dered as forming the original triangle, it follows that four triangles can be obtained each containing three cycles of six circles.

§ 7. Since ZC′S and DB′C are in perspective,

	RS	passes	through	C.
Similarly	ST	,,	,,	A,
and	TR	,,	,,	B.

Since ZC′E and DA′C are in perspective,

	D′E	passes	through	C.
Similarly	F′D	,,	,,	B,
and	E′F	,,	,,	A.

Since ZC is perpendicular to AB, it is bisected perpendicularly by A′B′, and as CD is parallel to ZX, the figure CDZE′ is a rhombus, as are BD′YF and AEXF′.

D′ESR forms a complete quadrilateral two of whose diagonals ZC′ and ZC bisect their corresponding angles and are perpendicular to each other.

Sixth Meeting, April 10, 1891.

J. S. Mackay, Esq., M.A., LL.D., ex-President, in the Chair.

On some properties of a triangle of given shape inscribed in a given triangle.

By R. E. Allardice, M.A.

It is well known that in a given triangle a one-fold infinity of triangles may be inscribed similar to a given triangle This becomes at once obvious on consideration of the converse problem; for we may circumscribe about a given triangle (A), a triangle similar to a second triangle (B), and having its sides parallel to the sides of (B).

We may also show in the following manner that, in a given triangle, one triangle and only one can in general be inscribed having its sides parallel to given directions.

Let D (fig. 16) be a point in the side BC of a triangle ABC; and let DE, EF, FD′, be parallel to the given directions.

Now D and D′ trace out projective ranges on BC; and hence to get the inscribed triangle corresponding to the given directions, we

https://doi.org/10.1017/S0013091500030807 Published online by Cambridge University Press

must make D coincide with a double point of this double range But as D and D′ move in opposite directions along BC and reach infinity together, there is only one double point on this double range.

This way of looking at the problem is of advantage as it suggests a way of extending some of the following theorems to the case of polygons of any number of sides. It should be noted however that in the general case the inscribed polygons will have given angles but will not in general be of given shape.

Another way of looking at the question is to notice that there are only five conditions on the inscribed triangle and that therefore one condition still remains to be satisfied.

Let now (fig. 17), DEF, D′E′F′ be any two triangles of given shape inscribed in ABC; and let corresponding sides of these triangles intersect in the points P, Q, R. We shall show that the triangle PQR is also of constant shape, its angles depending only on those of ABC and DEF.

We have the following relations among the angles in the figures :—

$$\begin{aligned}
&\text{QPR} = \pi - \text{FPQ} - \text{EPR}\,;\\
&\text{FPQ} = \text{BF}'\text{D}' = \pi - \text{B} - \text{BD}'\text{F}'\,;\\
&\text{EPR} = \text{EE}'\text{R} = \pi - \text{C} - \text{CD}'\text{E}'.\\
\therefore\ &\text{QPR} = \pi - (\pi - \text{B} - \text{BD}'\text{F}') - (\pi - \text{C} - \text{CD}'\text{E}')\\
&\qquad = \text{B} + \text{C} + \text{BD}'\text{F}' + \text{CD}'\text{E}' - \pi\\
&\qquad = (\pi - \text{A}) + (\pi - \text{D}) - \pi = \pi - (\text{A} + \text{D}).
\end{aligned}$$

Now if we suppose the triangle DEF to remain fixed, while D′E′F′ varies its position, the various positions of PQR will give a series of triangles of given shape inscribed in DEF. If L, M, N, are the points of intersection of corresponding sides of two of the triangles PQR, the triangle LMN will also be of constant shape. In fact, $\text{L} = \pi - (\text{D} + \text{P}) = \pi - \text{D} - (\pi - \text{A} - \text{D}) = \text{A}$.

That is to say the triangle LMN is similar to the triangle ABC with which we started. We obviously get in the same manner an infinity of triangles of constant shape inscribed in any one of the series LMN, and so on *ad infinitum.*

In other words, we get an infinity of similar triangles DEF inscribed in ABC, an infinity of similar triangles PQR inscribed in any one of the set DEF, and so on; and corresponding angles of these triangles recur in the order A, D, P, A, D ......, where the sum of any three consecutive angles is two right angles.

https://doi.org/10.1017/S0013091500030807 Published online by Cambridge University Press

A similar set of propositions may easily be proved regarding triangles of given shape circumscribing a given triangle.

The following is a proof of the known theorem that the triangle DEF is a minimum (has minimum area and therefore also minimum perimeter) when the perpendiculars to the sides of ABC at the points DEF are concurrent; and also of the theorem that if P, Q, R, are the points of intersection of corresponding sides of the minimum triangle and the consecutive triangle, then the perpendiculars at P, Q, R, to the sides of DEF are concurrent.

Let $\angle AEF = x_1, \ \angle AFE = x_2, \ \angle BFD = y_1$, etc.;

then, by drawing perpendiculars from F and E′ on PF′ and PE respectively, we may easily show that

$$EP \cot x_1 = FP \cot x_2$$

$$\therefore \quad PR \sin x_1 \cot x_1 / \sin E = PQ \sin x_2 \cot x_2 / \sin F$$

$$\therefore \quad \frac{\cos x_1}{\cos x_2} \cdot \frac{\cos y_1}{\cos y_2} \cdot \frac{\cos z_1}{\cos z_2} \cdot = \frac{PQ \sin E}{PR \sin F} \cdot \frac{QR \sin F}{QP \sin D} \cdot \frac{RP \sin D}{RQ \sin E} = 1 ;$$

and hence the perpendiculars at D, E, F, to the sides of ABC are concurrent.

Another proof of this theorem, without the use of infinitesimals, will be given later.

From the fact that the area of DEF is stationary we get,

$$\Delta FPF' + \Delta DQD' + \Delta ERE' - \Delta FQF' - \Delta DRD' - \Delta EPE' = 0,$$

$$\therefore \quad FP^2 + DQ^2 + ER^2 - FQ^2 - DR^2 - EP^2 = 0 ;$$

and hence the perpendiculars at the points P, Q, R, to the sides of DEF are concurrent. (It may be shown that they concur in the same point).

It may easily be shown that the following construction* gives the minimum inscribed triangle of given shape :—

On the sides of the triangle ABC (fig. 18), described externally, triangles BCA′, B′CA, BCA′, of the required shape. The lines AA′, BB′, CC′ concur in a point Q. From Q draw perpendiculars QD′, QE′, QF′ to the sides of ABC; and from A,B,C, draw perpendiculars to the sides of D′E′F′. These perpendiculars concur in a point P such that AP and AQ, BP and BQ, CP and CQ are isogonal lines. [P and Q are frequently called inverse points; but as I shall have

* See a paper by Mr M. Jenkins in the Quarterly Journal of Mathematics, vol. xxi. (1886), p. 84.

occasion to speak of points inverse with reference to a circle, I shall call points such as P and Q reciprocal points. This name is appropriate since the trilinear co-ordinates of Q are the reciprocals of the co-ordinates of P.]

If perpendiculars PD, PE, PF be drawn from P to the sides BC, CA, AB, the triangle DEF will be similar to A′BC, the angles corresponding in this order.

It may easily be shown further that the angles A, D and D′ make up two right angles so that D′E′F′ is similar to the triangle PQR of fig. 17.

By means of the converse of the theorem that if a parabola touches the three sides of a triangle the circumcircle of the triangle passes through the focus, it may easily be proved that the envelopes of the sides of the inscribed triangle of a given shape are three parabolas. (Compare the paper by Mr Jenkins already referred to). For let P (fig. 19) be the point whose pedal triangle has the given shape; then, if PD, PE, PF be drawn so that PEAF, PFBD, PDCE are cyclic quadrilaterals, it may easily be shown that the triangle DEF is of invariable shape. Hence the sides of DEF envelope three parabolas of which P is the common focus.

We can now get the condition that the triangle DEF (fig. 19) be a minimum very simply by means of the propositions just proved. For if R be the (variable) radius of the circle circumscribing AEPF, we have $EF = 2R\sin A$; and therefore, as A is constant, EF is a minimum when R is a minimum, that is, when the circle is described on AP as diameter. Moreover, in this case, EF is the tangent to the parabola at the vertex; and therefore the foot of the perpendicular from P on FE is the vertex of the parabola, and is the point of intersection of two consecutive positions of EF, when EF is a minimum, as was proved above.

It may be noted that the above propositions, taken together, give the following property of the parabola :—

Let any three confocal parabolas be described so that each touches two sides of a triangle ABC. Take any point D in BC and draw a tangent to the first parabola meeting CA in E; draw a tangent from E to the second parabola meeting AB in F; then the tangent from F to the third parabola will pass through D, and the triangle DEF will be of invariable shapo whatever be the position of D. Further, if R, P, Q, be the points of contact of these tangents, the triangle PQR will also be of invariable shape, and the angles will satisfy the relations $A + D + P = B + E + Q = C + F + R =$ two right angles.

A construction was given above for the minimum inscribed triangle of a given shape and for what may be called the point corresponding to the series of triangles of a given shape.

In the course of the construction it came out that, in order to make the problem definite, it must be stated in which of the sides of ABC the different vertices of DEF are to lie. By varying the positions of these vertices in every possible way we get six series of triangles of an assigned shape, with, of course, a definite point corresponding to each series. Further, in that construction, the triangles were described *externally* on the sides of ABC. But if triangles be described *internally* on the sides of ABC another series of similar triangles is obtained with a corresponding point. The relation between the series of inscribed triangles obtained by describing triangles (1) externally and (2) internally on the sides of ABC is that any triangle of the one series is inversely similar to any triangle of the second series. This is a point of some interest as it shows that as regards the problem in hand, two triangles that are inversely similar are perfectly distinct and have to be considered separately. Of two such triangles one cannot of course be transformed into the other by continuous variation.

We thus get twelve series of triangles of an assigned shape with a point corresponding to each series; and the question naturally arises, What is the relation between these twelve points?* In order to answer this question it is necessary to consider some of the properties of the three circles, each of which has for diameter the line joining the points dividing one of the sides internally and externally in the ratio of the other two. [These circles are sometimes known as the circles of Apollonius; and it will be convenient to call them by that name, to avoid circumlocution]. The following are some of the properties of these circles :—

(1) They have two points in common; namely the points the distances of which from the vertices are inversely proportional to the opposite sides. They have therefore a common radical axis.

---

* A useful hint in this connection was supplied by the following question in the *Educational Times* :—10695 (Professor Neuberg.)—Soient P,P′ deux points inverses par rapport à un cercle d'Apollonius du triangle ABC. Démontrer que les triangles podaires de ces points par rapport au triangle ABC sont inversement semblable.

https://doi.org/10.1017/S0013091500030807 Published online by Cambridge University Press

(2) They cut the circumcircle orthogonally; and hence the circumcentre lies on their common radical axis.

This may be proved as follows:—

Let DD′ (fig. 20) be the diameter of one of the circles of Apollonius of the triangle ABC, and let O be the circumcentre.

$$\angle OAB = C - \Pi/2, \quad \angle DAP = \tfrac{1}{2}A + B;$$
$$\therefore \quad \angle OAP = (C - \Pi/2) + (\tfrac{1}{2}A + B) + \tfrac{1}{2}A = \Pi/2.$$

It follows from this that the points of intersection of the circles of Apollonius are inverse with respect to the circumcircle.

(3) One of the circles cuts each of the other two circles at an angle of 60°; and the two latter therefore intersect at an angle of 120° (v. *Lady's and Gentleman's Diary*, 1845, p. 59).

It may be noted that the central axis and the radical axis of the circles of Apollonius are two straight lines at right angles to one another which are symmetrically related to the triangle.

Returning to the question of accounting for the twelve points the pedal triangles of which are similar, we shall begin by proving the following proposition which has been already referred to:—

The pedal triangles of two points that are inverse to one another with respect to one of the circles of Apollonius are inversely similar.

Let P and P′ (fig. 21) be inverse points with reference to the circle of Apollonius passing through A, whose centre is L; and let DEF, D′E′F′, be the pedal triangles of P and P′, D and D′ lying in BC, etc. It may easily be seen from fig. 19 that $\angle D = PBA \pm PCA$ (the sign being taken according as the two angles, read in this way, are traced out in opposite directions or in the same direction).

In the figure we have

$$\begin{aligned} PCB &= \pi - PP'B = AP'P + P'BA + P'AB \\ &= PAL + P'BA + P'AB \\ &= PAC + CBA + P'BA + P'AB \end{aligned}$$
$$\begin{aligned} \therefore \quad PCB + PAB &= (CBA + P'BA) + (PAC + P'AB + PAB) \\ &= P'BC + P'AC; \end{aligned}$$

that is E = F′.

Similarly we may show that F = E′, D = D′.

Hence when we take points inverse with respect to the circle passing through A, the vertices lying on BC are corresponding vertices, but those lying on CA and AB are not.

If therefore we start with any point, take its inverse with reference to each of the circles of Apollonius, take the inverses of these points again, and so on as long as we get any new points, the pedal triangles of all the points we get in this way must be similar. An upper limit to the number of points that may be so obtained, is given by the fact proved above that there are only twelve such triangles; and it might be shown *à priori* without much difficulty that more than six points cannot be obtained in this way. In fact it is obvious that the two points whose pedal triangles are inversely similar, but have corresponding vertices on the same sides of the triangle ABC, cannot be derived from one another by this method.

We shall now show that there are in general six points and only six that may be derived from one another by repeated inversion with reference to the circles of Apollonius.

Given two circles (fig. 22) with centres A and B; let B′ be the inverse of the point B with reference to the circle A and A′ be the inverse of the point A with reference to the circle B. To find the condition that the inverses of B′ with reference to the circle B and of A′ with reference to the circle A coincide (in a point T, say).

Let the circles intersect in P.

Since B and B′ are inverse points with reference to A, $APB' = PBA$; similarly $BPA' = PAB$, and $TPB = PB'B = PAB' + APB'$. $\therefore$ $A'PT = B'PA$ and $B'PT = A'PB$.

Hence $APT = BPT = PAB + PBA = 60°$; and $APB = 120°$.

In fig. 23 the point T does not lie between A and B and APB is 60°.

The condition of this theorem is satisfied by the circles of Apollonius, as these intersect at angles of 60° and 120°. In this case the point T is the centre of the third circle.

Take now (fig. 24) two circles intersecting at 120° (or at 60°); and let P be any point.

Let Q be the inverse of P with reference to the circle A,
„ R „ „ „ P „ „ „ „ B
„ T „ „ „ Q „ „ „ „ B,
„ S „ „ „ R „ „ „ „ A

Let AT and BS intersect in O; to show that O is the inverse of T with reference to the circle A and also of S with reference to the circle B.

The points PQSR and the points PQTR are concyclic; and hence the five points PQSTR are concyclic.

Let now B′ be the inverse of B with reference to the circle A
„ „ A′ „ „ „ A „ „ „ „ B
„ „ U „ „ „ B′ „ „ „ „ B
and also U „ „ „ A′ „ „ „ „ A.

The four points BB′QP are concyclic. Invert the circle on which these points lie, with reference to the circle B. The straight line which is the inverse of this circle passes through U, T and R; and hence the straight line RT passes through U. Similarly QS passes through U.

Hence by the properties of Pascal's Hexagram O must lie on the circle passing through P,Q,S,T,R.

No new point can be got by inverting any of these six points with reference to any of the circles of Apollonius.

Hence we see that six of the twelve points the pedal triangles of which are similar, are got by inverting any one of the points, in every possible way, with reference to the circles of Apollonius, and that the six points obtained in this way all lie on the circumference of a circle which cuts the circles of Apollonius orthogonally.

We shall next proceed to account for the other six points and shall begin by proving the following theorem:—

If P and P′ (fig. 25) are inverse points with reference to the circumcircle of a triangle ABC, then the pedal triangles of P and P′ are inversely similar and have corresponding vertices on the same sides of ABC.

Let O be the circumcentre; DEF, D′E′F′, the pedal triangles of P and P′.

Then

$$\angle E = \angle PAB + \angle PCB = \angle PAO + \angle PCO + \angle B = \angle AP'C + \angle B;$$

and

$$\angle E' = \angle P'AF' + \angle P'CD' = \angle AP'C + \angle B = \angle E.$$

Hence the remaining six of the twelve points under consideration are got by inverting the six already obtained, with reference to the circumcircle. The new set of six points will of course also lie on a circle cutting the circles of Apollonius orthogonally; and the two circles on which the two sets of six points lie are inverse to one another with reference to the circumcircle. It may be noted that all the six points belonging to one set are obtained, according to the construction given above, by constructing triangles of the required shape on the sides of ABC *externally;* while those belonging to the other set are obtained by constructing the triangles on the sides of ABC *internally.*

It may be of interest now to consider the positions of these twelve points in certain particular cases.

In the first instance, let the pedal triangle of any one of the twelve points be similar to the given triangle. In the set obtained by constructing the triangles externally one of the points is the circumcentre and two others are the Brocard points; and hence the circle on which the six points lie is the Brocard circle. The other three points lie on the symmedians and are therefore the points where the symmedians meet the Brocard circle. [It is known that the symmedian point lies on the Brocard circle; but its pedal triangle is not similar to ABC.] As a consequence of previous theorems, these six points are all inverses of one another with reference to the circles of Apollonius; and the Brocard circle cuts the circle of Apollonius orthogonally.

Considering now the other set of six points we may easily show that three of them are the centres of the circles of Apollonius and hence the circle on which these six points lie becomes in this case *a straight line, namely the central axis of the circles of Apollonius.* [This may easily be seen otherwise, as follows. Since the circle on which the six points lie is the inverse of the Brocard circle with reference to the circumcircle, the centre of which lies on the Brocard circle, it must be a straight line; and the only straight line that cuts the circles of Apollonius orthogonally is their central axis.]

One of the three remaining points is at infinity, being the inverse of the circumcentre; and the other two are the inverses of the centres of two of the circles of Apollonius with reference to the third circle. These two points are the inverses of the Brocard points with reference to the circumcircle. It may easily be shown that the central axis of the circles of Apollonius is the radical axis of the circumcircle and the Brocard circle.

A few other results may be stated without proof. The demonstrations will be found to be in all cases very simple.

Let P be any point in the plane and DEF its pedal triangle, D lying in BC, E in CA and F in AB. We may impose any conditions (as regards shape) on DEF and look for the corresponding locus of P.

If the ratio DE : DF is constant $(=l : m)$ the locus of P is a circle on QR as diameter where Q and R are points dividing BC internally and externally in the ratio $mc : lb$. It should be noted that the twelve points which have similar pedal triangles will not all lie on this circle.

If DEF is isosceles (with DE = DF) the locus of P is the circle of Apollonius passing through A. Since all the points whose pedal triangles are isosceles lie in one or other of the circles of Apollonius, it is evident that all the six points that give isosceles triangles of a given shape lie on the circles of Apollonius, two on each. [The twelve points of the general case reduce to six when the pedal triangle is isosceles.] From this it may easily be seen that of the three circles of Apollonius any two are inverse to one another with reference to the third. If further any point on one of the circles of Apollonius be inverted in every possible way with reference to these circles, only three points in all can be obtained ; and these together with their inverses with reference to the circumcircle are the six points that have pedal triangles isosceles and of given shape.

As a particular case of the above, the points whose pedal triangles are equiangular must lie on all the circles of Apollonius and must be one or other of the two points common to these three circles. This is a well-known property of these two points.

If in the triangle DEF the angle D is constant, the locus of P will be a circle passing through B and C. If we make D equal to two right angles, the angles E and F will also be constant, each being zero, and the locus of P will be the circumcircle. Hence we get the property of the pedal line (or so-called Simson line) as a particular case of the above theory. A flat triangle is the only triangle the angles of which may be assigned without determining the shape of the triangle The general method given above for determining the twelve points whose pedal triangles have a given shape may be applied even in the case where the triangles are flat ones. The twelve points reduce in this case to six.

A particular case of this again that may be specially considered is when the flat triangle is isosceles. There are three points on the circumcircle the pedal lines of which may be considered as flat isosceles triangles. The above method applied to the construction of these points shows that they are the points where the symmedians meet the circumcircle. Since their pedal triangles are isosceles these points must lie on the circles of Apollonius ; hence we get the theorem that the symmedians are the common chords of the circumcircle and the circles of Apollonius.