $G(K)$ and $G(k)$ have the same Bala-Carter labels. Since the dimensions of the centralizer of a unipotent element over an algebraically closed field depend only on the Bala-Carter label and not the characteristic, (2) implies (3) and (4). If u is pure and generically has Bala-Carter label σ, then u and u_σ are $G(K')$-conjugate for some extension K' of K. By [Cot22a, Theorem 5.11], u and u_σ are $G(\mathcal{O}_{K'})$-conjugate. Thus, (3) implies (1). Similarly, (4) implies (1). □

Remark 3.17. ∘ When defining the minimally ramified deformation condition in [Boo19a] and this paper, a key step is to restrict to deformations where a particular inertial element σ, whose mod-p image is unipotent, is sent to a conjugate of a particular pure unipotent lift. Proposition 3.16 shows that this condition is equivalent to controlling the conjugacy class of the lift in the generic fiber or to enforcing purity of the lift.

∘ There is, of course, an analogous statement for nilpotents. These can be cheaply deduced using an integral Springer isomorphism [Cot22b, Theorem 1.1] or proven directly using similar techniques.

We will also need the following lemma later.

Lemma 3.18. *Let K be a p-adic field and G a connected reductive group defined over $\mathcal{O}_K$ such that p is good for G. Suppose u is a fiberwise unipotent element of $G(\mathcal{O}_K)$ satisfying any of the equivalent conditions of Proposition 3.16. Then for any integer n relatively prime to n, the elements u and u^n generically have the same Bala-Carter label, and u^n is pure.*

Proof. Over an algebraically closed field, if X is nilpotent, then all nonzero multiples of X lie in the same nilpotent orbit [Jan04, 2.10 Lemma]. Thus, the same is true for powers of unipotents. In particular, u^n is therefore pure, and u^n has the same Bala-Carter labels as u generically and in the special fiber. □

4. Decomposition types

Throughout this section, let G be a weakly reductive group scheme over a DVR $\mathcal{O}$ with residue field k of characteristic p.

4.1. Definitions and examples

Definition 4.1. A *decomposition type* for G over $\mathcal{O}$ is a pair $(\mathfrak{C}, \Delta)$ where $\mathfrak{C}$ and Δ are closed subgroup schemes of G defined over $\mathcal{O}$ such that

∘ $\mathfrak{C}$ and Δ are weakly reductive (i.e., $\mathcal{O}$-smooth with (not necessarily connected) reductive fibers and finite étale component groups of order invertible on $\mathcal{O}$).

∘ Δ (resp. $\mathfrak{C}$) represents the scheme theoretic centralizer of $\mathfrak{C}$ (resp. Δ).

Definition 4.2. Let $\overline{\Lambda} \subset G(k)$ be a finite subgroup with order prime to p. We say a decomposition type $(\mathfrak{C}, \Delta)$ over $\mathcal{O}$ is *adapted to* $\overline{\Lambda}$ if $C_{G_k}(\overline{\Lambda}) = \mathfrak{C}_k$ (hence, also $C_{G_k}(C_{G_k}(\overline{\Lambda})) = \Delta_k$).

This definition is motivated by the goal of giving a group theoretic reformulation of the isotypic decomposition of a representation, which we now explain.

Example 4.3. This is an elaboration of the example of GL_n discussed in the introduction. Let M be a free $\mathcal{O}$-module of rank n, $G = \mathrm{Aut}(M)$, and $V = M_k$. As $\overline{\Lambda}$ has pro-order prime to p, V decomposes as a direct sum of irreducible $\overline{\Lambda}$-representations

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r},$$

where V_i and V_j are non-isomorphic for $i \neq j$, and the m_i's are multiplicities of the irreducibles. Letting $W_i = \mathrm{Hom}_{\overline{\Lambda}}(V_i, V)$ be the weight space associated to V_i, we have that $\dim W_i = m_i$ and

$$V = (V_1 \otimes_k W_1) \oplus \ldots \oplus (V_r \otimes_k W_r). \tag{4.1}$$

Enlarging k if necessary, we may and do assume that V_i is absolutely irreducible for all i. Let

$$L_k := \mathrm{GL}(V_1)^{m_1} \times \cdots \times \mathrm{GL}(V_r)^{m_r} \subset \mathrm{GL}(V)$$

be the standard Levi associated to the decomposition of $\overline{\tau}$, and let $d_i = \dim_k V_i$. By Schur's lemma,

$$\mathfrak{C}_k := C_{\mathrm{GL}(V)}(\overline{\Lambda}) = \mathrm{Aut}_{\Lambda}(V) = \prod_{i=1}^{r} \mathrm{GL}(W_i) \subset \mathrm{GL}(V).$$

A direct computation then shows that

$$\Delta_k := C_{\mathrm{GL}(V)}(\mathfrak{C}_k) = \prod_{i=1}^{r} \mathrm{GL}(V_i) \subset \mathrm{GL}(V),$$

where the embedding into $\mathrm{GL}(V)$ comes from acting on the first factors of the tensor products in (4.1). Note that $\Delta_k \subset L_k$, with $\mathrm{GL}(V_i)$ embedding in $\mathrm{GL}(V_i)^{m_i}$ diagonally. One checks that $(\mathfrak{C}_k, \Delta_k)$ is a decomposition type adapted to $\overline{\Lambda}$.

A decomposition type for GL_n over $\mathcal{O}$ adapted to $\overline{\Lambda}$ is just a version of this decomposition for the $\mathcal{O}$-module M phrased group-theoretically. We can specify a representation (of a group containing $\overline{\Lambda}$) on M compatible with the isotypic decomposition using the groups $\mathfrak{C}$ and Δ, which are the automorphisms of the weight spaces and irreducible constituents.

We also give some examples for the symplectic groups that are reminiscent of the 'isotypic decomposition with pairings' in [Boo19a, §6.1]. Fix the symmetric form so that

$$\mathrm{Sp}_{2n} = \left\{ M \in \mathrm{GL}_{2n} : M \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

For $A \in \mathrm{GL}_n$, we write A^* for $(A^t)^{-1}$. Notice that $L = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} : A \in \mathrm{GL}_n \right\}$ is a Levi subgroup of Sp_{2n}.

Let $\overline{\Lambda} \subset L(k)$ be an L-irreducible subgroup. This gives a natural $2n$-dimensional representation of the form $\tau \oplus \tau^*$ where τ is irreducible representation of $\overline{\Lambda}$ with dimension n.

Example 4.4. If τ and τ^* are non-isomorphic $\overline{\Lambda}$-representations, we have

$$\mathfrak{C}_k = \left\{ \begin{pmatrix} aI_n & 0 \\ 0 & a^{-1}I_n \end{pmatrix} : a \in \mathbf{G}_m \right\} = Z(L)_k, \quad \Delta_k = L_k.$$

Example 4.5. Suppose $\tau \cong \tau^*$. In this case, it is easy to see that there exists $J \in \mathrm{GL}_n(k)$, either symmetric or skew-symmetric, such that $\tau J \tau^t = J$. Let us suppose that $p \neq 2$ and J is skew-symmetric (the other case is similar but occurs inside an orthogonal group). Conjugating τ if necessary, we may assume that $J = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$, so τ takes values in $\mathrm{Sp}_n(k)$. We then have

$$\mathfrak{C}_k = \left\{ \begin{pmatrix} aI_n & 0 \\ 0 & a^{-1}I_n \end{pmatrix} : a \in \mathbf{G}_m \right\} \cup \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \left\{ \begin{pmatrix} aI_n & 0 \\ 0 & a^{-1}I_n \end{pmatrix} : a \in \mathbf{G}_m \right\},$$

$$\Delta_k = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} : A \in \mathrm{Sp}_n \right\}.$$

Definition 4.6. Let $(\mathfrak{C}, \Delta)$ be a decomposition type over $\mathcal{O}$. We say p is good for $(\mathfrak{C}, \Delta)$ if

1. p is pretty good for G.
2. $Z(\mathscr{D}(\Delta^0))_k$ is smooth.
3. p does not divide $\#N_{G_k}(\Delta_k)/\mathfrak{C}_k\Delta_k$.

4.2. Basic properties

As the group schemes constituting a decomposition type are weakly reductive, we begin by recalling what we know about their centers and abelianizations.

Proposition 4.7. *Let H be a weakly reductive group scheme over $\mathcal{O}$.*

1. *The center $Z(H)$ and abelianization H^{ab} both exist as group schemes of multiplicative type over $\mathcal{O}$.*
2. *H^{ab} is smooth.*
3. *If $Z(H)$ is smooth, then the natural map $Z(H)^0 \to H^{ab,0}$ is an isogeny of tori. If $Z(\mathscr{D}(H^0))$ is smooth, then this map is smooth.*

Proof. Combine Proposition 2.19, Proposition 2.27 and Proposition 2.30. □

Lemma 4.8. *Let $(\mathfrak{C}, \Delta)$ be a decomposition type over k. Then the quotient group scheme $N_G(\mathfrak{C})/\mathfrak{C}\Delta = N_G(\Delta)/\mathfrak{C}\Delta$ is finite étale.*

Proof. Note that the quotient is well defined since $C_G(\Delta) = \mathfrak{C}$ and $C_G(\mathfrak{C}) = \Delta$. Since $\mathfrak{C}$ and Δ are weakly reductive, the finiteness follows from Corollary 2.22. □

Lemma 4.9. *Let $(\mathfrak{C}, \Delta)$ be a decomposition type for G over $\mathcal{O}$. Then for any $\mathcal{O}$-algebra R, $N_{G(R)}(\mathfrak{C}(R)) = N_{G(R)}(\Delta(R))$. Moreover, $Z(\mathfrak{C}) = Z(\Delta)$.*

Proof. This is basic group theory using Definition 4.1. For $\mathcal{O}$-algebras R, if $g \in G(R)$ normalizes $\mathfrak{C}(R)$, then it normalizes $\Delta(R) = C_{G(R)}(\mathfrak{C}(R))$, so $N_{G(R)}(\mathfrak{C}(R)) \subset N_{G(R)}(\Delta(R))$, and hence, $N_{G(R)}(\mathfrak{C}(R)) = N_{G(R)}(\Delta(R))$ by symmetry. Note that for $\mathcal{O}$-algebras R, $Z(\Delta)(R) = \Delta(R) \cap \mathfrak{C}(R) = Z(\mathfrak{C})(R)$ by Definition 4.1. □

Lemma 4.10. *If $(\mathfrak{C}, \Delta)$ is a decomposition type adapted to $\overline{\Lambda}$, and Λ is as in Proposition 4.12, then $C_\Delta(\Lambda) = Z(\Delta)$.*

Proof. This is basic group theory using Definition 4.1: for $\mathcal{O}$-algebras R, we see that $C_{\Delta(R)}(\Lambda) = C_{G(R)}(\Lambda) \cap \Delta(R) = \mathfrak{C}(R) \cap \Delta(R) \subset Z(\Delta(R)) = Z(\Delta)(R)$. □

We also record a few useful properties about $C_G(\Lambda)$ for later use, where Λ is a constant group scheme whose order is invertible in $\mathcal{O}$ (equivalently, the order of Λ is prime to p). We know it is $\mathcal{O}$-smooth and weakly reductive by Corollary 3.7.

Lemma 4.11. *If p is pretty good for G, then p is pretty good for $C_G(\Lambda)$.*

Proof. It suffices to work over a field of characteristic p. By [Her13, Theorem 3.3], p is pretty good if and only if the centralizer of every closed subgroup is smooth. Note $(G, C_G(\Lambda))$ is a reductive pair in the sense of [Her13, 2.7] since the Λ-isotypic decomposition of Lie G is stable under $C_G(\Lambda)$. Using [Her13, Lemma 3.6], if H is a closed subgroup of $C_G(\Lambda)$ such that $C_G(H)$ is smooth, then $C_{C_G(\Lambda)}(H)$ is smooth as well. Since p is pretty good for G, $C_G(H)$ is always smooth, and therefore, $C_{C_G(\Lambda)}(H)$ is always smooth. □

Proposition 4.12. *Let $(\mathfrak{C}, \Delta)$ and $\overline{\Lambda}$ be as in Definition 4.2. Suppose that $\mathcal{O}$ is a complete DVR. Then there exists a subgroup $\Lambda \subset G(\mathcal{O})$ reducing to $\overline{\Lambda}$ such that $C_G(\Lambda) = \mathfrak{C}$ (hence, also $C_G(C_G(\Lambda)) = \Delta$).*

Proof. Note that $\overline{\Lambda} \subset \Delta(k)$. Since Δ is $\mathcal{O}$-smooth and $\mathcal{O}$ is complete, there is a subgroup $\Lambda \subset \Delta(\mathcal{O})$ lifting $\overline{\Lambda}$. We have $C_G(\Lambda) \supset C_G(\Delta) = \mathfrak{C}$, which is an equality on special fibers. Since $\mathfrak{C}$ and $C_G(\Lambda)$ are smooth $\mathcal{O}$-group schemes of the same dimension, it follows that $\mathfrak{C}^0 = C_G(\Lambda)^0$: by the fibral isomorphism criterion, it is enough to show that if k is an algebraically closed field and $H \subset H'$ is a closed embedding of reduced connected k-schemes of the same dimension, then $H = H'$. For this, note that the natural map $H \to H'$ is dominant for dimension reasons, so it is surjective by closedness. Being a surjective closed embedding with reduced target, we find $H = H'$.

In view of the previous paragraph, there is a monomorphism $\pi_0 \mathfrak{C} \hookrightarrow \pi_0 C_G(\Lambda)$. Since $\pi_0 \mathfrak{C}_k = \pi_0 C_G(\Lambda)_k$ and both $\pi_0 \mathfrak{C}$ and $\pi_0 C_G(\Lambda)$ are finite étale (by Corollary 3.7), it follows that $\pi_0 \mathfrak{C} = \pi_0 C_G(\Lambda)$, and hence, $\mathfrak{C} = C_G(\Lambda)$ by the five lemma. □

Proposition 4.13. *Let $(\mathfrak{C}, \Delta)$ be a decomposition type adapted to $\overline{\Lambda}$. Suppose that p is pretty good for G and $Z(\mathscr{D}(\Delta^0))_k$ is smooth. Then $Z(\Delta)$ is smooth over $\mathcal{O}$, and the natural map $Z(\Delta)^0 \to \Delta^{ab,0}$ is a smooth isogeny of $\mathcal{O}$-tori.*

Proof. By hypothesis and Proposition 4.12, there is a group $\Lambda \subset G(\mathcal{O})$ such that $C_G(\Lambda) = \mathfrak{C}$. To show the smoothness of $Z(\Delta)$, first note that $Z(\Delta) = Z(\mathfrak{C}) = Z(C_G(\Lambda))$ by Lemma 4.9. By Proposition 2.19, the smoothness of $Z(\Delta)$ would then follow from the smoothness of $Z(C_G(\Lambda)^0)$. As $Z(\Delta)$ is of multiplicative type and hence $\mathcal{O}$-flat, it suffices to check smoothness on fibers. But since p is pretty good for G, Lemma 4.11 and Remark 1.13 imply that $Z(C_G(\Lambda)^0)$ is smooth. Thus, $Z(\Delta)$ is smooth. By Proposition 4.7(3), it follows that the natural map $Z(\Delta)^0 \to \Delta^{ab,0}$ is an isogeny of $\mathcal{O}$-tori. But $Z(\mathscr{D}(\Delta^0))_k$ is smooth by assumption, so the isogeny is smooth by *loc. cit.* □

Proposition 4.14. *Let $\overline{\Lambda}$ be as in Definition 4.2. Suppose that p is pretty good for G. Then there exists a decomposition type $(\mathfrak{C}, \Delta)$ over $\mathcal{O}$ adapted to $\overline{\Lambda}$.*

Proof. Fix a lift $\Lambda \subset G(\mathcal{O})$ of $\overline{\Lambda}$ such that the natural map $\Lambda \to \overline{\Lambda}$ is an isomorphism. Let $\mathfrak{C} = C_G(\Lambda)$, and let $\Delta = C_G(C_G(\Lambda))$. By Corollary 3.7, $(\mathfrak{C}, \Delta)$ is a decomposition type over $\mathcal{O}$ adapted to $\overline{\Lambda}$. □

5. Clifford theory

This section proves a result about lifting and extending representations that is the analog of Part 2 of the argument for GL_n sketched in Section 1.2.

5.1. Local Galois groups

Fix a local field F with residue characteristic $\ell \neq p$. Let the residue field of F have size q, a power of ℓ.

Lemma 5.1. *The maximal tame extension of F has inertia group isomorphic to $\prod_{\ell' \neq \ell} \mathbf{Z}_\ell$. The Galois group of the maximal tame extension is a semi-direct product of the inertia group with $\widehat{\mathbf{Z}}$, with $\widehat{\mathbf{Z}}$ acting via the cyclotomic character. In particular, conjugation by the Frobenius is multiplication by q on $\prod_{\ell' \neq \ell} \mathbf{Z}_\ell$.*

Proof. This is standard. □

Let Γ_F be the absolute Galois group of F, and let I_F be the inertia group. Using Lemma 5.1, we may fix a surjection $I_F \to \mathbf{Z}_p$ with kernel Λ_F.

Lemma 5.2. *The group Λ_F is normal in Γ_F. The quotient Γ_F / Λ_F is isomorphic to $T_q := \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$, where conjugating by $1 \in \widehat{\mathbf{Z}}$ is multiplication by q on $\mathbf{Z}_p$.*

Proof. This is also well known. □

Lemma 5.3. *The exact sequence*

$$1 \to \Lambda_F \to \Gamma_F \to \Gamma_F / \Lambda_F \to 1$$

is topologically split, so Γ_F is a semi-direct product.

Proof. This is basically [CHT08, 2.4.10]. □

For the rest of this paper, fix a preimage σ of a topological generator of $\mathbf{Z}_p$ under the surjection $I_F \to \mathbf{Z}_p$ and a Frobenius ϕ satisfying

$$\phi \sigma \phi^{-1} = \sigma^q.$$

5.2. *ν-Tame extensions*

Let $\mathcal{O}$ be the ring of integers in a *p*-adic field with residue field k and Δ be as in Definition 4.1. Recall Proposition 4.7. We write Z for $Z(\Delta)$ and write S for Δ^{ab}, which are both multiplicative groups. The natural morphism $\nu : \Delta \to S$ restricts to an isogeny $Z \to S$ (also denoted by ν). Let R be a complete local noetherian $\mathcal{O}$-algebra with residue field k.

Definition 5.4. We say a homomorphism $\rho : I_F \to \Delta(R)$ is *ν-tame* if for any $\sigma \in I_F$, $(\nu \circ \rho)(\sigma) \in S(R)$ is of finite prime-to-*p* order.

We will prove the following:

Proposition 5.5. *Given Δ as above. Assume that $\nu : Z \to S$ is smooth. Given a continuous homomorphism $\tau : \Lambda_F \to \Delta(R)$ such that*

1. $\ker \tau = \ker \overline{\tau}$ *(in particular, the image of τ is finite);*
2. $\overline{\tau}^{\sigma} \cong \overline{\tau}$ *for any $\sigma \in I_F$ (i.e., the representations are conjugate by an element in $\Delta(k)$);*
3. $C_{\Delta(R)}(\tau(\Lambda_F)) = Z(R)$;

then τ admits a unique continuous, ν-tame extension to I_F.

We first establish some useful lemmas. Recall that for a group scheme H over $\mathcal{O}$, $\widehat{H}(R)$ is the kernel of the reduction map $H(R) \to H(k)$.

Lemma 5.6. *For any $\sigma \in I_F$, $\tau^{\sigma} \cong \tau$ (i.e., they are conjugate by an element in $\Delta(R)$).*

Proof. Note that $\mathrm{H}^1(\Lambda_F, \mathrm{ad}\,\overline{\tau}) = 0$ since Λ_F has pro-order prime to *p* and $\mathrm{ad}\,\overline{\tau}$ has order a power of *p*. Therefore, if $\tau' : \Lambda_F \to \Delta(R)$ is another continuous lift of $\overline{\tau}$, then τ' is $\widehat{\Delta}(R)$-conjugate to τ. (See [Til96, §3] for deformation theory in this level of generality.) By assumption for $\sigma \in I_F$, we have $\overline{\tau}^{\sigma} \cong \overline{\tau}$ (i.e., $\overline{\tau}^{\sigma} = \overline{g}\overline{\tau}\overline{g}^{-1}$ for some $\overline{g} \in \Delta(k)$). As Δ is $\mathcal{O}$-smooth, there exists a lift $g \in \Delta(R)$ of $\overline{g}$. Now both τ^{σ} and $g\tau g^{-1}$ reduce to $\overline{\tau}^{\sigma}$, and hence, they are $\widehat{\Delta}(R)$-conjugate. Thus, $\tau^{\sigma} \cong \tau$. $\square$

Lemma 5.7. *If H is a smooth group scheme over $\mathcal{O}$, then $\widehat{H}(R) = \widehat{H^0}(R)$ for any $\mathcal{O}$-algebra R.*

Proof. If an R-point reduces to the identity, it lies in the identity component of H. $\square$

Lemma 5.8. *The group $S(R)$ is the product of $\widehat{S}(R)$ and a finite group whose order is prime to p.*

Proof. Recall that S is smooth multiplicative by Proposition 4.7. In particular, $S(k)$ has order prime to *p*. Consider the exact sequence of groups

$$1 \to \widehat{S}(R) \to S(R) \to S(k) \to 1.$$

Lemma 5.7 implies that $\widehat{S}(R) = \widehat{S^0}(R)$, the latter is pro-*p*. Then since S is commutative, $S(R)$ is the product of the pro-*p* group $\widehat{S^0}(R)$ and the Teichmuller lifts of elements in $S(k)$. $\square$

Lemma 5.9. *The map $\nu : \widehat{Z}(R) \to \widehat{S}(R)$ is an isomorphism.*

Proof. By the completeness of R, it suffices to prove the following: suppose A is a local Artinian $\mathcal{O}$-algebra and $I \subset A$ is a nilpotent ideal. Let $z \in Z(A/I)$ and $s \in S(A/I)$ be such that $\nu(z) = s$. Then for any $\tilde{s} \in S(A)$ mapping to $s \in S(A/I)$, there exists a unique element $\tilde{z} \in Z(A)$ mapping to $z \in Z(A/I)$ such that $\nu(\tilde{z}) = \tilde{s}$. By assumption, $\nu : Z \to S$ is a smooth isogeny of multiplicative groups; in particular, it is étale, so the above follows immediately from the infinitesimal criterion for étale morphisms. $\square$

Proof of Proposition 5.5. A continuous extension of τ to I_F is determined by its value on σ, a chosen topological generator of the $\mathbf{Z}_p$-part of the tame inertia. By Lemma 5.6, there is an $A \in \Delta(R)$ such that for $g \in \Lambda_F$,

$$\tau^{\sigma}(g) = \tau(\sigma g \sigma^{-1}) = A\tau(g)A^{-1}.$$

As $\overline{\tau}$ is continuous, it factors through a finite quotient, so there is a power p^b such that $\overline{\tau}^{\sigma^{p^b}} = \overline{\tau}$ as σ has pro-p order.

Since $\ker \tau = \ker \overline{\tau}$, we see that for all $g \in \Lambda_F$,

$$A^{p^b} \tau(g) A^{-p^b} = \tau^{\sigma^{p^b}}(g) = \tau(g).$$

It follows that $A^{p^b} \in C_{\Delta(R)}(\tau(\Lambda_F))$, which equals $Z(R)$ by assumption. As k is perfect and Z is multiplicative, the p-th power map is an automorphism of $Z(k)$. We can then modify A by a lift of an appropriate element of $Z(k)$ so that A^{p^b} reduces to the identity. By Lemma 5.8 and (the surjectivity part of) Lemma 5.9, we can further multiply A by an element in $\widehat{Z}(R)$ so that $\nu(A)$ has finite prime-to-p order. We can now obtain a continuous, ν-tame extension $\tau : I_F \to \Delta(R)$ by sending σ to A.

It remains to show this extension is unique. Suppose another (continuous and ν-tame) extension sends σ to $B \in \Delta(R)$. Note that BA^{-1} commutes with $\tau(\Lambda_F)$ so $z := BA^{-1} \in Z(R)$. By continuity, there is a power p^b such that A^{p^b} and B^{p^b} reduce to the identity in $\Delta(k)$, and hence, z^{p^b} does as well. Hence, z reduces to the identity as the p-th power map is an automorphism of $Z(k)$. Since both extensions are ν-tame, we see that $\nu(z)$ has finite order relatively prime to p. Since $\nu(z)$ also reduces to the identity, we see that $\nu(z) = 1$. By Lemma 5.9, $\nu : \widehat{Z}(R) \to \widehat{S}(R)$ is injective. We conclude that $z = 1$, and hence, the extensions are the same. $\square$

Remark 5.10. We call this step in the argument 'Clifford theory' as the analogous step for GL_n [CHT08, Lemma 2.4.11] makes use of ideas from Clifford theory (see [CR81, §11]).

6. Lifts and minimally ramified deformations

As before, let F be a local field of residue characteristic $\ell \neq p$ and G be a weakly reductive group scheme over a p-adic ring of integers $\mathcal{O}$ with residue field k.

6.1. Lifting residual representations

Recall the terminology for decomposition types from Definitions 4.1 and 4.2. We will prove the following theorem over the course of this subsection.

Theorem 6.1. *Let $\overline{\rho} : \Gamma_F \to G(k)$ be a continuous homomorphism. Suppose there exists a decomposition type $(\mathfrak{C}, \Delta)$ over $\mathcal{O}$ adapted to $\overline{\rho}(\Lambda_F)$. Suppose that p is good for the decomposition type $(\mathfrak{C}, \Delta)$. Then there exists a continuous homomorphism $\rho : \Gamma_F \to G(\mathcal{O})$ that lifts $\overline{\rho}$ such that $C_G(\rho(I_F))$ is $\mathcal{O}$-smooth.*

Remark 6.2. The proof will also show that $\rho(\Lambda_F) \subset \Delta(\mathcal{O})$ and give some control over the 'inertial type' of the lift. We remark that the $\mathcal{O}$-smoothness of $C_G(\rho(I_F))$ is crucial for identifying a formally smooth component of the deformation ring; see Theorem 6.16. In the course of the proof, we will construct 'many' lifts parameterized by an element $z \in \mathfrak{C}^0(\mathcal{O})$. This is not strictly necessary for the proof, and in fact, one can take z to be the identity element throughout this subsection. However, we use this z to build flexibility into our argument and allow inertial types that are not minimally ramified, which correspond to other components of the deformation ring.

Set $\overline{\tau} := \overline{\rho}|_{\Lambda_F}$, and let $\overline{\Lambda} := \overline{\tau}(\Lambda_F)$. As the decomposition type is adapted to $\overline{\rho}(\Lambda_F)$, by Proposition 4.12, there exists $\Lambda \subset \Delta(\mathcal{O})$ lifting $\overline{\Lambda}$ such that $C_G(\Lambda) = \mathfrak{C}$ and $\Delta = C_G(\mathfrak{C})$. As the pro-order of Λ_F is prime to p, there is also a lift $\tau : \Lambda_F \to \Delta(\mathcal{O})$ of $\overline{\rho}|_{\Lambda_F}$ with $\ker(\tau) = \ker(\overline{\rho}|_{\Lambda_F})$. Thus, $\Lambda = \tau(\Lambda_F)$.

Lemma 6.3. *The order of $\pi_0 C_G(\Lambda)$ is prime to p.*

Proof. This follows from Corollary 3.7. $\square$

Lemma 6.4. *We have that $N_G(\Lambda) \subset N_G(\Delta)$.*

Proof. Any point of $N_G(\Lambda)$ normalizes Λ and hence normalizes $C_G(\Lambda)$ and therefore normalizes $C_G(C_G(\Lambda)) = \Delta$. $\qquad\square$

As in Section 5.1, fix a preimage σ of a topological generator of $\mathbf{Z}_p$ under the surjection $I_F \to \mathbf{Z}_p$ and a Frobenius ϕ satisfying

$$\phi\sigma\phi^{-1} = \sigma^q,$$

where q is the size of the residue field of F (a power of ℓ). As σ and ϕ normalize Λ_F, $\overline{\rho}(\sigma)$ and $\overline{\rho}(\phi)$ belong to $N_{G(k)}(\overline{\Lambda})$. Then Lemma 6.4 shows that $\overline{\rho}(\sigma)$ and $\overline{\rho}(\phi)$ are contained in $\mathcal{N}(k) := N_{G(k)}(\Delta(k))$. As I_F/Λ_F is pro-p and p does not divide the index of $\Delta(k)\mathfrak{C}(k)$ in $\mathcal{N}(k)$ as p is good for the decomposition type, it follows that $\overline{\rho}(\sigma) \in \Delta(k) \cdot \mathfrak{C}(k)$.

Recall the morphism $\nu : \Delta \to \Delta^{\mathrm{ab}}$ and the notion of ν-tameness from Definition 5.4.

Proposition 6.5. *There is a continuous, ν-tame extension of τ to a homomorphism*

$$\tau : I_F \to \Delta(\mathcal{O}).$$

We use τ and $\overline{\tau}$ to denote the extensions as well as the functions originally defined on Λ_F.

Proof. We will apply Proposition 5.5 with $R = \mathcal{O}$. First, since p is good for the decomposition type, Proposition 4.13 implies that $\nu : Z(\Delta)^0 \to \Delta^{\mathrm{ab},0}$ is smooth. It remains to check the three hypotheses. The first is built into the construction of τ. Since $\overline{\rho}(\sigma) \in \Delta(k) \cdot \mathfrak{C}(k)$ and Δ and $\mathfrak{C}$ commute, it follows that $\overline{\tau}^\sigma = \overline{\rho}(\sigma)\overline{\tau}\,\overline{\rho}(\sigma)^{-1}$ and $\overline{\tau}$ are isomorphic as $\Delta(k)$-valued representations, whence the second condition. Lemma 4.10 gives the third condition. $\qquad\square$

Corollary 6.6. *We have $\overline{\tau}^\phi = \overline{\rho}(\phi) \cdot \overline{\tau} \cdot \overline{\rho}(\phi)^{-1}$ on I_F.*

Proof. Since $\overline{\rho}(\phi) \in N_{G(k)}(\Delta(k))$, this is a consequence of the uniqueness part of Proposition 5.5. $\qquad\square$

We next analyze $\overline{\rho}$ as a combination of $\overline{\tau}$ (valued in Δ) and a representation valued in $\mathfrak{C}^0$.

Proposition 6.7. *Continuing the standing assumptions:*

1. *There is a continuous homomorphism $\overline{\omega} : I_F/\Lambda_F \to \mathfrak{C}^0(k)$ such that*

$$\overline{\rho}|_{I_F} = \overline{\tau} \cdot \overline{\omega} = \overline{\omega} \cdot \overline{\tau}.$$

2. *The element $\overline{\omega}(\sigma)$ is unipotent, and there exists a pure fiberwise unipotent $u \in \mathfrak{C}^0(\mathcal{O})$ lifting $\overline{\omega}(\sigma)$.*
3. *For any element $z \in \mathfrak{C}^0(\mathcal{O})$ which reduces to the identity in $\mathfrak{C}^0(k)$ and commutes with u, define a homomorphism $\omega_z : I_F/\Lambda_F \to \mathfrak{C}^0(\mathcal{O})$ sending σ to zu. The function $\tau \cdot \omega_z : I_F \to \Delta(\mathcal{O})\mathfrak{C}^0(\mathcal{O}) \subset G(\mathcal{O})$ is a continuous homomorphism lifting $\overline{\rho}|_{I_F}$.*

(At a first reading, it is fine to take z to be the identity.)

Proof. We define $\overline{\omega} := \overline{\tau}^{-1} \cdot \overline{\rho}|_{I_F}$. For any $g \in I_F$ and for any $h \in \Lambda_F$,

$$\overline{\omega}(g)\overline{\rho}(h)\overline{\omega}(g)^{-1} = \overline{\tau}(g^{-1})\overline{\rho}(ghg^{-1})\overline{\tau}(g) = \overline{\tau}(g^{-1})\overline{\tau}(ghg^{-1})\overline{\tau}(g) = \overline{\tau}(h) = \overline{\rho}(h),$$

where the second equality holds because $ghg^{-1} \in \Lambda_F$ and $\overline{\rho}|_{\Lambda_F} = \overline{\tau}|_{\Lambda_F}$. Thus, $\overline{\omega}$ is valued in $C_{G_k}(\overline{\Lambda})(k)$. Since I_F/Λ_F is pro-p and $\pi_0 C_{G_k}(\overline{\Lambda})$ has order prime to p by Lemma 6.3, $\overline{\omega}$ is valued in $C_G(\overline{\Lambda})^0(k) = \mathfrak{C}^0(k)$, with the equality following from Definition 4.2.

Moreover, for any $g, h \in I_F$, we see that

$$\overline{\omega}(gh) = \overline{\tau}(gh)^{-1}\overline{\rho}(gh) = \overline{\tau}(h)^{-1}\overline{\tau}(g)^{-1}\overline{\rho}(g)\overline{\rho}(h) = \overline{\tau}(g)^{-1}\overline{\rho}(g)\overline{\tau}(h)^{-1}\overline{\rho}(h) = \overline{\omega}(g)\overline{\omega}(h),$$

where the third equality uses that $\overline{\tau}(g)^{-1}\overline{\rho}(g) = \overline{\omega}(g) \in C_{G_k}(\overline{\Lambda})(k) = \mathfrak{C}(k)$ commutes with $\overline{\tau}(h)^{-1} \in \Delta(k)$ (Definition 4.1). So $\overline{\omega}$ is a group homomorphism. It is continuous because both $\overline{\rho}|_{I_F}$ and $\overline{\tau}$ are continuous.

Second, we show that $\overline{\omega}(\sigma)$ is unipotent. This element decomposes as a commuting product of semisimple and unipotent elements of $\mathfrak{C}^0(k)$. The order of a semisimple element in $\mathfrak{C}^0(k)$ is prime to p, while by continuity, there is an $r \geq 0$ such that $\sigma^{p^r} \in \ker\overline{\omega}$. Thus, $\overline{\omega}(\sigma)$ is unipotent. By Lemma 3.14, there exists a pure unipotent in $\mathfrak{C}^0(\mathcal{O})$ lifting $\overline{\rho}(\sigma)$.

For the last statement, note that ω_z is continuous as the image of a topological generator of $I_F/\Lambda_F \simeq \mathbf{Z}_p$ reduces to a unipotent element. The function $\tau \cdot \omega_z$ is a homomorphism as Δ and $\mathfrak{C}^0$ commute. □

Definition 6.8. For z as above, define $\theta_z : I_F \to \Delta(\mathcal{O})\mathfrak{C}^0(\mathcal{O})$ to be the product $\tau \cdot \omega_z$.

Note that $\overline{\theta}_z^{\phi} = \overline{\rho}^{\phi}|_{I_F} = \overline{\rho}(\phi) \cdot \overline{\theta}_z \cdot \overline{\rho}(\phi)^{-1}$. Corollary 6.6 implies that $\overline{\omega}^{\phi} = \overline{\rho}(\phi) \cdot \overline{\omega} \cdot \overline{\rho}(\phi)^{-1}$, hence using the structure of Γ_F/Λ_F from Lemma 5.2 that

$$\overline{\omega}(\sigma)^q = \overline{\rho}(\phi)\overline{\omega}(\sigma)\overline{\rho}(\phi)^{-1}. \tag{6.1}$$

Lemma 6.9. *Continuing the standing assumptions, there exists $n \in G(\mathcal{O})$ lifting $\overline{\rho}(\phi)$ such that*

1. $\tau^{\phi} = n\tau n^{-1}$ *on I_F and*
2. $u^q = nun^{-1}$.

Proof. There exists an element $n \in G(\mathcal{O})$ lifting $\overline{\rho}(\phi)$ as G is $\mathcal{O}$-smooth. Since the mod-p reductions of τ^{ϕ} and $n\tau n^{-1}$ agree on Λ_F and Λ_F is prime to p, we may and do multiply n by an element in $\widehat{G}(\mathcal{O})$ so that $\tau^{\phi} = n\tau n^{-1}$ on Λ_F. In particular, using Lemma 6.4, we see that $n \in N_{G(\mathcal{O})}(\Lambda) \subset N_{G(\mathcal{O})}(\Delta(\mathcal{O}))$.

Since τ is ν-tame, so is τ^{ϕ}. However, $\mathrm{Ad}\, n : \Delta_{\mathcal{O}} \to \Delta_{\mathcal{O}}$ induces an automorphism $A : \Delta_{\mathcal{O}}^{\mathrm{ab}} \to \Delta_{\mathcal{O}}^{\mathrm{ab}}$ with $\nu \circ \mathrm{Ad}\, n = A \circ \nu$ by natural properties of the abelianization map $\nu : \Delta \to \Delta^{\mathrm{ab}}$. So for any $\sigma \in I_F$, $\nu(n \cdot \tau(\sigma) \cdot n^{-1}) = A(\nu(\tau(\sigma)))$ has finite prime to p order. Therefore, $n\tau n^{-1}$ is ν-tame, and the uniqueness part of Proposition 5.5 then implies that $\tau^{\phi} = n\tau n^{-1}$ on I_F.

Note that u is pure by construction, and Lemma 3.18 implies that u^q is pure. Furthermore, $nun^{-1} \in \mathfrak{C}^0(\mathcal{O})$ is also pure. Now (6.1) shows that u^q and nun^{-1} agree in the special fiber. Recall that p is pretty good for G, so Lemma 4.11 implies that p is pretty good for $\mathfrak{C} = C_G(\Lambda)$. Hence, we can apply [Cot22a, Theorem 5.11] to conclude that u^q and nun^{-1} are $\mathfrak{C}^0(\mathcal{O})$-conjugate (i.e., $cnun^{-1}c^{-1} = u^q$ for some $c \in \mathfrak{C}^0(\mathcal{O})$).

We claim that we may choose c so that it reduces to the identity. We know that the reduction of c centralizes $\overline{u}^q = \overline{\rho}(\phi)\overline{u}\overline{\rho}(\phi)^{-1}$. As the centralizer $C_{\mathfrak{C}}(u^q)$ is smooth by Theorem 3.10 (recall that $\mathfrak{C}$ is weakly reductive and u^q is pure fiberwise unipotent), we may modify c so that it reduces to the identity. Making this choice, it is clear that cn reduces to $\overline{\rho}(\phi)$ and that $u^q = (cn)u(cn)^{-1}$. As $\tau^{\phi} = n\tau n^{-1}$ takes values in Δ and the groups Δ and $\mathfrak{C}$ commute, we have $\tau^{\phi} = cn\tau n^{-1}c^{-1}$ on I_F, as desired. □

We now have the necessary ingredients to complete the proof.

Proof of Theorem 6.1. Given $\overline{\rho} : \Gamma_F \to G(k)$, construct $\theta_z : I_F \to G(\mathcal{O})$ lifting $\overline{\rho}|_{I_F}$ as in Proposition 6.7, depending on the choice of pure fiberwise unipotent $u \in \mathfrak{C}^0(\mathcal{O})$ lifting $\overline{\omega}(\sigma)$ and choice of z. Then Lemma 6.9 gives an element $n \in G(\mathcal{O})$ lifting $\overline{\rho}(\phi)$ such that $\tau^{\phi} = n\tau n^{-1}$ on I_F and such that $u^q = nun^{-1}$.

We now fix a choice of $z \in \mathfrak{C}^0(\mathcal{O})$ so that

○ z reduces to the identity in $\mathfrak{C}^0(k)$;
○ z commutes with u;
○ $nzn^{-1} = z^q$.

(For example, we may take $z = 1$.) We define the lift $\rho : \Gamma_F \to G(\mathcal{O})$ to be θ_z on I_F and by setting $\rho(\phi) = n$. Note that

$$\omega_z^\phi(\sigma) = \omega_z(\sigma^q) = (zu)^q = z^q u^q.$$

Then ρ is a homomorphism as

$$\rho(\phi)\theta_z(\sigma)\rho(\phi)^{-1} = n\tau(\sigma)n^{-1}nzun^{-1} = \tau^\phi(\sigma)\omega_z^\phi(\sigma) = \theta_z^\phi(\sigma),$$

where the penultimate step uses that $nun^{-1} = u^q$ and that $nzn^{-1} = z^q$. It is continuous as θ_z is continuous, and $\overline{\rho}(\phi) = \bar{n}$ has finite order.

Finally, Lemma 6.11 implies that if, moreover, $z \in Z(\mathfrak{C})(\mathcal{O})$, then the centralizer of the inertia is $\mathcal{O}$-smooth. □

Example 6.10. Taking $z = 1$ is simplest and gives a minimally ramified lift. Other choices of z give lifts with different inertial types, which are of interest but are not the focus of the present work. We will just give an example.

After fixing the choice of τ, the restriction of ρ to I_F depends on u and z. Let $V = \mathrm{span}_{\mathcal{O}}\{e_1, e_2, e_3, e_4\}$ and $\mathfrak{C}$ be the Levi preserving the grading $V = \mathrm{span}\{e_1, e_2\} \oplus \mathrm{span}\{e_3, e_4\}$; observe $V \simeq \mathrm{GL}_2 \times \mathrm{GL}_2 \subset \mathrm{GL}_4$. Suppose u is the unipotent such that

$$u(e_1) = e_1, \quad u(e_2) = e_1 + e_2, \quad u(e_3) = e_3, \quad u(e_4) = e_3 + e_4.$$

One possible case is that $\overline{\rho}(\phi)$, and hence n, swap $\mathrm{span}\{e_1, e_2\}$ and $\mathrm{span}\{e_3, e_4\}$. In that case, for any p^a-th root of unity $\zeta \in \mathcal{O}$ where $p^a|(q - 1)$, we can take the scalar matrix $z = \zeta I_4$. It is clear that ζ reduces to the identity in k, that n and u commute with z, and that $z^q = z$. In this case, the centralizer of $\rho(I_F)$ equals $C_{\mathfrak{C}}(u)$ and is $\mathcal{O}$-smooth.

Lemma 6.11. *Under the standing assumptions, if $z \in Z(\mathfrak{C})(\mathcal{O})$ and satisfies the conditions on z in the proof of Theorem 6.1, then $C_G(\theta_z(I_F)) = C_{\mathfrak{C}}(\omega_z(\sigma))$ is $\mathcal{O}$-smooth.*

Proof. Note that $\theta_z(I_F)$ is generated by $\theta_z(\Lambda_F) = \tau(\Lambda_F)$ and $\theta_z(\sigma) = \tau(\sigma)\omega_z(\sigma)$. For any $\mathcal{O}$-algebra R, we have

$$C_{G(R)}(\theta_z(I_F)) = C_{G(R)}(\tau(\Lambda_F)) \cap C_{G(R)}(\theta_z(\sigma)) = \mathfrak{C}(R) \cap C_{G(R)}(\theta_z(\sigma)).$$

Now an element of $\mathfrak{C}(R)$ commutes with $\theta_z(\sigma) = \tau(\sigma)\omega_z(\sigma)$ if and only if it commutes with $\omega_z(\sigma)$ since $\tau(\sigma) \in \Delta(\mathcal{O})$ and Δ commutes with $\mathfrak{C}$, and hence, $C_{G(R)}(\theta_z(I_F)) = C_{\mathfrak{C}(R)}(\omega_z(\sigma))$. When z is central in $\mathfrak{C}$, we see that $C_{\mathfrak{C}(R)}(\omega_z(\sigma)) = C_{\mathfrak{C}(R)}(u)$. Thus, the smoothness follows from Theorem 3.10 (recall that $\mathfrak{C}$ is weakly reductive and u is pure fiberwise unipotent). □

Remark 6.12. If z is not central in $\mathfrak{C}$, the centralizer need not be smooth. In particular, the dimension of the centralizer of the semisimple part of $\omega_z(\sigma) = zu$ is not locally constant as would be needed to apply [Cot22a, Theorem 1.1] (since z reduces to the identity). See [Cot22a, Remark 5.5] for some examples of the failure of flatness for centralizers of pure sections.

6.2. The minimally ramified deformation condition

We now fix a residual representation $\overline{\rho} : \Gamma_F \to G(k)$. For a continuous homomorphism $\theta : I_F \to G(\mathcal{O})$ lifting $\overline{\rho}|_{I_F}$, let θ_R denote the composition $I_F \to G(\mathcal{O}) \to G(R)$ for an $\mathcal{O}$-algebra R. Recall that $\widehat{\mathcal{C}}_{\mathcal{O}}$ is the category of coefficient $\mathcal{O}$-algebras (complete local Noetherian rings with residue field k, with morphisms local homomorphisms inducing the identity map on k and with the structure morphism a map of coefficient rings). For the rest of this subsection, we suppose there exists a decomposition type $(\mathfrak{C}, \Delta)$ over $\mathcal{O}$ adapted to $\overline{\rho}(\Lambda_F)$ and that p is good for $(\mathfrak{C}, \Delta)$. By the paragraph below Remark 6.2, there

is a lift $\tau\colon \Lambda_F \to \Delta(\mathcal{O})$ of $\overline{\rho}|_{\Lambda_F}$ such that $C_G(\tau(\Lambda_F)) = \mathfrak{C}$. Moreover, τ extends to a homomorphism $\tau\colon I_F \to \Delta(\mathcal{O})$ by Proposition 6.5. We fix this homomorphism throughout this subsection.

Lemma 6.13. *Let* $\theta\colon I_F \to G(\mathcal{O})$ *be a continuous homomorphism lifting* $\overline{\rho}|_{I_F}$. *Then there exists* $g \in \widehat{G}(\mathcal{O})$ *and* $\omega\colon I_F/\Lambda_F \to \mathfrak{C}(\mathcal{O})$ *such that* $\theta^g = \tau \cdot \omega$. *Moreover,* ω *is unique up to* $\mathfrak{C}(\mathcal{O})$-*conjugacy.*

Proof. Since both $\theta|_{\Lambda_F}$ and $\tau|_{\Lambda_F}$ lift $\overline{\rho}|_{\Lambda_F}$, they are $\widehat{G}(\mathcal{O})$-conjugate. So after conjugating, we may assume that $\theta|_{\Lambda_F} = \tau|_{\Lambda_F}$. An argument similar to the first part of the proof of Proposition 6.7 then gives the first statement. For the second part, suppose $\theta^g = \tau \cdot \omega$ and $\theta^{g'} = \tau \cdot \omega'$ for $\omega, \omega'\colon I_F/\Lambda_F \to \mathfrak{C}(\mathcal{O})$. Restricting to Λ_F, we see that $g^{-1}g'$ centralizes $\tau(\Lambda_F)$, and hence, $c := g^{-1}g' \in \mathfrak{C}(\mathcal{O})$. So $\theta^{g'} = \theta^{gc} = \tau^c \cdot \omega^c = \tau \cdot \omega^c$, and hence $\omega' = \omega^c$. □

Corollary 6.14. *There is a unique* $\widehat{G}(\mathcal{O})$-*conjugacy class of lifts* $\theta\colon I_F \to G(\mathcal{O})$ *of* $\overline{\rho}|_{I_F}$ *such that the associated homomorphism* ω *satisfies that* $\omega(\sigma)$ *is a pure unipotent element in* $\mathfrak{C}^0(\mathcal{O})$. *Moreover,* θ *extends to a continuous homomorphism* $\Gamma_F \to G(\mathcal{O})$.

Proof. The existence follows from Section 6.1 (taking $z = 1$), which also shows that θ extends to Γ_F. For uniqueness, by Lemma 6.13, it suffices to show that if u and u' are pure unipotents in $\mathfrak{C}^0(\mathcal{O})$ lifting $\overline{\omega}(\sigma) \in \mathfrak{C}^0(k)$, then they are $\mathfrak{C}^0(\mathcal{O})$-conjugate. Recall that p is pretty good for G, so Lemma 4.11 implies that p is pretty good for $\mathfrak{C} = C_G(\tau(\Lambda_F))$. Since u and u' agree on the special fiber, [Cot22a, Theorem 5.11] implies that u and u' are $\mathfrak{C}^0(\mathcal{O})$-conjugate. □

Definition 6.15. Let $\theta\colon I_F \to G(\mathcal{O})$ be a continuous homomorphism lifting $\overline{\rho}|_{I_F}$. Let $\mathrm{Lift}_{\overline{\rho}}^{\theta}\colon \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Sets}$ be the functor whose set of R-points $\mathrm{Lift}_{\overline{\rho}}^{\theta}(R)$ is given by lifts $\rho\colon \Gamma_F \to G(R)$ of $\overline{\rho}$ such that there exists $g \in \widehat{G}(R)$ such that $\rho^g|_{I_F} = \theta_R$.

Let $\mathrm{Lift}_{\overline{\rho}}^{m.r.}$ be this functor when θ is a homomorphism as in Corollary 6.14. We call this the *minimally ramified lifting condition* for $\overline{\rho}$.

We will use the notions of lifting conditions and deformation conditions for Galois representations which are reviewed in [Boo19a, Definition 2.3] – in particular, the notion of liftability and its connection with formal smoothness of the representing object.

Theorem 6.16. *Let* $\theta\colon I_F \to G(\mathcal{O})$ *be a continuous homomorphism lifting* $\overline{\rho}|_{I_F}$. *Suppose that* $C_G(\theta(I_F))$ *is* $\mathcal{O}$-*smooth and* θ *extends to a continuous homomorphism* $\rho_0\colon \Gamma_F \to G(\mathcal{O})$. *Then* $\mathrm{Lift}_{\overline{\rho}}^{\theta}$ *is a well-defined lifting condition. Moreover, it is liftable, and the tangent space to the corresponding deformation functor* $\mathrm{Def}_{\overline{\rho}}^{\theta}$ *has dimension* $\dim_k \mathrm{H}^0(\Gamma_F, \overline{\rho}(\mathfrak{g}))$.

Proof. The functor $\mathrm{Lift}_{\overline{\rho}}^{\theta}$ is obviously closed under strict equivalence. The key to checking the second condition of [Boo19a, Definition 2.3] is that $C_G(\theta(I_F))$ is $\mathcal{O}$-smooth. In particular, suppose we have a Cartesian diagram in $\widehat{\mathcal{C}}_{\mathcal{O}}$

$$\begin{array}{ccc} R_1 \times_{R_0} R_2 & \longrightarrow & R_2 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_0 \end{array}$$

and $\rho_1 \in \mathrm{Lift}_{\overline{\rho}}^{\theta}(R_1)$ and $\rho_2 \in \mathrm{Lift}_{\overline{\rho}}^{\theta}(R_2)$ with common image in $\mathrm{Lift}_{\overline{\rho}}^{\theta}(R_0)$. There exists $g_1 \in \widehat{G}(R_1)$ and $g_2 \in \widehat{G}(R_2)$ such that $g_1\rho_1|_{I_F} g_1^{-1} = \theta_{R_1}$ and $g_2\rho_2|_{I_F} g_2^{-1} = \theta_{R_2}$. Looking at the images in R_0, we conclude that $g_1 g_2^{-1} \in C_{G(R_0)}(\theta(I_F))$. Since $C_G(\theta(I_F))$ is $\mathcal{O}$-smooth, there exists a lift $x \in C_{G(R_1)}(\theta(I_F))$. Then the element $(xg_1, g_2) \in G(R_1 \times_{R_0} R_2)$ conjugates (ρ_1, ρ_2) to $(\theta_{R_1}, \theta_{R_2})$, so $(\rho_1, \rho_2) \in \mathrm{Lift}_{\overline{\rho}}^{\theta}(R_1 \times_{R_0} R_2)$, as desired.

To check smoothness, let $R \to S$ be a small morphism in $\widehat{\mathcal{C}}_{\mathcal{O}}$, and let $\rho_S \in \mathrm{Lift}_{\overline{\rho}}^{\theta}(S)$. We need to show that there exists $\rho_R \in \mathrm{Lift}_{\overline{\rho}}^{\theta}(R)$ mapping to ρ_S under the morphism $R \to S$. Since G is

$\mathcal{O}$-smooth, we may assume that $\rho_S|_{I_F} = \theta_S$ by $G(S)$-conjugation. Since $\theta^\phi = \rho_0(\phi)\theta\rho_0(\phi)^{-1}$ and $\theta_S^\phi = \rho_S(\phi)\theta_S\rho_S(\phi)^{-1}$, $\rho_0(\phi)^{-1}\rho_S(\phi)$ centralizes $\theta_S(I_F)$. Using the $\mathcal{O}$-smoothness of $C_G(\theta(I_F))$, we obtain an element $c \in C_{G(R)}(\theta(I_F))$ lifting $\rho_0(\phi)^{-1}\rho_S(\phi)$. Then $\rho_0(\phi)c$ lifts $\rho_S(\phi)$ and on I_F,

$$(\rho_0(\phi)c)\theta_R(\rho_0(\phi)c)^{-1} = \rho_0(\phi)(\rho_0|_{I_F})\rho_0(\phi)^{-1} = \theta_R^\phi.$$

Define $\rho_R : \Gamma_F \to G(R)$ by $\rho_R|_{I_F} = \theta_R$ and $\rho_R(\phi) = \rho_0(\phi)c$, and observe that $\rho_R \in \mathrm{Lift}_{\overline{\rho}}^\theta(R)$ and that it maps to ρ_S.

Notice that the tangent space to $\mathrm{Def}_{\overline{\rho}}^\theta$ equals $\ker(\mathrm{H}^1(\Gamma_F, \overline{\rho}(\mathfrak{g})) \to \mathrm{H}^1(I_F, \overline{\rho}(\mathfrak{g})))$. The last claim then follows from inflation-restriction and the fact that $\dim \mathrm{H}^1(\Gamma_F/I_F, \overline{\rho}(\mathfrak{g})^{I_F}) = \dim \mathrm{H}^0(\Gamma_F, \overline{\rho}(\mathfrak{g}))$ [Was97, Lemma 1]. □

Corollary 6.17. *Let $\overline{\rho} : \Gamma_F \to G(k)$ be a continuous homomorphism. Suppose there exists a decomposition type $(\mathfrak{C}, \Delta)$ over $\mathcal{O}$ adapted to $\overline{\rho}(\Lambda_F)$, and that p is good for $(\mathfrak{C}, \Delta)$. Then $\mathrm{Lift}_{\overline{\rho}}^{\mathrm{m.r.}}$ is a formally smooth lifting condition. Moreover, the corresponding deformation condition $\mathrm{Def}_{\overline{\rho}}^{\mathrm{m.r.}}$ has dimension $\dim_k \mathrm{H}^0(\Gamma_F, \overline{\rho}(\mathfrak{g}))$. Equivalently, $\mathrm{Spf}\, R_{\overline{\rho}}^{\mathrm{m.r.},\square}$ is $\mathcal{O}$-formally smooth of relative dimension $\dim_k G_k$.*

Proof. Construct a lift ρ as in Section 6.1 with $z = 1$, and observe that $C_G(\theta(I_F))$ is $\mathcal{O}$-smooth by Lemma 6.11. Applying Theorem 6.16 with $\theta = \rho|_{I_F}$, we obtain that $\mathrm{Lift}_{\overline{\rho}}^\theta$ is formally smooth. Note that θ satisfies Corollary 6.14 by construction (since $z = 1$ in Proposition 6.7), and so $\mathrm{Lift}_{\overline{\rho}}^\theta = \mathrm{Lift}_{\overline{\rho}}^{\mathrm{m.r.}}$. Finally, note that adding a framing increases the relative dimension by $\dim_k G_k$ minus the dimension of the automorphisms compatible with the Γ_F-action, $\dim_k H^0(\Gamma_F, \overline{\rho}(\mathfrak{g}))$. □

Remark 6.18. Let $\nu_R : G \to G^{\mathrm{ab}} = G/\mathscr{D}(G)$ be the quotient map (see Proposition 2.27). We could also formulate a variant of the deformation condition where the morphism $\nu_R \circ \rho_R : \Gamma_F \to G(R) \to G^{\mathrm{ab}}(R)$ is a fixed lift of $\nu_k \circ \overline{\rho}$. This generalizes adding the requirement that the determinant of the lift be fixed when $G = \mathrm{GL}_n$.

Remark 6.19. The deformation conditions $\mathrm{Def}_{\overline{\rho}}^\theta$ where θ has different inertial types should be of interest when investigating $\ell \neq p$ versions of the Breuil-Mézard conjecture. Lifts where $C_G(\theta(I_F))$ is $\mathcal{O}$-smooth give formally smooth components of the lifting ring $\mathrm{Spf}\, R_{\overline{\rho}}^\square$. For example, we may take $\theta = \theta_z$ for $z \neq 1$ as in Lemma 6.11.

We can now prove our main theorem from the introduction. Recall that G is a weakly reductive group defined over $\mathcal{O}$.

Theorem 6.20. *Let $\overline{\rho} : \Gamma_F \to G(k)$ be a continuous homomorphism. Suppose that p is large enough relative to the root datum of G (the bound can be made effective). Then $\mathrm{Lift}_{\overline{\rho}}^{\mathrm{m.r.}}$ is a formally smooth lifting condition. Moreover, the corresponding deformation condition $\mathrm{Def}_{\overline{\rho}}^{\mathrm{m.r.}}$ has dimension $\dim_k \mathrm{H}^0(\Gamma_F, \overline{\rho}(\mathfrak{g}))$. Equivalently, $\mathrm{Spf}\, R_{\overline{\rho}}^{\mathrm{m.r.},\square}$ is $\mathcal{O}$-formally smooth of relative dimension $\dim_k G_k$.*

Proof. By Proposition 4.14, if p is pretty good for G, then there exists a decomposition type $(\mathfrak{C}, \Delta)$ over $\mathcal{O}$ adapted to $\overline{\Lambda} := \overline{\rho}(\Lambda_F)$. We need to ensure that p is good for $(\mathfrak{C}, \Delta)$ in the sense of Definition 4.6. Condition (1) is trivially satisfied. If $p > \mathrm{rank}\,\mathscr{D}(G^0) + 1$, condition (2) follows as well. In fact, for any connected semisimple group H and any prime p dividing the order of $Z(H)$, $p \leq \mathrm{rank}\, H + 1$. To check this, we may assume that H is simply-connected, and hence, it is a direct product of simple, simply-connected groups, in which case the claim follows from a simple case-checking. It remains to consider Definition 4.6, (3). By Proposition 2.24, this will hold if $p \nmid |W_G|$ and $p > |\pi_0\mathfrak{C}_k| = |\pi_0 C_{G_k}(\overline{\Lambda})|$. Theorem A.10 gives a bound for the last quantity which depends only on G. The theorem now follows from Corollary 6.17. □

Remark 6.21. By the proof of the above theorem, the condition that p is large enough relative to the root datum of G is equivalent to the following:

○ p is pretty good for G in the sense of Definition 1.12.
○ $p > \operatorname{rank} \mathscr{D}(G^0) + 1$.
○ p does not divide the order of W_G.
○ p is larger than the constant in Theorem A.10.

The last condition is explicit but not always pleasant. To illustrate, notice that the constant in Theorem A.10 must be at least the number of components of the centralizer of any particular prime-to-p solvable subgroup of G_k.

When $G = \operatorname{GL}_n$, Schur's lemma shows that centralizer of a completely reducible subgroup is a product of general linear groups. Hence, the component group is always trivial.

For the exceptional group of type G_2, the lower bound for p is 72; see Example A.5.

When $G = \operatorname{PGL}_n$, there is an example of a subgroup whose centralizer is finite of order n^2 [Lie23, Examples(3)]. Similarly, when $G = \operatorname{Sp}_{2n}$, there is an example of a subgroup whose centralizer is finite of order 2^n [Lie23, Examples(2)].

In particular, for $G = \operatorname{Sp}_{2n}$, the last example shows that we must require $p > 2^n$. This is far from optimal: the techniques from [Boo19a] work when $p > 2n$.

Remark 6.22. Note that the last bulleted point in Remark 6.21 is needed to ensure Definition 4.6, (3) for the decomposition type $(\mathfrak{C}, \Delta)$ adapted to the residual representation $\overline{\rho}$. This in turn implies that $\overline{\rho}(I_F) \subset \mathfrak{C}(k)\Delta(k)$, which is the starting point of the argument in Section 6.1. Currently, we do not know how to construct lifts of $\overline{\rho}$ without this condition.

Remark 6.23. A variant of the arguments in this section can be used to show the following mild strengthening of Theorem 6.20. Maintain all of the assumptions on F, $\mathcal{O}$, G, and so on. Let W be the Weil group of F, and let W^0 be the 'discretized version' from [DHKM24], obtained by choosing a topological generator of tame intertia. Let $\mathscr{H}_W$ be the moduli space $\underline{\operatorname{Hom}}_{\mathcal{O}\text{-gp}}(W^0, G)$, which is representable by [DHKM24]. Moreover, [DHKM24, Theorem 1.5] shows that the p-adic formal completion of $\mathscr{H}_W$ represents the functor on p-adically complete (and separated) $\mathcal{O}$-algebras given by

$$R \mapsto \operatorname{Hom}_{\mathrm{cts}}(W, G(R)).$$

There is a similar functor $\mathscr{H}_\Lambda$, obtained from $\mathscr{H}_W$ by replacing W by Λ_F. Attached to $\mathscr{H}_\Lambda$ are the universal centralizer $\mathfrak{C}$ and the universal double centralizer Δ.

For any continuous $f : W \to G(R)$ as above, the restriction $f|_{I_F}$ factors uniquely as $f|_{I_F} = \tau \cdot \omega$, where $\tau : I_F \to \Delta(R)$ is the unique tame extension of $f|_{\Lambda_F}$ (which exists by a variant of Proposition 5.5) and $\omega : I_F/\Lambda_F \to \mathfrak{C}(R)$ is a homomorphism. Let $\mathscr{H}_W^{\mathrm{pure}}$ be the subfunctor of $\mathscr{H}_W$ with $\mathscr{H}_W^{\mathrm{pure}}(R)$ consisting of those f as above such that $\omega(\sigma)$ is pure fiberwise unipotent, where σ is a generator of the pro-p part of I_F. (If R is not reduced, 'pure fiberwise unipotent' should be interpreted to mean that $f(\sigma)$ is fppf-locally on R conjugate to a pure fiberwise unipotent section of $G(\mathcal{O}')$, for some finite extension $\mathcal{O}'$ of $\mathcal{O}$.)

The arguments of this section can be used to show that $\mathscr{H}_W^{\mathrm{pure}}$ is formally smooth and the inclusion map $\mathscr{H}_W^{\mathrm{pure}} \to \mathscr{H}_W$ gives a stratification of $\mathscr{H}_W$ into smooth pieces, each of which is open in an irreducible component of $\mathscr{H}_W$ (for dimension reasons coming from Theorem 6.20). Because of this openness statement, it follows that every point f in the special fiber of $\mathscr{H}_W$ lies in an irreducible component X such that f is a smooth point of X_{red}.

A. Some group theory

The following Theorem is due to Liebeck [Lie23], which originated in an email correspondence between the third author (S.T.) and Martin Liebeck. S.T. would like to thank Martin Liebeck for his interest in our question and for answering it, which allows us to obtain an effective lower bound for p in Theorem 1.1.

Theorem A.1. *Let G be a connected semisimple group over an algebraically closed field k, and let H be a G-irreducible subgroup. Then there is a constant $c \leq 197$ such that $\#C_G(H) \leq c^{\operatorname{rank} G}\#Z(G)$.*

By [Lie23], the constant c is at most 16 if all the simple factors of G are classical, and [Lie23, Lemma 2.5] gives precise bounds for c when G is of exceptional type.

Corollary A.2. *Let G be a connected reductive group over an algebraically closed field k, and let H be a G-irreducible subgroup. Then $\#\pi_0 C_G(H) \leq c^{\operatorname{rank} G^{\operatorname{ad}}} \#\pi_0 Z(G)$.*

Proof. Consider the exact sequence

$$1 \to Z(G) \to C_G(H) \to C_{G^{\operatorname{ad}}}(\overline{H}),$$

where $\overline{H} \subset G^{\operatorname{ad}}$ is the image of H under the natural map $G \to G^{\operatorname{ad}}$. Note that $\overline{H}$ is G^{ad}-irreducible, so Theorem A.1 implies that $\#C_{G^{\operatorname{ad}}}(\overline{H}) \leq c^{\operatorname{rank} G^{\operatorname{ad}}}$. However, since H is G-irreducible, $C_G(H)^0 = Z(G)^0$, so the above exact sequence implies that $\pi_0 C_G(H)/\pi_0 Z(G)$ injects into $C_{G^{\operatorname{ad}}}(\overline{H})$. Therefore, $\#\pi_0 C_G(H) \leq \#\pi_0 Z(G) c^{\operatorname{rank} G^{\operatorname{ad}}}$. $\qquad\square$

Lemma A.3. *Let H be a closed, completely reducible subgroup of G. Choose a maximal torus S of $K := C_G(H)^0$, and let $L = C_G(S)$. Denote by $W(G, L)$ (resp. $W(K, S)$) the quotient $N_G(L)/L$ (resp. $N_K(S)/S$); the latter can be naturally identified as a subgroup of the former. Then there is a canonical injection*

$$N_G(H)/N_L(H) \cdot K \hookrightarrow N_{W(G,L)}(W(K,S))/W(K,S).$$

Proof. Observe that $Z_L^0 = S$, which implies $N_G(L) = N_G(S)$. So $N_G(L)/L = N_G(S)/C_G(S)$ contains $N_K(S)/C_K(S) = N_K(S)/S$ as a subgroup. Any $n \in N_G(H)$ normalizes $K = C_G(H)^0$, so S^n is another maximal tori of K. By the conjugacy of maximal tori, there is an element $c \in K$ such that $(S^n)^c = S$ (i.e. $nc \in N_G(S) = N_G(L)$). For any element $w \in N_G(L)$, denote by $\bar{w}$ its image in $W(G, L)$. Define a map

$$\varphi : N_G(H) \to N_{W(G,L)}(W(K,S))/W(K,S)$$

by $\varphi(n) = \overline{nc} W(K, S)$. First, note that $n \mapsto \overline{nc} W(K, S)$ is a well-defined map from $N_G(H)$ to the quotient set $W(G, L)/W(K, S)$, and that nc normalizes $N_K(S)$, which implies $\overline{nc}$ normalizes $W(K, S)$. It follows that φ defined above is a well-defined map. We check that φ is a group homomorphism. Let $n_1, n_2 \in N_G(H)$ with $c_1, c_2 \in K$ such that $n_1 c_1, n_2 c_2$ both normalize S. So $\varphi(n_i) = \overline{n_i c_i} W(K, S)$ for $i = 1, 2$. However, the product $(n_1 c_1)(n_2 c_2)$ normalizes S, and we have $(n_1 c_1)(n_2 c_2) = (n_1 n_2)(n_2^{-1} c_1 n_2) c_2$ with $(n_2^{-1} c_1 n_2) c_2 \in K$, so $\varphi(n_1 n_2) = \overline{(n_1 n_2)(n_2^{-1} c_1 n_2) c_2} W(K, S) = \overline{(n_1 c_1)(n_2 c_2)} W(K, S) = \overline{n_1 c_1} W(K, S) \overline{n_2 c_2} W(K, S) = \varphi(n_1)\varphi(n_2)$. We now compute $\ker \varphi$. It consists of $n \in N_G(H)$ for which $\overline{nc} \in W(K, S) = N_K(S)/S = N_K(S)/C_K(S) = N_K(S)/N_K(S) \cap C_G(S) = N_K(S) \cdot L/L$ for some $c \in K$. So $nc \in N_K(S) \cdot L$, which implies $n \in K \cdot L$. As n normalizes H and K commutes with H, it follows that $n \in K \cdot N_L(H)$. However, if $n = b \cdot c \in N_L(H) \cdot K$, then $\varphi(n) = \overline{nc^{-1}} W(K, S) = \bar{b} W(K, S) = W(K, S)$, where the first equality holds since $S^b = S$ ($b \in L = C_G(S)$), and the last equality holds since $b \in N_L(H) \subset L$, and hence, $\bar{b} = \bar{1} \in W(G, L)$. Thus, $\ker \varphi = K \cdot N_L(H) = N_L(H) \cdot K$ (the last equality holds since $N_L(H)$ normalizes $K = C_G(H)^0$). $\qquad\square$

Theorem A.4. *Let G be a connected reductive group over an algebraically closed field k, and let H be a G-completely reducible subgroup. Then the size of $\pi_0 C_G(H)$ is bounded by*

$$c_G := \#W_G \cdot \sup_{L \subset G} \left\{ c^{\operatorname{rank} L^{\operatorname{ad}}} \cdot \#\pi_0 Z(L) \right\},$$

where L runs over the finitely many conjugacy classes of Levi subgroups of G.

Proof. Let L be as in Lemma A.3. Note that H is an irreducible subgroup of L. There is a natural surjection from $\pi_0(C_G(H))$ to $C_G(H)/C_L(H) \cdot C_G(H)^0$, the latter injects into the group on the left

side of the inclusion in Lemma A.3, and the kernel is a quotient of $\pi_0 C_L(H)$. The theorem now follows from Corollary A.2 and Lemma A.3. □

Example A.5. The constant c_G is a very crude bound, which can be made much smaller for specific G by going through the proof of Theorem A.4. For example, for G the exceptional group of type G_2, if H is not G-ir, then the minimal Levi containing H (unique up to conjugacy) is either a maximal torus or isomorphic to GL_2, and $\pi_0 C_G(H)$ injects into the group on the left side of the inclusion in Lemma A.3, so $\#\pi_0 C_G(H) \leq W_G = 12$. Now if H is G-ir, then $\pi_0 C_G(H) \leq 8.5^2 = 72.25$ by Theorem A.1 and [Lie23, Lemma 2.5]. Thus, the bound c_G can be improved to 72 in this case.

For the remainder of this section, we will generalize Theorem A.4 to non-connected reductive groups in the case that H is solvable. In what follows, we make no attempt in optimizing the bounds for the component groups.

Lemma A.6. *Let $f : G' \to G$ be a central isogeny of connected reductive groups over an algebraically closed field k, and let λ' and λ be compatible automorphisms of G' and G, respectively. Then $\#\pi_0 C_G(\lambda) \leq \#\pi_0 C_{G'}(\lambda') \cdot \# \ker f$.*

Proof. Note that there is an exact sequence

$$1 \to C_{G'}(\lambda') \to f^{-1}(C_G(\lambda)) \to \ker f,$$

where the rightmost map is $x \mapsto \lambda'(x)x^{-1}$. The component group of $f^{-1}(C_G(\lambda))$ is thus an extension of a subgroup of $(\ker f)_{\mathrm{red}}$ by a quotient of the component group of $C_{G'}(\lambda')$.

Moreover, there is an obvious exact sequence

$$1 \to \ker f \to f^{-1}(C_G(\lambda)) \to C_G(\lambda) \to 1.$$

Thus, the component group of $C_G(\lambda)$ is a quotient of the component group of $f^{-1}(C_G(\lambda))$. The lemma follows. □

Lemma A.7. *Let T be a split torus, and let λ be an automorphism of T of order n. Then $\#\pi_0 C_T(\lambda)$ divides $n^{\dim T}$.*

Proof. This follows immediately from [DM18, Lemma 1.2(1)]. □

Lemma A.8. *Let G be a connected reductive group over an algebraically closed field k. Let λ be a semisimple automorphism of G of order n. Then $\#\pi_0 C_G(\lambda) \leq 4^{\mathrm{rank}\,\mathscr{D}(G)} \cdot n^{\dim Z(G)}$.*

Proof. By a theorem of Steinberg, if G is semisimple, then $\pi_0 C_G(\lambda)$ can be identified with a subgroup of $\pi_1(G)$. Moreover, $\#\pi_1(G) \leq 2^{\mathrm{rank}\,G}$ (realized by $G = (\mathrm{PGL}_2)^{\mathrm{rank}\,G}$). Now we suppose that $G = T \times H$ for a torus T and a semisimple group H. Then λ induces acts on T and H. By the above, $\#\pi_0 C_H(\lambda) \leq 2^{\mathrm{rank}\,H} = 2^{\mathrm{rank}\,\mathscr{D}(G)}$. By Lemma A.7, $\#\pi_0 C_T(\lambda) \leq n^{\dim T} = n^{\dim Z(G)}$. So $\#\pi_0 C_G(\lambda) \leq 2^{\mathrm{rank}\,\mathscr{D}(G)} \cdot n^{\dim Z(G)}$.

For the general case, note the canonical isogeny $Z(G)^0 \times \mathscr{D}(G) \to G$, whose kernel is contained in $Z(\mathscr{D}(G))$. By Lemma A.6 and the above, it follows that

$$\#\pi_0 C_G(\lambda) \leq 2^{\mathrm{rank}\,\mathscr{D}(G)} \cdot n^{\dim Z(G)} \cdot \#Z(\mathscr{D}(G)).$$

Finally, note that if H is semisimple, then $\#Z(H) \leq \#Z(H^{\mathrm{sc}}) = \#\pi_1 H^{\mathrm{ad}} \leq 2^{\mathrm{rank}\,H}$. So $\#Z(\mathscr{D}(G)) \leq 2^{\mathrm{rank}\,\mathscr{D}(G)}$. The lemma follows. □

Lemma A.9. *Let H be a (possibly nonconnected) reductive group over an algebraically closed field k. Let F be a solvable finite group with prime-to-p order acting on H. Then the size of $\pi_0 C_H(F)$ is bounded by a constant depending only on $\mathrm{rank}\,H^0$, $\#\pi_0 H$ and $\#F$.*

Proof. This will follow from the preceding lemma and induction. First, note the exact sequence

$$1 \to C_{H^0}(F) \to C_H(F) \to \pi_0(H)$$

and that $C_{H^0}(F)^0 = C_H(F)^0$. It follows that $\#\pi_0 C_H(F) \le \#\pi_0 H \cdot \#\pi_0 C_{H^0}(F)$.

Since F is solvable, there is a composition series

$$F = F_0 \supset F_1 \supset \cdots \supset F_n = \{1\}$$

such that F_{i+1} is normal in F_i and F_i/F_{i+1} is cyclic. Since F_{i+1} is normal in F_i, F_i acts on $C_H(F_{i+1})$, and in fact, $C_H(F_i) = C_{C_H(F_{i+1})}(F_i) = C_{C_H(F_{i+1})}(F_i/F_{i+1})$. So

$$\#\pi_0 C_H(F_i) \le \#\pi_0 C_H(F_{i+1}) \cdot \#\pi_0 C_{C_H(F_{i+1})^0}(F_i/F_{i+1}),$$

and hence,

$$\#\pi_0 C_H(F) \le \prod_{0 \le i < n} \#\pi_0 C_{C_H(F_{i+1})^0}(F_i/F_{i+1}) \cdot \#\pi_0 H.$$

We now conclude by applying Lemma A.8 to the group $C_{C_H(F_{i+1})^0}(F_i/F_{i+1})$. □

Theorem A.10. *Let G be a (possibly nonconnected) reductive group over an algebraically closed field k. Let $\Lambda \subset G(k)$ be a solvable finite group with prime-to-p order. Then the size of $\pi_0 C_G(\Lambda)$ is bounded by a constant depending only on c_{G^0}, rank G^0 and $\#\pi_0 G$. In the special case when G is connected, $\#\pi_0 C_G(\Lambda) \le c_G$ by Theorem A.4.*

Proof. By the first part of the proof of Lemma A.9, we have $\#\pi_0 C_G(\Lambda) \le \#\pi_0 C_{G^0}(\Lambda) \cdot \#\pi_0 G$. Let Λ' be $\Lambda \cap G^0(k)$, so that $\Lambda/\Lambda' \hookrightarrow G/G^0$. Note that $C_{G^0}(\Lambda) = C_{C_{G^0}(\Lambda')}(\Lambda/\Lambda')$. By Theorem A.4 (recall Λ has prime-to-p order by assumption), the size of $\pi_0 C_{G^0}(\Lambda')$ is at most c_{G^0}. Applying Lemma A.9 with $H = C_{G^0}(\Lambda')$ and $F = \Lambda/\Lambda'$ yields a bound for $\#\pi_0 C_{G^0}(\Lambda)$ in terms of c_{G^0}, rank G^0 and $\#\pi_0 G$. □

B. A curious application

This section is not relevant to the main aims of this paper, but it follows from the methods developed here, so we will give a (terse) proof.

Theorem B.1. *Let G be a connected reductive group over a field k, and let $\Lambda \subset G(k)$ be a finite subgroup of order n, prime to char k. The only prime numbers dividing the order of $\pi_0 C_G(\Lambda)$ also divide n.*

Proof. We may extend k to assume that G is split, and if k is of positive characteristic, we may then lift Λ to characteristic 0. Using Corollary 3.7, it therefore suffices to assume that char $k = 0$. By spreading out and specializing, we may and do assume that k is a number field. Let p be a prime number not dividing n, and let v be a place of k dividing p. The argument of [GR98, Lemma A.8] shows that after passing to a finite extension of k and conjugating, we may assume $\Lambda \subset \mathscr{G}(\mathcal{O}_v)$, where $\mathscr{G}$ is the split model of G over $\mathbf{Z}$ and $\mathcal{O}_v$ is the ring of integers of the completion of k at v. But now Corollary 3.5 shows that $C_{\mathscr{G}}(\Lambda)$ is weakly reductive, so in particular, p does not divide the order of $\pi_0 C_G(\Lambda)$, as desired. □

C. A Proof of Corollary 1.2

Proof. It is enough to check the third bulleted point in [FKP21, Theorem A]. By Theorem 1.1, for $v \ne p$, $\overline{\rho}|_{\Gamma_{\mathbf{Q}_v}}$ has a p-adic lift. However, by [Lin20b, Theorem C], $\overline{\rho}_p := \overline{\rho}|_{\Gamma_{\mathbf{Q}_p}}$ has a crystalline lift $\rho_p \colon \Gamma_{\mathbf{Q}_p} \to G(\overline{\mathbf{Z}}_p)$ for $p > 3$. We claim that the Hodge–Tate cocharacter of the lift can be chosen to be regular. If $\overline{\rho}_p$ is irreducible, this follows from [Lin20a, Theorem 2]. Otherwise, $\overline{\rho}_p$ factors through a maximal parabolic subgroup P of G with Levi factor $L \cong \mathrm{GL}_2$, and there is a

corresponding representation $\bar{r}_p : \Gamma_{\mathbf{Q}_p} \to L(\overline{\mathbf{F}}_p)$. By [Lin20b, §7.2.1-7.2.2], r_p has a crystalline lift r^o with regular Hodge–Tate cocharacter such that $\phi^{\mathrm{Lie}}(r^o)$ has Hodge–Tate weights slightly less than $\underline{0}$ (in the terminology of loc. cit.). Now the second half of [Lin20b, Theorem C] implies that ρ_p can be chosen such that it factors through P, and its associated L-valued representation lies on the same irreducible component of the spectrum of the crystalline lifting ring that r^o does; in particular, its Hodge–Tate cocharacter is the same as that of r^o. Thus, [FKP21, Theorem A] gives the desired lift of $\overline{\rho}$. □

Remark C.1. The lower bound for p in Corollary 1.2 has to do with the global lifting theorem [FKP21, Theorem A], the local lifting theorem in the $\ell = p$ case [Lin20b, Theorem C], and Theorem 1.1. The bound for [FKP21, Theorem A] can be made explicit; see [FKP21, Remark 6.17]. The bound for [Lin20b, Theorem C] is 3, and the bound for Theorem 1.1 in the G_2 case is 72 (Remark 6.21).

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