HAMILTONIAN THEORY OF THE LIBRATION OF THE MOON

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ABSTRACT

The feasibility of applying the Lie transform method to the problem of the physical libration of the Moon is investigated. By a succession of canonical transformations, the Hamiltonian of the problem is brought under a form suitable for perturbation technique. The mean value of the inclination of the angular momentum upon the ecliptic and the frequencies of the free libration are computed.

1. INTRODUCTION

One of the main tools needed to analyze the data of the laser ranging of the Moon is a precise theory of the libration of the Moon.

Several authors (Eckhardt 1973, Cook 1976, Migus 1976) have worked and are still working on the improvement of such a theory. All of them use a technique of successive approximations to the solution of a set of differential equations.

We felt it would be interesting to investigate whether the problem could be treated by an Hamiltonian perturbation method such as the Lie transform method. This method has been used successfully in several problems of celestial mechanics and presents some advantages. One of them is that it enables (or forces) the scientist to take one difficulty of the system at a time and thus gives him a better understanding of it, especially in case of resonance. On the other hand, this technique is often more difficult to implement and requires more care in choosing coordinate systems.

Whatever the advantages or drawbacks of these two methods, it seems to us very interesting to compare their results. Indeed their philosophy and especially the way they treat resonance is quite different. Thus, if the results of both of them agree, one can feel confident that the problem has no hidden traps and that the solution is valid.

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In this paper, we study the feasibility of applying the Lie transform method to the problem of the physical libration of the Moon. This task is not a trivial one as the libration of the Moon is not obviously the perturbation of an integrable problem or, if it is (one can think of the constant rotation around the axis of inertia), the integrable problem is too degenerate to be of any use.

2. THE PHASE SPACE AND THE FREE ROTATION PROBLEM

To describe the problem of the libration of the Moon as an Hamiltonian system, we choose the Tisserand's canonical variables (Deprit 1967). They are the three angles μ_1 , μ_2 , μ_3 (see figure 1) and their conjugate momenta :

$M_1 = M_2 \cos I$

M_2 = norme of the angular momentum of the Moon

$M_3 = M_2 \cos b$

where the angles I and b are defined in figure 1. The frame of reference X , Y , Z is an inertial frame (the plane X , Y being the ecliptic) and the frame x , y , z is the frame of the principal axis of inertia of the Moon.

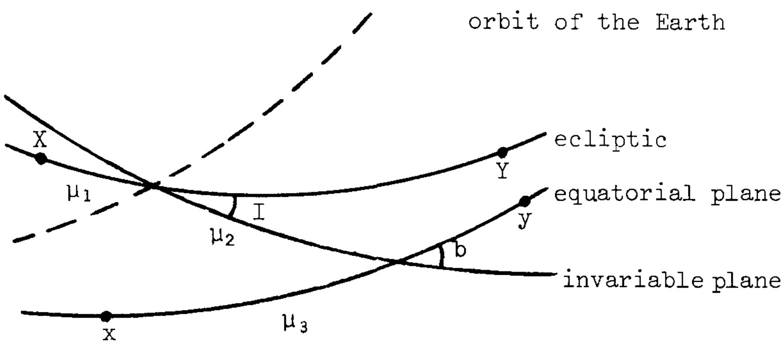

Figure 1. Geometry of the libration of the Moon.

The invariable plane is the plane perpendicular to the angular momentum of the Moon. Note that we have taken the angle I as negative so that the mean value of the longitude μ_1 , will be equal to the mean value of the node of the orbit of the Earth as seen from the Moon.

The Hamiltonian of the free rotation is then written as (Deprit 1967) :

$$H = \frac{1}{2}\frac{M_3^2}{C} + \frac{1}{2}\,(M_2^2 - M_3^2)\,\left[\frac{\sin^2 \mu_3}{A} + \frac{\cos^2 \mu_3}{B}\right] \qquad (1)$$

where A , B , C $(A \leqslant B \leqslant C)$ are the moments of inertia of the Moon with respect to the axis x , y , z .

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The Tisserand's canonical variables are singular when I and b are equal to zero (or to π). When $I = 0$ (resp. $b = 0$) the angles μ_1 and μ_2 (resp. μ_2 and μ_3) are undefined (although their sum is well defined). This is a situation very similar to the one appearing when one uses the Delaunay's variables for small eccentricities or inclinations in the two body problem.

As usual, this situation can be dealt with easily by using the properties of d'Alembert characteristic if the singularity can be shown to be of the polar-coordinates type (Henrard 1974). To do this, we introduce the modified Tisserand's elements :

$$\Lambda_1 = M_2 \qquad \lambda_1 = \mu_1 + \mu_2 + \mu_3 \tag{2.1}$$

$$\Lambda_2 = M_2 - M_3 \qquad \lambda_2 = -\mu_3 \tag{2.2}$$

$$\Lambda_3 = M_2 - M_1 \qquad \lambda = -\mu_1 \tag{2.3}$$

The angle λ_2 (resp. λ_3) is undefined when $\Lambda_2 = 0$ (resp. $\Lambda_3 = 0$) but the Poincaré-type variables associated with (2.2) and (2.3) :

$$\xi = \sqrt{2\ \Lambda_2}\ \sin\ \lambda_2 \qquad \Xi = \sqrt{2\ \Lambda_2}\ \cos\ \lambda_2 \tag{3.1}$$

$$\eta = \sqrt{2\ \Lambda_3}\ \sin\ \lambda_3 \qquad H = \sqrt{2\ \Lambda_3}\ \cos\ \lambda_3 \tag{3.2}$$

are non singular. Thus, the virtual singularities of the modified Tisserand's variables (2) will not present any difficulty if the functions we shall be dealing with present the d'Alembert characteristic with respect to the couple (λ_2, Λ_2), (λ_3, Λ_3).

The Hamiltonian of the free rotation now reads :

$$H = \frac{1}{2}\frac{(\Lambda_1 - \Lambda_2)^2}{C} + \frac{1}{2}\Lambda_2\ (2\ \Lambda_1 - \Lambda_2)\ \left[\frac{\sin^2\ \lambda_2}{A} + \frac{\cos^2\ \lambda_2}{B}\right] \tag{4}$$

or

$$H = \frac{1}{2}\frac{\Lambda_1^2}{C} + 2\ \frac{\Lambda_1^2}{C}\sin^2\frac{b}{2}\cos^2\frac{b}{2}\left[\frac{C-A}{A}\sin^2\ \lambda_2 + \frac{C-B}{B}\cos^2\ \lambda_2\right] \tag{5}$$

From (4), it is obvious that this Hamiltonian function presents the d'Alembert characteristic with respect to the couples (λ_2, Λ_2) and (λ_3, Λ_3). The expression (5) with the auxiliary quantities :

$$\sin^2\frac{b}{2} = \frac{\Lambda_2}{2\ \Lambda_1} \qquad \sin^2\frac{I}{2} = \frac{\Lambda_3}{2\ \Lambda_1} \tag{6}$$

which are geometrically meaningfull and undimensional, will be prefered. It makes obvious that, when $A = B = C$ (which is almost the case for the Moon), the problem is trivial.

3. THE PERTURBATION FROM THE FREE ROTATION PROBLEM

The function which has to be added to the Hamiltonian of the free

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rotation to take into account the attraction of the Earth on the rigid Moon is :

$$V = - G \int_{Earth} \int_{Moon} \frac{d\mu \, d\mu'}{r'}$$

where r' is the distance between an element of mass $d\mu'$ in the Earth and an element of mass $d\mu$ in the Moon. We assume that the Earth is a mass-point, thus neglecting terms of the order of :

$$[J_2 \left(\frac{R}{r}\right)^2]_{Earth} \simeq 10^{-7}$$

with respect to the mean terms of the perturbation (r being the distance of the centers of mass and R the equatorial radius of the body).

Furthermore, in this preliminary computation, we shall neglect the terms of the third order in the expansion of the potential of the Moon, thus neglecting terms of the order of :

$$[\frac{J_3}{J_2} \left(\frac{R}{r}\right)]_{Moon} \simeq 5.10^{-4}$$

with respect to the mean terms of the perturbation.

Neglecting terms independant of our phase space, we are thus considering as perturbing potential :

$$V = - \frac{3}{2} \frac{G E}{r^3} [(C - A) \xi_1^2 + (C - B) \xi_2^2] \tag{7}$$

where ξ_1 and ξ_2 are the first two coordinates of the unit vector pointing to the Earth in the frame of the principal axis of inertia of the Moon.

Defining the mean semi-major axis of the orbit of the Earth by :

$$a^3 = G (E + M) / n^2 \tag{8}$$

where n is the mean notion in longitude of this orbit, we write the equation (7) under the form :

$$V = - \frac{3}{4} \frac{n^2 C}{1 + \kappa} \left(\frac{a}{r}\right)^3 \{\delta (\xi_1^2 + \xi_2^2) + \gamma (\xi_1^2 - \xi_2^2)\} \tag{9}$$

where γ and κ are defined as usual by $\kappa = M / E$ and $\gamma = (B - A) / C$. The quantity δ is defined by :

$$\delta = (2 C - A - B) / C \tag{10}$$

and is related to the usual quantity $\beta = (C - A) / B$ by :

$$\delta = \frac{2 \beta - \gamma (1 - \beta)}{1 + \beta} = 2 \beta - \gamma - 2 \beta^2 + 2 \gamma \beta + \vartheta(\beta^3) \tag{11}$$

The expressions of ξ_1 and ξ_2 in (9) are obtained by a succession of rotations (of angles $\mu_1, I, \mu_2, b, \mu_3$) from the expressions of X_1, X_2, X_3, the components of the unit vector pointing to the Earth in the

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ecliptical frame. In our program, the quantities :

$$s = \sin (b / 2) \qquad k = \cos (b / 2)$$
$$S = \sin (I / 2) \qquad K = \cos (I / 2) \tag{12}$$

are used to express the rotation of angles I and b .

The expressions of (a / r) , X_1 , X_2 , X_3 in turn are obtained from the Theory of the Moon; they are multiple Fourier series in the arguments λ , D , F , l , l' (resp. the mean longitude of the Moon, the difference of longitude between Moon and Sun, the arguments of latitude, the mean anomaly of the Moon and the mean anomaly of the Sun). In this computation, we have used the solution of the main problem called ALE (Deprit et al. 1971a, 1971b, Henrard 1972) truncated at 50" for longitude and latitude and at 0"5 for the sine-parallax. This corresponds to a relative accuracy of 2.10^{-4} in the computation of V .

Note that if we use in the expression (a / r) the definition (8) for the mean semi-major axis and not the inverse of the constant term of sine parallax, we do not have to take into account the correcting factor $\lambda = 1.0027$ of Jeffreys (Jeffreys 1961). Actually, our computation corresponds to $\lambda = 1.002726$ as the value given by Jeffreys is truncated.

The system we are considering is a system with three degrees of freedom but it depends explicitely on the time through the frequencies n (of the longitude of the Moon), n' (of the longitude of the Sun), n_g (of the perigee of the Moon) and n_h (of the longitude of the node of the Moon). We transform it into an autonomous system with seven degrees of freedom by introducing artificial momenta L , L' , G , H , conjugated respectively to the angles λ , l' , g , h (note that L , G , H are not the Delaunay's momenta).

Eventually, taking into account (5) and (9), we obtain as Hamiltonian function for the problem of the libration of the Moon :

$$\begin{aligned} H = n L + n' L' + n_g G + n_h H + \Lambda_1^2 / 2 C + \\ + \frac{\Lambda_1^2}{C} s^2 k^2 (\delta - \gamma \cos 2 \lambda_2) - \\ - \frac{3}{4} \frac{n^2 C}{1+K} \delta P_1 - \frac{3}{4} \frac{n^2 C}{1+K} \gamma P_2 \end{aligned} \tag{13}$$

where P_1 and P_2 are the expansions respectively of $(a / r)^3 (\xi_1^2 + \xi_2^2)$ and $(a / r)^3 (\xi_1^2 - \xi_2^2)$.

In writing (13), we have neglected a term :

$$\frac{\Lambda_1^2}{C} s^2 k^2 [\frac{\delta^2 + \gamma^2}{2} - \gamma \delta \cos 2 \lambda_2] + \mathcal{O}(\delta^3) \tag{14}$$

of the order of δ^2 in the expression of the Hamiltonian of the free rotation.

The principal terms in P_1 and P_2 are approximatively :

$$\begin{aligned} P_1 = & (K^4 + S^4)(k^4 + s^4) + 8 K^2 S^2 k^2 s^2 + \\ & + 2 K^4 k^2 s^2 \cos(2\lambda - 2\lambda_1 - 2\lambda_2) + \\ & + 2 \sin i\, K S (K^2 - S^2)(k^4 + s^4 - 4 k^2 s^2) \cos(\lambda_3 + h) - \\ & - 0.015 K^4 k s (k^2 - s^2) \cos(\lambda - \lambda_1 - \lambda_2 - g) + \\ & + \ldots \end{aligned} \tag{15}$$

$$\begin{aligned} P_2 = & 2 k^2 s^2 (K^4 + S^4 - 4 K^2 S^2) \cos(2\lambda_2) + \\ & + K^4 k^4 \cos(2\lambda - 2\lambda_1) - \\ & - 0.015 K^4 k^3 s \cos(\lambda - \lambda_1 + \lambda_2 - g) + \\ & + \ldots \end{aligned} \tag{16}$$

4. CASSINI'S LAWS AND THE FREQUENCIES OF THE SYSTEM

In our notations, Cassini's laws which provide an approximation for the physical libration of the Moon, can be written :

$$\begin{aligned} & <\lambda_1> = \lambda \\ & <\lambda_3> = -h \\ & <2S> \simeq -\sin(1^\circ\ 30') \end{aligned} \tag{17}$$

where $<\ldots>$ stands for "the mean value of".

To take into account those laws and bring forward the librations around these mean values, we propose the following canonical transformation :

$$x_1 = \lambda_1 - \lambda \qquad y_1 = \frac{\Lambda_1}{n\,C} - \nu \tag{18.1}$$

$$x_2 = \sqrt{\frac{2\,\Lambda_2}{n\,C}} \sin\lambda_2 \qquad y_2 = \sqrt{\frac{2\,\Lambda_2}{n\,C}} \cos\lambda_2 \tag{18.2}$$

$$x_3 = \sqrt{\frac{2\,\Lambda_3}{n\,C}} \sin(\lambda_3 + h) \qquad y_3 = \sqrt{\frac{2\,\Lambda_3}{n\,C}} \cos(\lambda_3 + h) - 2\mu \tag{18.3}$$

$$\lambda^+ = \lambda \qquad L^+ = (L + \Lambda_1)/n\,C \tag{18.4}$$

$$h^+ = h \qquad H^+ = (H - \Lambda_3)/n\,C \tag{18.5}$$

$$g^+ = g \qquad G^+ = G/n\,C \tag{18.6}$$

$$l'^+ = l' \qquad L'^+ = L'/n\,C \tag{18.7}$$

The constant ν (close to one) which appears in (18.1) comes from the fact that the mean value of Λ_1 is close but not quite **equal to n C** and the constant μ which appears in (18.3) is close to the constant

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<-S> . These constants will be computed so that the coordinates (x_i, y_i) will be cartesian-like coordinates centered at a mean equilibrium. The multiplier of the transformation has been made equal to $1/nC$ in order to obtain undimensional (x_i, y_i) .

The Hamiltonian function (13) now reads :

$$\begin{aligned} H = {} & n L + n' L' + n_g G + n_h H + \\ & + n (\nu - 1) y_1 + 2 \mu n_h y_3 + \\ & + \frac{n}{2} y_1^2 + \frac{n}{4} [\delta (x_2^2 + y_2^2) + \gamma (x_2^2 - y_2^2)] \nu + \\ & + \frac{n_h}{2} (x_3^2 + y_3^2) + \dots - \\ & - \frac{3}{4} \frac{n \delta}{1+K} P_1 - \frac{3}{4} \frac{n \gamma}{1+K} P_2 \end{aligned} \qquad (19)$$

where we have dropped the stars. The principal terms in P_1 and P_2 are approximately :

$$\begin{aligned} P_1 = {} & [-2\sqrt{\mu^2/\nu} - \sin i + \vartheta(\mu^2)] y_3 + \\ & + [\sqrt{\mu^2/\nu} \sin i + \vartheta(\mu^2)] y_1 + \\ & + \frac{1}{2} [-(x_3^2 + y_3^2) - 2 x_2^2 + 0.15 y_1 y_3 + \\ & + 0.16 \cos g (y_2 y_3 - x_2 x_3) - \\ & - 0.16 \sin g (y_2 x_3 + x_2 y_3)] + \vartheta(\mu) + \\ & + \text{cubic terms in } (x_i, y_i) \end{aligned} \qquad (20)$$

$$\begin{aligned} P_2 = {} & [-2\sqrt{\mu^2/\nu} - \sin i + \vartheta(\mu^2)] y_3 + \\ & + [\sqrt{\mu^2/\nu} \sin i + \vartheta(\mu^2)] y_1 + \\ & + \frac{1}{2} [-(x_3^2 + y_3^2) - 2 x_2^2 - 4 x_1^2 + \\ & + 0.15 y_1 y_3 + 0.32 y_1 x_3 + \\ & + 0.16 \sin g (y_2 x_3 - x_2 y_3) - \\ & - 0.16 \cos g (y_2 y_3 + x_2 x_3)] + \vartheta(\mu) + \\ & + \text{cubic terms in } (x_i, y_i) \end{aligned} \qquad (21)$$

where $\sin i$ is the sine of the inclination of the orbit of the Earth.

The origin of the phase space will be a mean equilibrium for the system if μ and ν are such that the mean values of the coefficients of the linear terms in (x_i, y_i) are zero.

Then, in order to compute the basic frequencies of the system around this mean equilibrium, we should, by a linear canonical transformation, bring the quadratic part in (x_i, y_i) of the Hamiltonian under the form of three uncoupled harmonic oscillators.

$$\frac{n_1}{2}(x_1'^2 + y_1'^2) + \frac{n_2}{2}(x_2'^2 + y_2'^2) + \frac{n_3}{2}(x_3'^2 + y_3'^2) \tag{22}$$

As we have defined only a mean equilibrium in the first step, the frequencies n_i obtained would be only approximations of the true frequencies around the true equilibrium. In view of this, we shall simplify this second step by asking only that the quadratic part of the Hamiltonian becomes approximatively (22) and restrict ourselves to linear canonical transformations of the scaling-type :

$$x_i = \alpha_i \, x_i' \qquad y_i = \alpha_i^{-1} \, y_i' \tag{23}$$

Eventually, the transformation :

$$\begin{aligned} x_1 &= \sqrt{2P} \sin p & y_1 &= \sqrt{2P} \cos p \\ x_2 &= \sqrt{2Q} \sin q & y_2 &= \sqrt{2Q} \cos q \\ x_3 &= \sqrt{2R} \sin r & y_3 &= \sqrt{2R} \cos r \end{aligned} \tag{24}$$

will introduce the action-angles coordinates used in the following step of the theory.

5. NUMERICAL VALUES

The constants μ , ν , α_i and n_i $(1 \leqslant i \leqslant 3)$ of the preceding paragraph depend upon the values of κ , n_h , δ , γ . We have taken :

$$\begin{aligned} \delta &= 0.00103 & \kappa &= 1/81.30 \\ \gamma &= 0.00023 & n_h/n &= -0.00402133375326 \end{aligned} \tag{25}$$

Of course, the constants δ and γ are not well determined and we should allow for their variation by computing the derivative with respect to them of the final solution.

With the above values, we find :

$$\begin{aligned} \mu &= 0.013499866212 \\ \nu &= 1.00000146 5746 \end{aligned} \tag{26}$$

which, as we check, can be compared with the values given for the mean inclination of the axis of rotation of the Moon on the ecliptic. Assuming (x_i, y_i) to be zero, we find :

$$I = -2 \sin^{-1}(\sqrt{\mu^2/\nu}) = -\ 1^\circ\ 32'\ 49'' \tag{27}$$

which is to be compared to $-\ 1^\circ\ 32'\ 28''$ given by Eckhardt (Eckhardt 1965) or to $-\ 1^\circ\ 32'\ 57''$ given by Migus (Migus 1976). Not too much emphasis should be placed upon this comparison at this stage. Indeed, if both the quoted authors give as data the final mean value of I , it

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should be compared to the mean value of :

$$2 \sin^{-1} \left(\frac{1}{2}\left[\frac{(2\mu + y_3)^2 + x_3^2}{\nu + y_1}\right]^{1/2}\right) \qquad (28)$$

which is close to (because y_1, x_3, y_3 are small and of zero mean value) but not quite equal to (27).

Figure 2 shows the variation of this last constant when δ and γ varie. The level lines are lines of constant offset from the value $-1° 32' 49''$.

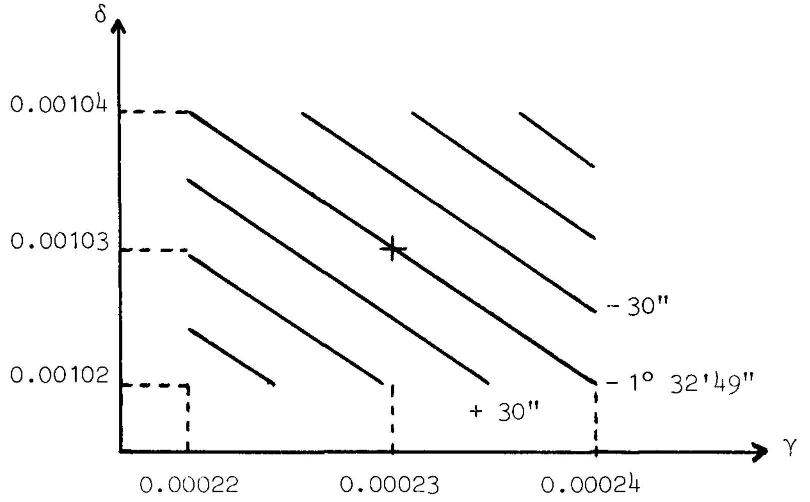

Figure 2. Variation of I with δ and γ.

For the values of α_i $(1 \leq i \leq 3)$ (see equation (23)), we find :

$$\alpha_1 = 6.207191342059$$
$$\alpha_2 = 0.633750505567 \qquad (29)$$
$$\alpha_3 = 1.001132806819 .$$

The approximations of the frequencies at the equilibrium are thus :

$$n_1 / n = 0.025954386109$$
$$n_2 / n = 0.000993516224 \qquad (30)$$
$$n_3 / n = -0.003098977293$$

The corresponding periods are respectively : 2.88 years, 75.23 years and 24.14 years.

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6. CONCLUSIONS

The Hamiltonian function of the problem of the libration of the Moon has now been transformed into a form suitable for perturbation theory :

$$\begin{aligned} H = {} & n\,L + \\ & + n'\,L' + n_g\,G + n_1\,P + \\ & + n_2\,Q + n_3\,R + n_h\,H + \\ & + n\,H_3\,(P, Q, R, p, q, r, \lambda, l', l, F, D) \end{aligned} \tag{31}$$

where the principal terms in H_3 are approximately :

$$\begin{aligned} H_3 = {} & -0.00020\sqrt{P\,R}\cos(p - r) + \\ & + 0.00018\sqrt{P\,R}\cos(p + r) - \\ & - 0.00013\sqrt{Q\,R}\cos(q + r + g) - \\ & - 0.00007\sqrt{Q\,R}\cos(q - r - g) + \\ & + 0.00004\sqrt{P\,Q}\cos(p - q - g) - \\ & - 0.00004\sqrt{P\,Q}\cos(p + q + g) - \\ & - 0.000004\sqrt{P}\cos(p - 2\,g) + \\ & + \ldots \end{aligned} \tag{32}$$

As suggested by the way we have written equation (31), the elimination of the periodic terms could be done in three steps, according to their frequencies. First, one could eliminate the monthly terms, then the terms of a period of a few years. They are the terms in l', g, p with the exception of the resonant term in $p - 2g$. In the last step, one could eliminate the terms of a period of twenty years and more, i.e. the terms in q, r and the resonant term in $p - 2g$.

In the near future, we plan to implement this elimination of periodic terms and thus compute the generator of the canonical transformation which brings the Hamiltonian (31) into an integrable one. This transformation known, the series describing the libration of the Moon can be obtained easily.

Before obtaining what we hope will be a usefull theory, we shall have to include several neglected terms in the Hamiltonian and compute the derivatives of the series with respect to δ and γ .

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