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Symmetric Tessellations on Euclidean Space-Forms

With best wishes to H.S.M. Coxeter for his 90th birthday.

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Abstract. It is shown here that, for $n \geqslant 2$, the n-torus is the only n-dimensional compact euclidean space-form which can admit a regular or chiral tessellation. Further, such a tessellation can only be chiral if $n = 2$.

1 Introduction

The study of regular (reflexible) or chiral (irreflexible) maps on closed surfaces is a classical branch of topology which has seen many applications (Coxeter & Moser [3]). Such maps on orientable surfaces of genus $g \leqslant 6$ have been completely enumerated (Sherk [18], Garbe [7]). It is well-known that there are infinitely many regular or chiral maps on the 2-torus, but that an orientable surface of genus $g \geqslant 2$ can only admit finitely many such maps. Each regular or chiral map on a non-orientable surface is doubly covered by a map of the same kind on an orientable surface (Wilson [19]). However, in contrast to the 2-torus, the only non-orientable surface of Euler characteristic zero, the Klein bottle, does not admit any regular or chiral tessellation [3].

In this paper, we shall investigate tessellations on n-dimensional euclidean space-forms. We shall prove that the n-torus is the only compact euclidean space-form which can admit a regular or chiral tessellation, and that chirality can only occur if $n = 2$. For $n = 2$, this gives another proof that such tessellations cannot exist on the Klein bottle.

For $n \geqslant 3$, the regular toroids of rank $n + 1$, that is, the regular tessellations on the n-torus, were completely enumerated in [15]; see [3] for the case $n = 2$. It was also proved there that there are no chiral toroids of rank greater than 3; that is, an n-torus can admit a chiral tessellation only if $n = 2$. Together with the results of the present paper, this now completes the classification of all regular or chiral tessellations on compact euclidean space-forms.

In Section 2, we recall some basic facts about polytopes and tessellations, and discuss sparse subgroups. In Section 3, we give a detailed proof of our result for those tessellations which are regular. Chiral tessellations are then treated in Section 4.

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2 Polytopes and Tessellations

Following [15], [16], an (*abstract*) *polytope of rank* n, or simply an n-*polytope*, satisfies the following properties. It is a partially ordered set $\mathcal{P}$ with a strictly monotone *rank* function whose range is $\{-1, 0, \dots, n\}$. The elements of rank j are called the j-*faces* of $\mathcal{P}$; the set of j-faces of $\mathcal{P}$ is denoted $\mathcal{P}_j$. For $j = 0$, 1 or $n-1$, we also refer to j-faces as *vertices*, *edges* or *facets*, respectively. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ each contain exactly $n+2$ faces, including the unique minimal face F_{-1} and unique maximal face F_n of $\mathcal{P}$. Further, $\mathcal{P}$ is *strongly flag-connected*, meaning that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$, which are such that Φ_{i-1} and Φ_i are *adjacent* (differ by one face), and such that $\Phi \cap \Psi \subseteq \Phi_i$ for each $i = 1, \dots, k$. Finally, if F and G are a $(j-1)$-face and a $(j+1)$-face with $F < G$, then there are exactly *two* j-faces H such that $F < H < G$.

When F and G are two faces of a polytope $\mathcal{P}$ such that $F \leqslant G$, we call $G/F := \{H \mid F \leqslant H \leqslant G\}$ a *section* of $\mathcal{P}$. The conditions ensure that this section is itself a polytope, whose rank is $\dim G - \dim F - 1$. It is usually safe to identify a face F with the section F/F_{-1}. When F is a vertex, then the section F_n/F is called the *vertex-figure of* $\mathcal{P}$ *at* F.

An n-polytope $\mathcal{P}$ is *regular* if its (*automorphism*) *group* $\Gamma(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_{-1}, F_0, \dots, F_{n-1}, F_n\}$ be a fixed or *base* flag of $\mathcal{P}$. The group $\Gamma(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \dots, \rho_{n-1}$ (*with respect to* Φ), where ρ_j is the unique automorphism which keeps all but the j-face of Φ fixed. These generators satisfy relations

$$(2.1) \qquad (\rho_i\rho_j)^{p_{ij}} = \varepsilon \quad (i, j = 0, \dots, n-1),$$

with

$$(2.2) \qquad p_{ii} = 1, \quad p_{ij} = p_{ji} \geqslant 2 \ (i \neq j), \quad p_{ij} = 2 \ (|i-j| \geqslant 2).$$

The numbers $p_j := p_{j-1,j}$ $(j = 1, \dots, n-1)$ determine the (*Schläfli*) *type* $\{p_1, \dots, p_{n-1}\}$ of $\mathcal{P}$. Further, $\Gamma(\mathcal{P})$ has the *intersection property* (with respect to the distinguished generators), namely

$$(2.3) \qquad \langle \rho_i \mid i \in I\rangle \cap \langle \rho_i \mid i \in J\rangle = \langle \rho_i \mid i \in I \cap J\rangle \quad \text{for all } I, J \subset \{0, \dots, n-1\}.$$

Observe that, in a natural way, the group of the facet of $\mathcal{P}$ is $\langle \rho_0, \dots, \rho_{n-2}\rangle$, while that of the vertex-figure is $\langle \rho_1, \dots, \rho_{n-1}\rangle$.

By a *string C-group*, we mean a group which is generated by involutions such that (2.1), (2.2) and (2.3) hold. The group of a regular polytope is a string C-group. Conversely, given a string C-group Γ, there is an associated regular polytope $\mathcal{P}(\Gamma)$ whose automorphism group is Γ.

We denote by $\{p_1, \dots, p_{n-1}\}$ the (universal) regular n-polytope whose group is the Coxeter group $[p_1, \dots, p_{n-1}]$ which is abstractly defined by the relations (2.1) and (2.2).

An n-polytope $\mathcal{P}$ is called (*globally*) *spherical* if it is isomorphic to the face-lattice of a convex n-polytope. (We shall ignore here the rather less interesting case where the Schläfli symbol has an entry 2, when the group $\Gamma(\mathcal{P})$ is an internal direct product.) Then each facet and vertex-figure, and, more generally, proper section of $\mathcal{P}$ is again a spherical polytope. If a spherical polytope is regular, then it is isomorphic to a convex regular polytope ([12]). Further, we say that a polytope $\mathcal{P}$ is *locally spherical* if all its proper sections are spherical polytopes, or equivalently, if all its facets and vertex-figures are spherical. However, we do not require here that $\mathcal{P}$ itself be spherical. Thus, if $\mathcal{P}$ is regular, then its facets and vertex-figures are isomorphic to convex regular polytopes.

Let $\mathcal{P}$ be a regular n-polytope with group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and let $N \leqslant \Gamma(\mathcal{P})$ be any subgroup. We write $\mathcal{P}/N$ for the poset whose elements are the orbits of the faces of $\mathcal{P}$ under N (with the induced partial order); this is the *quotient* of $\mathcal{P}$ by N. Under suitable conditions on N, this is again an n-polytope [14]. We are interested here in quotients which preserve the facets and vertex-figures of $\mathcal{P}$, so that the quotient map acts in a "global" rather than "local" fashion; this property is assured using an interesting class of subgroups N known as sparse (compare Lemma 2.6 below). The term "sparse" was introduced in [10], but the groups themselves occur earlier, for example in [4], [11], [14] (we are indebted to Wolfgang Kühnel and Jörg Wills for drawing our attention to the first two of these references).

A subgroup N of $\Gamma(\mathcal{P})$ is called *sparse* if

$$\varphi N \varphi^{-1} \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{n-2} \rangle = \{\varepsilon\} \quad \text{for each } \varphi \in \Gamma(\mathcal{P}). \tag{2.4}$$

We shall establish three lemmas about sparse subgroups; our main results do not actually need them in full generality. The first describes a simple characterization in terms of the action of N on $\mathcal{P}$, and gives a combinatorial interpretation of sparseness.

Lemma 2.5 *Let $\mathcal{P}$ be a regular polytope, and let $N \leqslant \Gamma(\mathcal{P})$. Then N is sparse if and only if each orbit of N meets each proper section of $\mathcal{P}$ in at most one face.*

Proof Let $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$. First assume that the condition on the orbits holds. Let $\varphi \in \Gamma(\mathcal{P})$, and let $\tau \in N$ be such that $\varphi\tau\varphi^{-1} \in \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{n-2} \rangle$. Then $F_0\varphi\tau \leqslant F_{n-1}\varphi$ in $\mathcal{P}$. Since $F_0\varphi$ is the only element in its orbit which is a vertex of the facet $F_{n-1}\varphi/F_{-1}$ of $\mathcal{P}$, we must have $F_0\varphi\tau = F_0\varphi$. It follows that τ maps the whole vertex-figure $F_n/F_0\varphi$ of $\mathcal{P}$ onto itself. Since an orbit meets this vertex-figure in at most one face, τ must fix each face of $F_n/F_0\varphi$. But then $\tau = \varepsilon$, as required.

It is sufficient to prove the converse for facets and vertex-figures, because every proper section of $\mathcal{P}$ is a section of a facet or a vertex-figure. Taking duality into account, we need only consider the case of a facet $F_{n-1}\varphi/F_{-1}$, with $\varphi \in \Gamma(\mathcal{P})$. An i-face of this facet has the form $F_i\alpha\varphi$, for some $\alpha \in \langle \rho_0, \ldots, \rho_{n-2} \rangle$. Now, if an orbit of N meets $F_{n-1}\varphi/F_{-1}$ in two i-faces, then there are $\alpha, \beta \in \langle \rho_0, \ldots, \rho_{n-2} \rangle$ and $\tau \in N$ such that $F_i\alpha\varphi = F_i\beta\varphi\tau$. If necessary, we replace φ by $\beta\varphi$ and α by $\alpha\beta^{-1}$, and then we may assume that $\beta = \varepsilon$. Thus $F_i\alpha\varphi = F_i\varphi\tau$, and hence

$$\varphi\tau\varphi^{-1}\alpha^{-1} \in \langle \rho_j \mid j \neq i \rangle = \langle \rho_j \mid j > i \rangle \langle \rho_j \mid j < i \rangle \subseteq \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{n-2} \rangle.$$

But $\alpha \in \langle \rho_0, \ldots, \rho_{n-2} \rangle$, and so we also have $\varphi\tau\varphi^{-1} \in \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{n-2} \rangle$. Since N is sparse, this implies that $\varphi\tau\varphi^{-1} = \varepsilon$, so that $\tau = \varepsilon$. It follows that the two faces in the same orbit must actually coincide. ∎

We are particularly interested in the case when $N \lhd \Gamma(\mathcal{P})$, since the quotient is then a candidate to be a regular polytope.

Lemma 2.6 *Let $\mathcal{P}$ be a regular n-polytope with group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and let N be a normal subgroup of $\Gamma(\mathcal{P})$ such that $\Gamma(\mathcal{P})/N$ is a string C-group. Then N is sparse if and only if*

$$N \cap \langle \rho_0, \ldots, \rho_{n-2} \rangle = \{\varepsilon\} = N \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle.$$

Proof If N is sparse then, since $\varphi N \varphi^{-1} = N$ for each $\varphi \in \Gamma(\mathcal{P})$, the claimed property obviously holds. For the converse, let $\tau \in N \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{n-2} \rangle$, say $\tau = \alpha\beta^{-1}$ with $\alpha \in \langle \rho_1, \ldots, \rho_{n-1} \rangle$ and $\beta \in \langle \rho_0, \ldots, \rho_{n-2} \rangle$. Since $\Gamma(\mathcal{P})/N$ is a string C-group, from (2.3) we have

$$N\alpha = N\beta \in \langle N\rho_1, \ldots, N\rho_{n-1} \rangle \cap \langle N\rho_0, \ldots, N\rho_{n-2} \rangle = \langle N\rho_1, \ldots, N\rho_{n-2} \rangle,$$

so that $N\alpha = N\beta = N\gamma$ with $\gamma \in \langle \rho_1, \ldots, \rho_{n-2} \rangle$. But then $\alpha\gamma^{-1} \in N \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle$, and hence $\alpha = \gamma$. Similarly, $\beta = \gamma$, and therefore $\tau = \alpha\beta^{-1} = \varepsilon$. ∎

Lemma 2.7 *Let $\mathcal{P}$ be a locally spherical regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, and let $\mathcal{T} := \{p_1, \ldots, p_{n-1}\}$. Then $\mathcal{P} = \mathcal{T}/N$, where N is a sparse normal subgroup of $\Gamma(\mathcal{T})$.*

Proof Let $\Gamma(\mathcal{T}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ $(= [p_1, \ldots, p_{n-1}])$ and $\Gamma(\mathcal{P}) = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$, with respect to appropriate distinguished generators. Then the mappings $\rho_i \mapsto \sigma_i$ $(i = 0, \ldots, n-1)$ induce a homomorphism $\kappa\colon \Gamma(\mathcal{T}) \to \Gamma(\mathcal{P})$. Let $N := \ker(\kappa)$, so that $\Gamma(\mathcal{P}) = \Gamma(\mathcal{T})/N$. Then we know that $\Gamma(\mathcal{T})/N$ is a string C-group. Since the facets and vertex-figures of $\mathcal{T}$ and $\mathcal{P}$ are of the same kind (recall the definition of "locally spherical"; indeed, this is all what we really need of the assumptions), we must also have

$$N \cap \langle \rho_0, \ldots, \rho_{n-2} \rangle = \{\varepsilon\} = N \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle.$$

Lemma 2.6 then implies that N is sparse. Clearly we also have $\mathcal{P} = \mathcal{T}/N$ (see [14]). ∎

We next discuss tessellations on real manifolds (see [16]). We shall only consider tessellations whose tiles are homeomorphic images of convex polytopes, and which thus come equipped with a natural face structure.

Let X be any n-dimensional real manifold; we shall always assume here that manifolds are without boundary. A family $\mathcal{P}$ of subsets of X (including $\varnothing$ and X itself) is called a (*locally finite*) *tessellation* in X if the following three conditions are satisfied. First, for each proper subset $F \in \mathcal{P}$ there exist a convex polytope F' and a homeomorphism $\gamma\colon F \to F'$ such that $G\gamma^{-1} \in \mathcal{P}$ for each face G of F'. The subsets in $\mathcal{P}$ are called the *faces* of $\mathcal{P}$, and the subsets $G\gamma^{-1}$ of F the *faces* of F. In particular, F is a j-*face* of $\mathcal{P}$ if F' is a j-polytope, and $G\gamma^{-1}$ is a j-*face* of F if G is a j-face of F'. The n-faces of $\mathcal{P}$ are also called the *tiles* or *facets* of $\mathcal{P}$. Second, if $F_1, F_2 \in \mathcal{P}$, then $F_1 \cap F_2 \in \mathcal{P}$ also (possibly this is $\varnothing$). Third, each point in X is contained in a tile of $\mathcal{P}$, and has a neighbourhood which meets only finitely many tiles (this last is what is meant by local finiteness). In other words, X is the underlying polyhedron (in the topological sense) of a possibly infinite cell-complex.

We shall usually identify a tessellation $\mathcal{P}$ with the poset consisting of its faces ordered by inclusion. This context explains why it was convenient to adjoin to $\mathcal{P}$ the underlying manifold X as an (improper) $(n+1)$-face. It is then straightforward to check that $\mathcal{P}$ becomes

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an abstract $(n+1)$-polytope. (In the terminology of [1], the manifold X is then *associated* with the polytope $\mathcal{P}$.) A tessellation $\mathcal{P}$ on X is called (*combinatorially*) *regular* if, as an abstract polytope, $\mathcal{P}$ is regular.

In our applications, X will be a compact euclidean space-form. Recall that an n-dimensional *euclidean space-form* is the quotient (or orbit) space $\mathbb{E}^n/N$ of euclidean n-space $\mathbb{E}^n$ by a discrete group N of euclidean isometries which acts freely on $\mathbb{E}^n$. Spherical or hyperbolic space-forms are defined similarly, with $\mathbb{E}^n$ replaced by the unit n-sphere $\mathbb{S}^n$ or hyperbolic n-space $\mathbb{H}^n$, respectively [20].

There is a close connexion between the classes of locally spherical polytopes and of tessellations on space-forms [4], [11], [16].

Theorem 2.8 *Let $\mathcal{P}$ be a locally spherical regular $(n+1)$-polytope of type $\{p_1,\dots,p_n\}$.*

(a) Combinatorially, $\mathcal{P}$ is a quotient $\mathcal{T}/N$ of the regular tessellation $\mathcal{T}=\{p_1,\dots,p_n\}$ in spherical, euclidean or hyperbolic n-space E by a (sparse) normal subgroup N of $\Gamma(\mathcal{T})$, which, when considered as a group of isometries of E, is discrete and acts freely on E.

(b) Topologically, $\mathcal{P}$ can be viewed as a regular tessellation on the corresponding spherical, euclidean or hyperbolic space-form E/N (whose fundamental group is isomorphic to N). This tessellation is regular, in the strong sense that its symmetry group in E/N is simply flag-transitive and isomorphic to $\Gamma(\mathcal{P})$.

Theorem 2.9 *Let $\mathcal{P}$ be a regular tessellation on an n-dimensional real manifold X (without boundary). Then X is homeomorphic to a space-form X', and $\mathcal{P}$ can be viewed as a tessellation on X'. More precisely:*

(a) if $\mathcal{P}$ has only two tiles, then $X'=\mathbb{S}^n$ and $\mathcal{P}$ is a ditope (polytope with only two facets) on $\mathbb{S}^n$;

(b) if $\mathcal{P}$ has more than two tiles, then $\mathcal{P}$ is a locally spherical regular $(n+1)$-polytope (to which Theorem 2.8 applies).

In either case, X is homeomorphic to $|\mathcal{C}(\mathcal{P})|$, the underlying space of the order complex $\mathcal{C}(\mathcal{P})$ of $\mathcal{P}$.

Recall that $\mathcal{C}(\mathcal{P})$ is the simplicial n-complex whose simplices are the totally ordered subsets of $\mathcal{P}$ which do not contain the minimal or maximal face.

By Theorem 2.9, a regular tessellation on a space-form X determines X up to homeomorphism. It is a standard result in topology that, for compact space-forms, the type of geometry is uniquely determined by the topology. (Spherical space-forms are distinguished topologically from euclidean and hyperbolic space-forms by the finiteness of their fundamental groups. The Gromov norm for manifolds distinguishes topologically between euclidean and hyperbolic space-forms (see [8]); it takes a positive value for hyperbolic space-forms of finite volume, but it is zero for all euclidean space-forms.)

Lemma 2.10 *Let $n\geqslant 2$, and let $\mathcal{P}$ be a regular tessellation of type $\{p_1,\dots,p_n\}$ on an n-dimensional compact space-form X. Let $\mathcal{T}:=\{p_1,\dots,p_n\}$, the universal polytope of which $\mathcal{P}$ is a quotient. Then the geometry of X determines the geometry of the space E of $\mathcal{T}$; that is, $E=\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ according as X is a spherical, euclidean or hyperbolic space-form.*

In fact, if $\mathcal{P} = \mathcal{T}/N$, we may view $\mathcal{P}$ as a tessellation on the space-form E/N which is homeomorphic to X. Hence the two space-forms must have the same type of geometry. For example, if X is euclidean, then $\mathcal{T}$ must be a euclidean tessellation.

3 Regular Tessellations

We prefer to work with $(n+1)$-polytopes here, because it is natural to take the underlying space-forms to be n-dimensional. We begin with a result about sparse subgroups, which is the key to our non-existence result for regular or chiral tessellations. Its proof is geometric. For the cubical tessellation $\{4, 3^{n-2}, 4\}$, this result was also obtained in [10], with a more algebraic proof. We shall also find it convenient to assume that a regular tessellation $\mathcal{T}$ in $\mathbb{E}^n$ is strongly regular in the sense of Theorem 2.8(b), so that its euclidean symmetry group acts flag-transitively. We can then identify this symmetry group with the automorphism group $\Gamma(\mathcal{T})$, and write $\Gamma^+(\mathcal{T})$ for its subgroup (of index 2) consisting of all orientation preserving symmetries; the latter is the *rotation subgroup* of $\Gamma(\mathcal{T})$.

Theorem 3.1 *Let $\mathcal{T}$ be a regular tessellation in euclidean n-space $\mathbb{E}^n$, and let Λ be the translation subgroup of its symmetry group $\Gamma(\mathcal{T})$. Then a sparse subgroup N of $\Gamma(\mathcal{T})$ whose normalizer contains $\Gamma^+(\mathcal{T})$ is a subgroup of Λ.*

Proof First observe that, if necessary, we may replace the given regular tessellation $\mathcal{T}$ in $\mathbb{E}^n$ by its dual. In fact, dual tessellations have the same translation subgroups and rotation subgroups, and the concept of sparseness is also invariant under duality. In particular, of a pair of dual tessellations, we shall take $\mathcal{T}$ to be the one with a vertex-transitive translation subgroup Λ (see [2] for the reason why such a choice can always be made). We may then identify the vertex-set $\mathcal{T}_0$ with the translation vectors in Λ whenever it is convenient, and so write $\mathcal{T}_0 = \Lambda$; in particular, the base vertex of $\mathcal{T}$ is the zero vector o. More precisely, $\mathcal{T} = \{4, 3^{n-2}, 4\}$ (for $n \geqslant 2$), $\{\infty\}$, $\{3, 6\}$ or $\{3, 3, 4, 3\}$, as appropriate.

Let $\Gamma(\mathcal{T}) = \langle \rho_0, \dots, \rho_n \rangle$, so that $\Gamma_0 := \langle \rho_1, \dots, \rho_n \rangle$ is the stabilizer of the base vertex o of $\mathcal{T}$. The corresponding rotation subgroups are thus $\Gamma^+(\mathcal{T}) = \langle \rho_0\rho_1, \rho_1\rho_2, \dots, \rho_{n-1}\rho_n \rangle$ and $\Gamma_0^+ = \langle \rho_1\rho_2, \dots, \rho_{n-1}\rho_n \rangle$. For $t \in \mathbb{E}^n$, we write $\tau(t)$ for the translation defined by $x\tau(t) := x + t$ for $x \in \mathbb{E}^n$.

If $\varphi \in \Gamma(\mathcal{T})$, then for $x \in \mathbb{E}^n$ we have $x\varphi = x\omega + t$, with $t \in \mathbb{E}^n$ and ω a linear mapping; in fact, $t = o\varphi$, which is thus a vertex of $\mathcal{T}$, and $\omega = \varphi\tau(-t)$. Since Λ is vertex-transitive, we also know that $\tau(t) \in \Lambda$, and therefore $\omega \in \Gamma(\mathcal{T})$. Then $\omega \in \Gamma_0$, because $o\omega = o$. This proves that $\Gamma(\mathcal{T})$ is the semi-direct product of Λ by Γ_0.

Now let N be a sparse subgroup of $\Gamma(\mathcal{T})$ which is normalized by $\Gamma^+(\mathcal{T})$. Let $\varphi \in N$, and for $x \in \mathbb{E}^n$ let $x\varphi = x\omega + t$, as above. Note that φ^{-1} is given by $x\varphi^{-1} = (x - t)\omega^{-1}$ for $x \in \mathbb{E}^n$. We claim that $\omega = \varepsilon$, the identity mapping on $\mathbb{E}^n$, and therefore that $\varphi = \tau(t) \in \Lambda$.

Assume, if possible, that $\omega \neq \varepsilon$. We define

$$\mathcal{D} := \{F \in \mathcal{T}_n \mid o \in F\},$$

the set of facets of $\mathcal{T}$ which contain the initial vertex o. Our strategy is to prove that, if $\Lambda' := \Lambda \cap N$ denotes the subgroup consisting of the translations in N, then $\mathcal{D} + \Lambda'$ covers

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the vertex-set $\mathcal{T}_0 = \Lambda$. We then show that $\mathcal{D}$ contains an image of o under an element $\varphi \in N \setminus \{\varepsilon\}$, which will contradict the assumption that N is sparse.

So, let v be any neighbouring vertex of o in $\mathcal{T}$. By our assumption on Λ, we know that $\tau(v) \in \Lambda \leqslant \Gamma^+(\mathcal{T})$. Set $\psi(v) := \tau(-v)\varphi^{-1}\tau(v)\varphi$. Writing this as $\psi(v) = \tau(v)^{-1}\varphi^{-1}\tau(v) \cdot \varphi$, which is a product of elements of N because N is normalized by $\Gamma^+(\mathcal{T})$, we see that $\psi(v) \in N$. For $x \in \mathbb{E}^n$, we have

$$\begin{aligned} x\psi(v) &= x\tau(-v)\varphi^{-1}\tau(v)\varphi = (x - v - t)\omega^{-1}\tau(v)\varphi \\ &= \big((x - v - t)\omega^{-1} + v\big)\omega + t \\ &= x + (v\omega - v). \end{aligned}$$

Hence $\psi(v) = \tau(v\omega - v)$. Since $\omega \neq \varepsilon$, there is a neighbour v of o such that $v\omega \neq v$. We define $w := v\omega - v$; then $o\psi(v) = w$, and hence $w \in \mathcal{T}_0$.

Since $v\omega$ is also a neighbour of o and $\mathcal{T}$ is centrally symmetric, w is a neighbour of $-v$. It follows that N contains a translation, namely $\tau(w)$, which maps o onto some *second neighbour*, by which we mean a neighbour (different from o) of one of its neighbours.

Now N is normalized by $\Gamma^+(\mathcal{T})$, and so it is invariant under conjugation by the elements in the subgroup Γ_0^+. It follows that $\tau(w\sigma) = \sigma^{-1}\tau(w)\sigma \in N$ for all $\sigma \in \Gamma_0^+$, so that, by definition, $\tau(w\sigma) \in \Lambda'$ $(= \Lambda \cap N$, as defined above). We perform a similar analysis to that used in [15] in the construction of the regular toroids of rank $n+1$, working through the individual cases. As in [15], we define $\Lambda_a \leqslant \Lambda$ to be the subgroup generated by a vector $a \in \Lambda$ and its conjugates under $\Gamma(\mathcal{T})$ (or under Γ_0).

First let $\mathcal{T} = \{4, 3^{n-2}, 4\}$, with vertex-set $\mathcal{T}_0 = \Lambda = \mathbb{Z}^n$. The second neighbours of o are $\pm e_i \pm e_j$ (with $1 \leqslant i < j \leqslant n$) and $\pm 2e_i$ (with $i = 1, \dots, n$), where $e_1, \dots, e_n$ are the standard unit vectors in $\mathbb{Z}^n$. Since the generating vector w of $\Lambda_w \leqslant \Lambda'$ is such a second neighbour, then we can argue as in [15] that $\Lambda' \geqslant \Lambda_{(1,1,0^{n-2})}$ or $\Lambda_{(2,0^{n-1})}$ (note that the case $n = 2$ is not exceptional, even though rotations only may be employed). In either case, $\mathcal{D} + \Lambda'$ covers $\mathcal{T}_0$.

Next let $\mathcal{T} = \{3,3,4,3\}$, with vertex set $\Lambda = \mathbb{Z}^4 \cup \big((\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) + \mathbb{Z}^4\big)$. The second neighbours of o now consist of the neighbours themselves, comprising the vectors $\pm e_i$ (with $i = 1, \dots, 4$) and $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$, the vectors obtained from $(\pm 1, \pm 1, 0, 0)$, $(\pm 1, \pm 1, \pm 1, 0)$ or $(\pm\frac{3}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ by permutations of the coordinates, and $2v$ for each neighbour v. Again arguing as in [15], we see that $\Lambda' \geqslant \Lambda_{(1,0,0,0)}, \Lambda_{(1,1,0,0)}$ or $\Lambda_{(2,0,0,0)}$, as the generating vector w is a neighbour, one of the vectors $(\pm 1, \pm 1, 0, 0)$, $(\pm 1, \pm 1, \pm 1, 0)$ or $(\pm\frac{3}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$, or twice a neighbour. (For the second case, note the typical calculations $(0,0,1,1) = (1,1,1,0) - (1,1,0,-1)$ and $(0,0,1,1) = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) - (\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$, which only use rotations. If we can employ the full symmetry, we actually obtain $\Lambda_a = \Lambda_{(1,0,0,0)}$ $(= \Lambda)$ when $a = (\pm 1, \pm 1, \pm 1, 0)$ or $(\pm\frac{3}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$.) As before, in each case $\mathcal{D} + \Lambda'$ covers $\mathcal{T}_0$.

For $\mathcal{T} = \{\infty\}$ with vertex set $\mathbb{Z}$, the second neighbours are $\pm 2e_1$, and trivially $\mathcal{D} + \Lambda'$ covers $\mathcal{T}_0$.

If $\mathcal{T} = \{3,6\}$, then Λ is generated by two unit vectors v_1 and v_2 which are inclined at an angle $\pi/3$. The second neighbours then comprise the neighbours, the six images under rotation through multiples of $\pi/3$ of $v_1 + v_2$, and twice the neighbours. Then we have

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$\Lambda' \geqslant \Lambda_{(1,0)}, \Lambda_{(1,1)}$ or $\Lambda_{(2,0)}$, and once again $\mathcal{D} + \Lambda'$ covers $\mathcal{T}_0$. (Here, a suffix $a = (a_1, a_2)$ is shorthand for $a = a_1 v_1 + a_2 v_2$.)

We are now able to finish the proof. Consider the translation vector $t \in \Lambda$ of the element $\varphi \in N$ with which we began. Since $\mathcal{D} + \Lambda'$ covers $\mathcal{T}_0 = \Lambda$, there exists a translation $\tau(u) \in \Lambda' \leqslant N$ such that $t_0 := t\tau(u)$ $(= t + u)$ is a vertex in $\mathcal{D}$. Define $\varphi_0 := \varphi\tau(u) \in N$, so that $x\varphi_0 = x\omega + t_0$ for $x \in \mathbb{E}^n$. Now o and t_0 $(= o\varphi_0)$ are vertices of some common facet $F \in \mathcal{D}$. Since they belong to the same orbit of N, we must necessarily have $t_0 = o$; here we have used Lemma 2.5. But then $\omega = \varphi_0 \in N \setminus \{\varepsilon\}$. This is a contradiction, because $N \cap \Gamma_0 = N \cap \langle \rho_1, \ldots, \rho_n \rangle = \{\varepsilon\}$ when N is sparse. It therefore follows that $\omega = \varepsilon$, and hence N consists of translations alone, as was claimed. ■

Our main result is a straightforward consequence of Theorem 3.1.

Theorem 3.2 *For $n \geqslant 2$, the n-tori are the only (topological types of) n-dimensional compact euclidean space-forms which admit regular tessellations.*

Proof Let $\mathcal{P}$ be a regular tessellation on an n-dimensional compact euclidean space-form X, and let $\mathcal{P}$ be of type $\{p_1, \ldots, p_n\}$. By Theorem 2.9, $p_n > 2$ and $\mathcal{P}$ is locally spherical. Let $\mathcal{T} := \{p_1, \ldots, p_n\}$. By Theorem 2.8 (or Lemma 2.7), $\mathcal{P} = \mathcal{T}/N$ for some sparse normal subgroup $N \leqslant \Gamma(\mathcal{T})$. By Lemma 2.10, $\mathcal{T}$ is a euclidean tessellation. In particular, we may view $\mathcal{P}$ as a tessellation on the euclidean space-form $\mathbb{E}^n/N$. Since N is normalized by $\Gamma^+(\mathcal{T})$, Theorem 3.1 applies, and shows that N can only consist of translational symmetries of $\mathcal{T}$. It follows that $\mathbb{E}^n/N$ is an n-torus and that $\mathcal{P}$ is a regular toroid on it. Since X is homeomorphic to $\mathbb{E}^n/N$, this completes the proof. ■

4 Chiral Tessellations

We begin the discussion of chiral toroids with an immediate consequence of Theorem 3.1.

Theorem 4.1 *Under the conditions of Theorem 3.1, if $n > 2$, then the sparse subgroup N is normal in $\Gamma(\mathcal{T})$.*

Proof Since we now know that N consists only of translations, we may employ the straightforward argument of [15, Theorem 9.1]; we shall not repeat it here. ■

Recall that a polytope $\mathcal{P}$ is *chiral* if its group $\Gamma(\mathcal{P})$ has two orbits on the flags, such that adjacent flags are in distinct orbits ([17]). Let $\mathcal{P}$ be a chiral $(n+1)$-polytope, and let $\Psi := \{G_{-1}, G_0, \ldots, G_{n+1}\}$ be a fixed or *base* flag of $\mathcal{P}$. For $i = 1, \ldots, n$, let G_i' denote the i-face of $\mathcal{P}$ with $G_{i-1} < G_i' < G_{i+1}$ and $G_i' \neq G_i$, and let β_i be the automorphism of $\mathcal{P}$ which fixes each face G_j for $j \neq i-1, i$, and cyclically permutes consecutive i-faces of $\mathcal{P}$ in the (polygonal) section G_{i+1}/G_{i-2} of rank 2, mapping G_i' to G_i. Then $\Gamma(\mathcal{P}) = \langle \beta_1, \ldots, \beta_n \rangle$, where the "rotations" β_j satisfy the relations

$$\beta_j^{p_j} = \varepsilon \quad \text{for } 1 \leqslant j \leqslant n, \qquad (4.2)$$
$$(\beta_j \beta_{j+1} \cdots \beta_k)^2 = \varepsilon \quad \text{for } 1 \leqslant j < k \leqslant n;$$

here, $\{p_1, \dots, p_n\}$ is again the (*Schläfli*) *type* of $\mathcal{P}$.

Let $\mathcal{T} := \{p_1, \dots, p_n\}$ and $\Gamma(\mathcal{T}) = [p_1, \dots, p_n] = \langle \rho_0, \dots, \rho_n \rangle$, where the generators are defined with respect to the base flag $\Phi = \{F_{-1}, F_0, \dots, F_{n+1}\}$ of $\mathcal{T}$. Let $\alpha_j := \rho_{j-1}\rho_j$ for $j = 1, \dots, n$. Then the *rotation subgroup* $\Gamma^+(\mathcal{T}) := \langle \alpha_1, \dots, \alpha_n \rangle$ has index 2 in $\Gamma(\mathcal{T})$. If $\mathcal{P}$ is a chiral $(n+1)$-polytope with group $\Gamma(\mathcal{P}) = \langle \beta_1, \dots, \beta_n \rangle$, then the mappings $\alpha_j \mapsto \beta_j$ for $j = 1, \dots, n$ induce a surjective homomorphism $\kappa\colon \Gamma^+(\mathcal{T}) \to \Gamma(\mathcal{P})$, so that $\Gamma(\mathcal{P}) \cong \Gamma^+(\mathcal{T})/N$ with $N := \ker(\kappa)$. In fact, $\mathcal{P}$ is isomorphic to the quotient $\mathcal{T}/N$, and an isomorphism $\mathcal{T}/N \to \mathcal{P}$ is given by

$$F_i\varphi \cdot N \to G_i(\varphi\kappa) \quad \text{for } i = -1, 0, \dots, n+1 \text{ and } \varphi \in \Gamma^+(\mathcal{T});$$

see [17] for the tools needed in the proof, and [14] for a similar proof for regular polytopes.

Now suppose that $\mathcal{P}$ is a chiral tessellation of type $\{p_1, \dots, p_n\}$ on an n-dimensional manifold X (as usual, without boundary). Then the tiles must be isomorphic to convex n-polytopes which are actually regular. In fact, the subgroup $\langle \beta_1, \dots, \beta_{n-1} \rangle$ of $\Gamma(\mathcal{P})$ is now a subgroup of index 1 or 2 in the (finite) group of a tile of $\mathcal{P}$, and since convex polytopes cannot be chiral, it must actually have index 2. Hence the tiles have a flag-transitive group, and thus are regular.

On the other hand, the vertex-figures must also be isomorphic to convex regular polytopes. In fact, since X is a manifold, each vertex-figure of $\mathcal{P}$ is now an n-polytope each of whose sections of rank at least 2 has an order complex which is topologically a sphere. But this implies isomorphism with a convex regular polytope, because the vertex-figure has a Schläfli symbol [5], [12], [13]. In particular, chirality implies that $p_i > 2$ for each i.

It follows that $\mathcal{P}$ is a locally spherical chiral polytope, and that $\mathcal{T} = \{p_1, \dots, p_n\}$ is a regular tessellation in $E = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ (we could have obtained this directly from [4], [11]).

Now write $\mathcal{P} = \mathcal{T}/N$ as above. Then we can generalize Lemma 2.7 and show that N is sparse. Indeed, since the facets and vertex-figures of $\mathcal{P}$ are isomorphic to those of $\mathcal{T}$ (and again, this is all that we need), we must have

$$N \cap \langle \alpha_1, \dots, \alpha_{n-1} \rangle = \{\varepsilon\} = N \cap \langle \alpha_2, \dots, \alpha_n \rangle.$$

Further, since $\Gamma^+(\mathcal{T})/N$ is the group of a chiral polytope (namely $\mathcal{P}$), it must satisfy the intersection property for such groups, and so we can conclude, as in the proof of Lemma 2.6, that

$$N \cap \langle \alpha_1, \dots, \alpha_{n-1} \rangle \langle \alpha_2, \dots, \alpha_n \rangle = \{\varepsilon\}.$$

It is now immediate that N is sparse, because the groups in (2.4) are obtained from those here by adjoining ρ_1, and $\rho_1 N \rho_1$ is the only subgroup (distinct from N) which is conjugate to N.

Just as for regular polytopes, we can now view $\mathcal{P}$ as a chiral tessellation on the corresponding space-form E/N, which is homeomorphic to the original manifold X (and to $|\mathcal{C}(\mathcal{P})|$).

Now suppose that X is a compact euclidean space-form. Then again, $E = \mathbb{E}^n$, $\mathcal{T}$ is euclidean and N is a sparse subgroup of $\Gamma(\mathcal{T})$ which is a normal subgroup of $\Gamma^+(\mathcal{T})$. By Theorem 3.1, N can only consist of translational symmetries, so that X is an n-torus and $\mathcal{P}$

a chiral tessellation on it. Since by Theorem 4.1 (see also [15, Theorem 9.1]), N is normal in $\Gamma(\mathcal{T})$ if $n > 2$, then there are no chiral tessellations on an n-torus with $n > 2$, and so we must have $n = 2$.

Summarizing, we see that we have proved

Theorem 4.3 *If an n-dimensional compact euclidean space-form X admits a chiral tessellation, then $n = 2$ and X is the 2-torus.*

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