CATEGORIES OF CERTAIN MINIMAL TOPOLOGICAL SPACES

MANUEL P. BERRI[1]

(received 9 September 1963)

The main purpose of this paper is to discuss the categories of the minimal topological spaces investigated in [1], [2], [7], and [8]. After these results are given, an application will be made to answer the following question: If $\mathfrak{L}$ is the lattice of topologies on a set X and $\mathfrak{T}$ is a Hausdorff (or regular, or completely regular, or normal, or locally compact) topology does there always exist a minimal Hausdorff (or minimal regular, or minimal completely regular, or minimal normal, or minimal locally compact) topology weaker than $\mathfrak{T}$?

The terminology used in this paper, in general, will be that found in Bourbaki: [3], [4], and [5]. Specifically, T_1 and Fréchet spaces are the same; T_2 and Hausdorff spaces are the same; regular spaces, completely regular spaces, normal spaces, locally compact spaces, and compact spaces all satisfy the Hausdorff separation axiom. An open (closed) filter-base is one composed exclusively of open (closed) sets. A regular filter-base is an open filter-base which is equivalent to a closed filter-base. Finally in the lattice of topologies on a given set X, a topology is said to be minimal with respect to some property P (or is said to be a minimal P-topology) if there exists no other weaker (= smaller, coarser) topology on X with property P.

The following theorems which characterize minimal Hausdorff topologies and minimal regular topologies are proved in [1], [2], and [5].

THEOREM 1. *A Hausdorff space is minimal Hausdorff if, and only if,* (i) *every open filter-base has an adherent point*; (ii) *if such an adherent point is unique, then the filter-base converges to it.*

THEOREM 2. *A regular space is minimal regular if, and only if,* (i) *every regular filter-base has an adherent point*; (ii) *if such an adherent point is unique, then the filter-base converges to it.*

Since a compact space satisfies the conditions of both theorems 1 and 2, then a compact space is both minimal Hausdorff and minimal

[1] This research was partially supported by NSF Grant Nos. GE 257 and GP 1634.

https://doi.org/10.1017/S1446788700022758 Published online by Cambridge University Press

regular. Now it is well-known that a compact space is of second category. In fact if the space is denumerably infinite, then it contains infinitely many isolated points. In this paper, it will be shown that the latter result can be extended to all minimal Hausdorff topologies and all minimal regular topologies defined on denumerable spaces. Also, it will be shown that the former result can be extended to all minimal regular spaces.

THEOREM 3. (i) *Every countably infinite minimal Fréchet space is of first category*; (ii) *every uncountably infinite minimal Fréchet space is of second category.*

PROOF. Let X be any infinite set. From [1] and [6], the minimal Frechet topology $\mathfrak{T}$ defined on X is given by the family

$$\mathfrak{T} = \{A \subset X | X - A \text{ is finite}\} \cup \{\phi\}.$$

(i) For each $x \in X$, $\{x\}$ is nowhere dense. Since X is countable and $X = \bigcup_{x \in X} \{x\}$, then $(X, \mathfrak{T})$ is of first category.

(ii) Let $X = \bigcup_{i=1}^{\infty} B_i$. Since X is uncountable, then at least one B_i is infinite and hence dense in X. Thus X is of second category.

Remark 1. (i) The example of the space of rational numbers with the natural topology will demonstrate that on denumerable Fréchet spaces minimal Frechet and first category are not equivalent properties. For this space is of first category but many open sets have infinite complement.

(ii) The example of any uncountably infinite discrete space will demonstrate that on uncountably infinite Fréchet spaces minimal Fréchet and second category are not equivalent properties.

THEOREM 4. *Any minimal Hausdorff topology defined on a denumerable infinite set has an isolated point.*

PROOF. Let $\mathfrak{T}$ be a minimal Hausdorff topology defined on X, a denumerably infinite set. Assume $(X, \mathfrak{T})$ has no isolated point. We will now construct an open filter-base on X without any adherent points, thus contradicting a necessary condition for X to be minimal Hausdorff. List the points of X: $X = \{x_1, x_2, \cdots\}$. Since X has no isolated points and since X is Hausdorff, then every open set is infinite. Hence any neighborhood at any point is infinite. Take and fix a point $a_1 \in X$ such that $a_1 \neq x_1$. Now there exist disjoint open neighborhoods G_1 and V_1 of a_1 and x_1 respectively. Since G_1 is infinite, then there exist a point $a_2 \in G_1$, $a_2 \neq x_2$, and disjoint open neighborhoods G_2 and V_2 of a_2 and x_2 respectively such that $G_1 \supset G_2$. Repeating this argument for any element x_n, we can find an element $a_n \in G_{n-1}$, $a_n \neq x_n$, and disjoint open neighborhoods G_n and V_n of a_n and x_n respectively such that

$$G_1 \supset G_2 \supset \cdots \supset G_n.$$

Thus we obtain a descending chain of non-empty open sets of X which of course forms an open filter-base; call it $\mathfrak{D}$. Now for any element $x_k \in X$, we have by construction $V_k \cap G_k = \phi$. Thus $\mathfrak{D}$ has no adherent point. Hence X is not minimal Hausdorff which contradicts our hypothesis.

COROLLARY 1. *A minimal Hausdorff space X, where X is denumerably infinite is of second category.*

Remark 2. The example of any denumerably infinite space with the discrete topology will demonstrate that on denumerable Hausdorff spaces minimal Hausdorff and second category are not equivalent properties.

In [1], it is shown that any open and closed subspace of a minimal Hausdorff space is minimal Hausdorff. Using theorem 4, we can easily establish the following result.

COROLLARY 2. *A minimal Hausdorff space $(X, \mathfrak{T})$, where X is denumerably infinite, contains infinitely many isolated points.*

PROOF. Let A be the set of all isolated points of X. Assume A is finite. Then A is both open and closed. Thus $X-A$ is both open and closed. Since $X-A$ is minimal Hausdorff, then $X-A$ contains an isolated point which is also isolated in X. Thus A does not contain all the isolated points of X which contradicts our definition of A.

COROLLARY 3. *A compact space X where X is denumerably infinite, contains infinitely many isolated points and hence is of second category.*

THEOREM 5. *Any minimal regular topology defined on a countably infinite set has an isolated point.*

PROOF. Let $\mathfrak{T}$ be a minimal regular topology defined on a countably infinite set X. Assume $(X, \mathfrak{T})$ has no isolated point. By theorem 2, a necessary condition for a regular space to be minimal regular is that every regular filter-base, i.e., an open filter-base equivalent to a closed filter-base, have an adherent point. We will now construct a regular filter-base which has no adherent point. List the points of X: $X = \{x_1, x_2, \cdots\}$. Since X has no isolated points and since X is Hausdorff, then every open set is infinite. Hence any neighborhood of any point is infinite. Take and fix a point $a_1 \in X$ such that $a_1 \neq x_1$. Since X is regular, there exist disjoint open neighborhoods G_1 and V_1 of a_1 and x_1 respectively and a closed neighborhood F_1 of a_1 such that $G_1 \supset F_1$. Since $\mathring{F}_1$ is infinite, then there exist a point $a_2 \in F_1$, $a_2 \neq x_2$, and disjoint open neigborhoods G_2 and V_2 of a_2 and x_2 respectively and a closed neighborhood F_2 of a_2 such that $G_1 \supset F_1 \supset G_2 \supset F_2$. Repeating this argument for any element x_n, we can find an element $a_n \in G_{n-1}$, $a_n \neq x_n$, and disjoint open neighborhoods G_n and V_n of a_n and x_n respectively and a closed neighborhood F_n of a_n such that

$$G_1 \supset F_1 \supset G_2 \supset F_2 \supset \cdots \supset G_n \supset F_n.$$

Let $\mathfrak{O} = \{G_i | \ i = 1, 2, \cdots\}$ and $\mathfrak{F} = \{F_i | \ i = 1, 2, \cdots\}$. Thus $\mathfrak{O}$ is an open filter-base, $\mathfrak{F}$ is a closed filter-base, and $\mathfrak{O}$ and $\mathfrak{F}$ are equivalent. Since $\mathfrak{O}$ has no adherent point, then X is not minimal regular which contradicts our hypothesis.

From theorem 5, one may conclude that any minimal regular space X, where X is denumerable, is of second category. We shall now prove that all minimal regular spaces possess this property.

THEOREM 6. *If* $(X, \mathfrak{T})$ *is a minimal regular space, then* $(X, \mathfrak{T})$ *is of second category.*

PROOF. Let $\{O_n\}_{n=1}^{\infty}$ be a denumerable family of open, dense sets in X. We shall now show that $\bigcap_{n\geq 1} O_n \neq \phi$. Let $V_1 = O_1 \cap O_2$. Now V_1 is open and dense on X. Take and fix $a_1 \in V_1$. Since X is regular, there exist neighborhoods G_1 and F_1 of a_1 such that G_1 is open, F_1 is closed and $V_1 \supset F_1 \supset G_1$. Now let $V_2 = G_1 \cap O_3$. Since G_1 is non-empty and O_3 is dense, then V_2 is open and non-empty. Take and fix $a_2 \in V_2$. Since X is regular, there exist neighborhoods G_2 and F_2 of a_2, such that G_2 is open, F_2 is closed and $V_2 \supset F_2 \supset G_2$. Thus $V_1 \supset F_1 \supset V_2 \supset F_2 \supset G_2$. Now for $n \geqq 2$, we may repeat this last argument. Let $V_n = G_{n-1} \cap O_{n+1}$. Both G_{n-1} and O_{n+1} are open. Since G_{n-1} is non-empty and O_{n+1} is dense, then V_n is open and non-empty. Take and fix $a_n \in V_n$. Since X is regular, there exist neighborhoods G_n and F_n of a_n. Thus $V_1 \supset F_1 \supset G_1 \supset V_2 \supset F_2 \supset G_2 \supset \cdots \supset V_n \supset F_n \supset G_n$. Let $\mathfrak{O} = \{G_n | \ n \geqq 1\}$ and $\mathfrak{F} = \{F_n | \ n \geqq 1\}$. From the above remarks, we see that $\mathfrak{O}$ is a regular filter-base equivalent to the closed filter-base $\mathfrak{F}$. By theorem 2, $\mathfrak{O}$ has an adherent point. Let p be such a point. Since $p \in \bigcap_{n\geq 1} F_n$, then $p \in \bigcap_{n\geq 1} G_n$. But $\bigcap_{n\geq 1} G_n \subset \bigcap_{n\geq 1} O_n$. Hence $\bigcap_{n\geq 1} O_n \neq \phi$. Thus $(X, \mathfrak{T})$ is of second category.

Remark 3. (i) On regular spaces, minimal regularity and second category are not equivalent conditions. As an example, let X be any uncountable space with the discrete topology.

(ii) For an example of a non-compact minimal Hausdorff space, the reader is referred to [1], [7], [8] and [9].

(iii) For an example of a non-compact minimal regular space, the reader is referred to [2].

From [1] and [8], it is proved that all minimal completely regular spaces, minimal normal spaces, and minimal locally compact spaces are compact. Hence we have the following result.

THEOREM 7. *All minimal completely regular spaces, minimal normal spaces, minimal locally compact spaces are of second category.*

COROLLARY 4. *Let* $(X, \mathfrak{T})$ *be a topological space such that* X *is denumerably*

https://doi.org/10.1017/S1446788700022758 Published online by Cambridge University Press

infinite and $\mathfrak{T}$ is minimal completely regular (minimal normal, minimal locally compact). Then $(X, \mathfrak{T})$ contains isolated points.

As an application of theorems 4 and 5 and corollary 3, we shall now answer (in part) the following general question concerning the lattice of topologies on a set X. *Given a P-topology $\mathfrak{T}$ on X. Does there exist a minimal P-topology $\mathfrak{T}'$ such that $\mathfrak{T}'$ is weaker than $\mathfrak{T}$?* We shall answer this question for the different type of P-topologies discussed in this paper. If $P =$ Fréchet or locally compact, from [1] and [6] the answer to the question is always affirmative. If $P =$ Hausdorff, or regular, or completely regular, or normal, the answer is negative. Consider the following example. Let $(X, \mathfrak{T})$ be the set of all rational numbers with the natural topology. Then X is Hausdorff, regular, completely regular, and normal. But X has no isolated points. Hence by theorems 4 and 5 and corollary 3, there exist no corresponding minimal P-topologies on X weaker than $\mathfrak{T}$.

Question: Are all minimal Hausdorff spaces of second category?

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Tulane University of Louisiana

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