FOOTNOTE TO A FORMULA OF GIOIA AND SUBBARAO

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Recently Gioia and Subbarao [2] studied essentially the following problem: If $g(n)$ is an arithmetic function, and $h(n) = \sum_{d|n} g(d)$, then what is the behaviour of $H(a, n)$ defined for each fixed integer $a \geq 2$ by

$$(1) \qquad \sum_{d|n} H(a, d)g(d) = h(an)\,?$$

By using Vaidyanathaswamy's formula [e.g., 1], they obtain an explicit formula for $H(a, n)$ in case $g(n)$ is positive and completely multiplicative (Formula 2.2 of [2]). However, Vaidyanathaswamy's formula is unnecessary to the proof of this result, which indeed follows more simply without its use, by exploiting a simple idea used earlier by Subbarao [3] (referred to also in the course of [2]).

At Professor Subbarao's suggestion, this proof is communicated here.

Let r be the largest divisor of a relatively prime to n, and let $s = a/r$. Then, as shown in [3],

$$(2) \qquad H(a, n)g(n) = h(r)g(sn).$$

Hence since g is positive and completely multiplicative,

$$(3) \qquad H(a, n) = h(r)g(s) = h(r) \sum_{d|s} \mu(d)h(s/d)$$

$$= \sum_{d|s} \mu(d)h(rs/d) = \sum_{d|s} \mu(d)h(a/d);$$

by Mobius inversion and the definitions of r, s, $h(n)$, and since $h(n)$ is multiplicative (since $g(n)$ is multiplicative). But since only square-free divisors are relevant in the last sum in (3), it becomes (where (a, n) as usual denotes the

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g.c.d. of a and n),

$$\sum_{d \mid (a, n)} \mu(d)h(a/d)$$

which, when written out, is the formula 2.2 of [2] for H(a, n) in this case.

Just as easily, it may be seen similarly that the converse is also true, namely, if

$$H(a, n) = \sum_{d \mid (a, n)} \mu(d)h(a/d)$$

for all $a \geq 2$ and $n \geq 1$, then a positive multiplicative g(n) is completely multiplicative.

REFERENCES

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