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NON-NEHARI-MANIFOLD METHOD FOR ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION

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Abstract

We consider the semilinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $f(x,u)$ is asymptotically linear with respect to u, $V(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and $\sup[\sigma(-\triangle + V) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\triangle + V) \cap (0, \infty)]$. We develop a direct approach to find ground state solutions of Nehari–Pankov type for the above problem. The main idea is to find a minimizing Cerami sequence for the energy functional outside the Nehari–Pankov manifold $\mathcal{N}^-$ by using the diagonal method.

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1. Introduction

Consider the following semilinear Schrödinger equation:

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions, respectively:

(V) $V \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and

$$\sup[\sigma(-\triangle + V) \cap (-\infty, 0)] := \underline{\Lambda} < 0 < \overline{\Lambda} := \inf[\sigma(-\triangle + V) \cap (0, \infty)];$$

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(F1) $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, $F(x,t) := \int_0^t f(x,s)\,ds \geq 0$ and $f(x,t) = o(|t|)$, as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(F2) $f(x,t) = V_\infty(x)t + f_\infty(x,t)$, where $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and $\inf V_\infty > \bar{\Lambda}$, $f_\infty(x,t) = o(|t|)$ as $|t| \to \infty$, uniformly in $x \in \mathbb{R}^N$.

Let $\mathcal{A} = -\triangle + V$. Then $\mathcal{A}$ is self adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^N)$ (see [4, Theorem 4.26]). Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [3, Theorem IV 3.3]). Let

$$E = \mathfrak{D}(|\mathcal{A}|^{1/2}), \quad E^- = \mathcal{E}(0)E, \quad E^+ = [\mathrm{id} - \mathcal{E}(0)]E.$$

For any $u \in E$, it is easy to see that $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0)u \in E^-, \quad u^+ := [\mathrm{id} - \mathcal{E}(0)]u \in E^+$$

and

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|u^+ \quad \forall u \in E \cap \mathfrak{D}(\mathcal{A}). \tag{1.2}$$

Define an inner product

$$(u,v) = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_{L^2}, \quad u, v \in E$$

and the corresponding norm

$$\|u\| = \| \, |\mathcal{A}|^{1/2}u\|_2, \quad u \in E, \tag{1.3}$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^N)$, $\|\cdot\|_s$ denoting the norm of $L^s(\mathbb{R}^N)$. By (V), $E = H^1(\mathbb{R}^N)$ with equivalent norms. Therefore, E embeds continuously in $L^s(\mathbb{R}^N)$ for all $2 \leq s \leq 2^*$. In addition, one has the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

Under assumptions (V), (F1) and (F2), the solutions of problem (1.1) are critical points of the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx \quad \forall u \in E;$$

Φ is of class $C^1(E,\mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx \quad \forall u, v \in E. \tag{1.4}$$

In view of (1.2) and (1.3),

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N} F(x,u)\,dx \tag{1.5}$$

and

$$\langle \Phi'(u), u\rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{\mathbb{R}^N} f(x,u)u\,dx \quad \forall u = u^- + u^+ \in E.$$

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In [11], Szulkin and Zou firstly studied problem (1.1) with asymptotically linear term f by variational methods, and proved the existence of a nontrivial solution provided that V and f satisfy (V), (F1), (F2) and the following weak assumption.

(F3) $\tilde{F}(x,t) := \frac{1}{2}tf(x,t) - F(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and there exists a constant $\delta_0 \in (0, \lambda_0)$ such that if $f(x,t)/t \geq \lambda_0 - \delta_0$ then $\tilde{F}(x,t) \geq \delta_0$, where $\lambda_0 = \min\{-\underline{\Lambda}, \overline{\Lambda}\}$.

Similar results can be found in [7]. We point out that condition (F3) was firstly used by Jeanjean [5]. Under the same assumptions given above, moreover with $f(x,t)$ odd in t, Ding and Lee [1, 2] proved that problem (1.1) has infinitely many geometrically distinct solutions.

If $u_0 \in E$ is a nontrivial solution of problem (1.1), then $u_0 \in \mathcal{N}^-$, where

$$\mathcal{N}^- = \{u \in E \setminus E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0 \quad \forall v \in E^-\}.$$

The set $\mathcal{N}^-$, first introduced by Pankov [9], is a subset of the Nehari manifold

$$\mathcal{N} = \{u \in E \setminus \{0\} : \langle \Phi'(u), u\rangle = 0\}.$$

In general, the set $\mathcal{N}^-$ contains infinitely many elements of E. In fact, we can demonstrate that for any $u \in [\mathcal{E}(\mu_1) - \mathcal{E}(\overline{\Lambda})]E \setminus \{0\}$ with $\mu_1 = \inf V_\infty$, there exist $t = t(u) > 0$ and $w = w(u) \in E^-$ such that $w + tu \in \mathcal{N}^-$; see Lemma 2.12.

Set $m := \inf_{u \in \mathcal{N}^-} \Phi(u)$. Now a natural question arises: whether m is attained? or whether there exists $\bar{u} \in \mathcal{N}^-$ such that $\Phi'(\bar{u}) = 0$ and $\Phi(\bar{u}) = m$? Since $\bar{u}$ is a solution at which Φ has the least 'energy' in the set $\mathcal{N}^-$, we shall call it a ground state solution of Nehari–Pankov type. In [10], based on the Nehari-manifold method, Szulkin and Weth developed an approach to find ground state solutions of Nehari–Pankov type for problem (1.1) with superlinear term f. In fact, they proved that problem (1.1) has a solution $\bar{u} \in \mathcal{N}^-$ such that $\Phi(\bar{u}) = m > 0$ provided that (V) and the Nehari-type assumption

(Ne) $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$ and some other standard assumptions on f are satisfied.

We point out that the Nehari-type assumption (Ne) is very crucial in Szulkin and Weth [10].

In this paper, we will develop a direct approach to find ground state solutions of Nehari–Pankov type for problem (1.1) with asymptotically linear term f. The main idea is to find a minimizing Cerami sequence for Φ outside $\mathcal{N}^-$ by using the diagonal method, part of which comes from recent papers [12–15] of the author; see Lemma 2.13. From the work of Szulkin and Weth [10], it seems very difficult to find ground state solutions of Nehari–Pankov type for problem (1.1), but this can be made more concise by using our approach.

Before presenting our theorem, in addition to (V) and (F1), we make the following assumptions.

(F2′) $f(x,t) = V_\infty(x)t + f_\infty(x,t)$, where $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots x_N$ and $\inf V_\infty > 0$, and there exists a $u_0 \in E^+ \setminus \{0\}$ such that

$$\|u_0\|^2 - \|w\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(u_0+w)^2\,dx < 0 \quad \forall w \in E^-; \tag{1.6}$$

$tf_\infty(x,t) \le 0$, $f(x,t)f_\infty(x,t) < 0$ for $0 < |t| \le \alpha_0$ for some $\alpha_0 > 0$, $f_\infty(x,t) = o(|t|)$ as $|t| \to \infty$, uniformly in $x \in \mathbb{R}^N$;

(F2″) $f(x,t) = V_\infty(x)t + f_\infty(x,t)$, where $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and $\inf V_\infty > \bar\Lambda$, $f_\infty(x,t) = o(|t|)$ as $|t| \to \infty$, uniformly in $x \in \mathbb{R}^N$, and $0 < tf(x,t) < V_\infty(x)t^2$ for $x \in \mathbb{R}^N$ and $t \ne 0$;

(WN) $t \mapsto f(x,t)/|t|$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$.

Obviously, if $\inf V_\infty > \bar\Lambda$, take $\bar\mu \in (\bar\Lambda, \inf V_\infty)$; then $u_0 \in (\mathcal{E}(\bar\mu) - \mathcal{E}(0))E$ satisfies (1.6). Therefore, (F2″) implies (F2′); (WN) is a weak version of the Nehari-type assumption (Ne), and it yields that $\tilde F(x,t)$ is nondecreasing on $t \in [0,\infty)$ and nonincreasing on $t \in (-\infty, 0]$. Hence, it is slightly stronger than (F3).

We are now in a position to state the main results of this paper.

Theorem 1.1. *Assume that V and f satisfy (V), (F1), (F2′) and (WN). Then problem (1.1) has a solution $\bar u \in E$ such that $\Phi(\bar u) = \inf_{\mathcal{N}^-} \Phi > 0$. Moreover,*

$$\int_{\mathbb{R}^N} \{|\nabla \bar u|^2 + [V(x) - V_\infty(x)]\bar u^2\}\,dx < 0.$$

Corollary 1.2. *Assume that V and f satisfy (V), (F1), (F2″) and (WN). Then problem (1.1) has a solution $\bar u \in E$ such that $\Phi(\bar u) = \inf_{\mathcal{N}^-} \Phi > 0$. Moreover,*

$$\int_{\mathbb{R}^N} \{|\nabla \bar u|^2 + [V(x) - V_\infty(x)]\bar u^2\}\,dx < 0.$$

We point out that, as a consequence of Theorem 1.1, the least energy value m has a minimax characterization given by

$$m = \Phi(\bar u) = \inf_{v \in E_0^+ \setminus \{0\}} \max_{u \in E^- \oplus \mathbb{R}^+ v} \Phi(u),$$

where E_0^+ is defined by (2.6). Note that this minimax principle is much simpler than the usual characterizations related to the concept of linking.

The following functions satisfy all assumptions of Corollary 1.2.

Example 1.3. $f(x,t) = V_\infty(x)\min\{|t|^\varrho, 1\}t$, where $\varrho > 0$ and $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and $\inf V_\infty > \bar\Lambda$.

Example 1.4. $f(x,t) = V_\infty(x)[1 - (1/\ln(e + |t|))]t$, where $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and $\inf V_\infty > \bar\Lambda$.

Example 1.5. $f(x,t) = \zeta(x,|t|)t$, where $\zeta(x,s)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and nondecreasing for $s \in [0,\infty)$, $\zeta(x,s) \to 0$ as $s \to 0$ and $\zeta(x,s) \to V_\infty(x)$ as $s \to \infty$, uniformly in $x \in \mathbb{R}^N$, $\inf V_\infty > \bar\Lambda$ and $0 < \zeta(x,s) < V_\infty(x)$ for $s \ne 0$.

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2. Proof of main result

Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. For a functional $\varphi \in C^1(X, \mathbb{R})$, φ is said to be weakly sequentially lower semicontinuous if for any $u_n \rightharpoonup u$ in X one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$, and φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for each $v \in X$.

Lemma 2.1 ([1, Theorem 4.5], [6], [7, Theorem 2.1]). *Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X, \mathbb{R})$ be of the form*

$$\varphi(u) = \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

- (LS1) *$\psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semicontinuous;*
- (LS2) *ψ' is weakly sequentially continuous;*
- (LS3) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{w + se : w \in X^-, s \ge 0, \|w + se\| \le r\}.$$

Then, for some $c \in [\kappa, \sup \Phi(Q)]$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$ sequence.

We set

$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u)\, dx \quad \forall u \in E.$$

Lemma 2.2. *Suppose that (F1) and (F2′) are satisfied. Then Ψ is nonnegative, weakly sequentially lower semicontinuous, and Ψ' is weakly sequentially continuous.*

Using Sobolev's embedding theorem, one can check the above lemma easily, so we omit the proof.

The following lemma is interesting and shows an important behavior of nondecreasing functions.

Lemma 2.3. *Suppose that $h(x, t)$ is nondecreasing in $t \in \mathbb{R}$ and $h(x, 0) = 0$ for any $x \in \mathbb{R}^N$. Then*

$$\Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big)h(x, \tau)|\tau| \ge \int_{\theta\tau+\sigma}^{\tau} h(x, s)|s|\, ds \quad \forall \theta \ge 0, \tau, \sigma \in \mathbb{R}. \tag{2.1}$$

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Proof. Since $h(x,t)$ is nondecreasing in $t \in \mathbb{R}$, then, for any $x \in \mathbb{R}^N$,

$$h(x,s) \le h(x,\tau) \quad \forall s \le \tau; \quad h(x,s) \ge h(x,\tau) \quad \forall s \ge \tau. \tag{2.2}$$

To show (2.1), we consider four possible cases. Since $sh(x,s) \ge 0$, these follow from (2.2):

Case 1. If $0 \le \theta\tau + \sigma \le \tau$ or $\theta\tau + \sigma \le \tau \le 0$, then

$$\int_{\theta\tau+\sigma}^{\tau} h(x,s)|s|\,ds \le h(x,\tau)\int_{\theta\tau+\sigma}^{\tau} |s|\,ds \le \Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big) h(x,\tau)|\tau|;$$

Case 2. If $\theta\tau + \sigma \le 0 \le \tau$, then

$$\begin{aligned}\int_{\theta\tau+\sigma}^{\tau} h(x,s)|s|\,ds \le \int_{0}^{\tau} h(x,s)|s|\,ds \le h(x,\tau)\int_{0}^{\tau} |s|\,ds \\ \le \Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big) h(x,\tau)|\tau|;\end{aligned}$$

Case 3. If $0 \le \tau \le \theta\tau + \sigma$ or $\tau \le \theta\tau + \sigma \le 0$, then

$$\int_{\tau}^{\theta\tau+\sigma} h(x,s)|s|\,ds \ge h(x,\tau)\int_{\tau}^{\theta\tau+\sigma} |s|\,ds \ge -\Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big) h(x,\tau)|\tau|;$$

Case 4. If $\tau \le 0 \le \theta\tau + \sigma$, then

$$\begin{aligned}\int_{\tau}^{\theta\tau+\sigma} h(x,s)|s|\,ds \ge \int_{\tau}^{0} h(x,s)|s|\,ds \ge h(x,\tau)\int_{\tau}^{0} |s|\,ds \\ \ge -\Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big) h(x,\tau)|\tau|.\end{aligned}$$

The above four cases show that (2.1) holds. □

Lemma 2.4. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then*

$$\Phi(u) \ge \Phi(\theta u + w) + \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(u), u\rangle - \theta\langle \Phi'(u), w\rangle \qquad \forall \theta \ge 0, u \in E, w \in E^-. \tag{2.3}$$

Proof. For any $x \in \mathbb{R}^N$, it follows from (WN) and Lemma 2.3 that

$$\Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big) f(x,\tau) \ge \int_{\theta\tau+\sigma}^{\tau} f(x,s)\,ds \quad \forall \theta \ge 0, \tau, \sigma \in \mathbb{R}. \tag{2.4}$$

By virtue of (1.4), (1.5) and (2.4),

$$\begin{aligned}\Phi(u) - \Phi(\theta u + w) &= \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(u), u\rangle - \theta\langle \Phi'(u), w\rangle \\ &\quad + \int_{\mathbb{R}^N}\Big[\frac{1-\theta^2}{2} f(x,u)u - \theta f(x,u)w - \int_{\theta u+w}^{u} f(x,s)\,ds\Big]dx \\ &\ge \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(u), u\rangle - \theta\langle \Phi'(u), w\rangle \quad \forall \theta \ge 0, w \in E^-.\end{aligned}$$

This shows that (2.3) holds. □

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From Lemma 2.4, we have the following two corollaries.

COROLLARY 2.5. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then, for $u \in \mathcal{N}^-$,*

$$\Phi(u) \geq \Phi(tu + w) \quad \forall t \geq 0, w \in E^-.$$

COROLLARY 2.6. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then*

$$\Phi(u) \geq \frac{t^2}{2}(\|u^+\|^2 + \|u^-\|^2) - \int_{\mathbb{R}^N} F(x, tu^+)\,dx + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle + t^2\langle \Phi'(u), u^-\rangle \quad \forall u \in E, t \geq 0.$$

The following lemma is crucial to obtain a linking structure.

LEMMA 2.7. *Suppose that (V), (F1) and (F2′) are satisfied. If* $\inf V_\infty > 0$, *then*

$$\begin{aligned}
&\tau\langle \Phi'(u), \tau u + 2v\rangle \\
&\quad \geq \tau^2\|u^+\|^2 - \|\tau u^- + v\|^2 + \|v\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(\tau u + v)^2\,dx \\
&\qquad + \tau^2 \int_{\mathbb{R}^N} \frac{uf(x,u)V_\infty(x) - [f(x,u)]^2}{V_\infty(x)}\,dx \quad \forall u \in E, \tau \in \mathbb{R}, v \in E^-. \qquad (2.5)
\end{aligned}$$

PROOF. In view of (1.2), (1.3), (1.4) and $\inf V_\infty > 0$,

$$\begin{aligned}
&\tau\langle \Phi'(u), \tau u + 2v\rangle \\
&\quad = \tau^2\|u^+\|^2 - \tau^2\|u^-\|^2 - 2\tau(u^-, v) - \tau\int_{\mathbb{R}^N} f(x,u)(\tau u + 2v)\,dx \\
&\quad = \tau^2\|u^+\|^2 - \|\tau u^- + v\|^2 + \|v\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(\tau u + v)^2\,dx \\
&\qquad + \int_{\mathbb{R}^N} [V_\infty(x)(\tau u + v)^2 - \tau f(x,u)(\tau u + 2v)]\,dx \\
&\quad = \int_{\mathbb{R}^N} \{V_\infty(x)v^2 + 2[V_\infty(x)u - f(x,u)]\tau v + [V_\infty(x)u^2 - uf(x,u)]\tau^2\}\,dx \\
&\qquad + \tau^2\|u^+\|^2 - \|\tau u^- + v\|^2 + \|v\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(\tau u + v)^2\,dx \\
&\quad \geq \tau^2\|u^+\|^2 - \|\tau u^- + v\|^2 + \|v\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(\tau u + v)^2\,dx \\
&\qquad + \tau^2\int_{\mathbb{R}^N} \frac{uf(x,u)V_\infty(x) - [f(x,u)]^2}{V_\infty(x)}\,dx \quad \forall u \in E, \tau \in \mathbb{R}, v \in E^-,
\end{aligned}$$

which shows that (2.5) holds. □

COROLLARY 2.8. *Suppose that (V), (F1) and (F2′) are satisfied, and that* $\inf V_\infty > 0$. *Then*

$$\begin{aligned}
&\|u^+\|^2 - \|u^- + v\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(u + v)^2\,dx \\
&\quad \leq -\|v\|^2 - \int_{\mathbb{R}^N} \frac{uf(x,u)V_\infty(x) - [f(x,u)]^2}{V_\infty(x)}\,dx \qquad \forall u \in \mathcal{N}^-, v \in E^-.
\end{aligned}$$

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Applying Corollary 2.5, we can prove the following lemma in the same way as [10, Lemma 2.4].

Lemma 2.9. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then:*

(i) *there exists* $\rho > 0$ *such that*

$$m = \inf_{\mathcal{N}^-} \Phi \geq \kappa := \inf\{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0;$$

(ii) $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2m}\}$ *for all* $u \in \mathcal{N}^-$.

Define a set E_0^+ as follows:

$$E_0^+ = \Big\{u \in E^+ \setminus \{0\} : \|u\|^2 - \|w\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(u + w)^2\, dx < 0 \quad \forall w \in E^-\Big\}. \tag{2.6}$$

Obviously, (F2′) shows that the set E_0^+ is not empty.

Lemma 2.10. *Suppose that (V), (F1) and (F2′) are satisfied. Then, for any* $e \in E_0^+$, $\sup \Phi(E^- \oplus \mathbb{R}^+ e) < \infty$ *and there is* $R_e > 0$ *such that*

$$\Phi(u) \leq 0 \quad \forall u \in E^- \oplus \mathbb{R}^+ e, \|u\| \geq R_e.$$

This result is essentially contained in [11], see also [7, Lemma 3.1], but for the reader's convenience we choose to write it in detail.

Proof. It is sufficient to show that $\Phi(w + te) \leq 0$ for $t \geq 0$, $w \in E^-$ and $\|w + te\| \geq R$ for large $R > 0$. Arguing indirectly, assume that, for some sequence $\{w_n + s_n e\} \subset E^- \oplus \mathbb{R}^+ e$ with $\|w_n + s_n e\| \to \infty$, $\Phi(w_n + s_n e) \geq 0$ for all $n \in \mathbb{N}$. Set

$$v_n = \|w_n + t_n e\|/(w_n + t_n e) = v_n^- + \tau_n e;$$

then $\|v_n^- + \tau_n e\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E (v_n weakly convergences to v in E); then $v_n \to v$ almost everywhere on $\mathbb{R}^N$, $v_n^- \rightharpoonup v^-$ in E, $\tau_n \to \tau$ and

$$\frac{\Phi(w_n + t_n e)}{\|w_n + t_n e\|^2} = \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, w_n + t_n e)}{\|w_n + t_n e\|^2}\, dx \geq 0. \tag{2.7}$$

Clearly, (2.7) yields that $\tau > 0$. Since $e \in E_0^+$, there exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$\tau^2\|e\|^2 - \|v^-\|^2 - \int_\Omega V_\infty(x)(\tau e + v^-)^2\, dx < 0. \tag{2.8}$$

Let $F_\infty(x, t) = \int_0^t f_\infty(x, s)\, ds$; then $F(x, t) = \frac{1}{2} V_\infty(x) t^2 + F_\infty(x, t)$. It follows from (2.7) that

$$\begin{aligned} 0 &\leq \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega \frac{F(x, w_n + t_n e)}{\|w_n + t_n e\|^2}\, dx \\ &= \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2}\int_\Omega V_\infty(x) v_n^2\, dx - \int_\Omega \frac{F_\infty(x, w_n + t_n e)}{\|w_n + t_n e\|^2}\, dx. \end{aligned}$$

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Clearly, $|F_\infty(x,t)| \le c_0 t^2$ for some $c_0 > 0$ and $F_\infty(x,t)/t^2 \to 0$ as $|t| \to \infty$. Since $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^2(\Omega)$ and it is easy to see from the Lebesgue dominated convergence theorem that

$$\int_\Omega \frac{F_\infty(x, w_n + t_n e)}{\|w_n + t_n e\|^2}\, dx = \int_\Omega \frac{F_\infty(x, w_n + t_n e)}{|w_n + t_n e|^2} |v_n|^2\, dx = o(1).$$

Hence,

$$0 \le \tau^2 \|e\|^2 - \|v^-\|^2 - \int_\Omega V_\infty(x)(\tau e + v^-)^2\, dx,$$

which is a contradiction to (2.8). □

Corollary 2.11. *Suppose that (V), (F1) and (F2′) are satisfied. Let $e \in E_0^+$ with $\|e\| = 1$. Then there is a $r_0 > \rho$ such that $\sup \Phi(\partial Q) \le 0$ for $r \ge r_0$, where*

$$Q = \{w + se : w \in E^-, s \ge 0, \ \|w + se\| \le r\}. \tag{2.9}$$

Lemma 2.12. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then, for any $u \in E_0^+$, $\mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ u) \ne \emptyset$, that is, there exist $t(u) > 0$ and $w(u) \in E^-$ such that $t(u)u + w(u) \in \mathcal{N}^-$.*

Proof. By view of Lemma 2.10, there exists a constant $R > 0$ such that $\Phi(v) \le 0$ for $v \in (E^- \oplus \mathbb{R}^+ u) \setminus B_R(0)$. By Lemma 2.9(i), $\Phi(tu) > 0$ for small $t > 0$. Thus we have, $0 < \sup \Phi(E^- \oplus \mathbb{R}^+ u) < \infty$. It is easy see that Φ is weakly upper semicontinuous on $E^- \oplus \mathbb{R}^+ u$; therefore, $\Phi(u_0) = \sup \Phi(E^- \oplus \mathbb{R}^+ u)$ for some $u_0 \in E^- \oplus \mathbb{R}^+ u$. This u_0 is a critical point of $\Phi|_{E^- \oplus \mathbb{R} u}$, so $\langle \Phi'(u_0), u_0 \rangle = \langle \Phi'(u_0), v \rangle = 0$ for all $v \in E^- \oplus \mathbb{R}\, u$. Consequently, $u_0 \in \mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ u)$. □

Lemma 2.13. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then there exist a constant $c \in [\kappa, \sup \Phi(Q)]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0,$$

where Q is defined by (2.9).

Proof. Lemma 2.13 is a direct corollary of Lemmas 2.1, 2.2, 2.9(i) and Corollary 2.11. □

The following lemma is crucial to demonstrate the existence of ground state solutions of Nehari–Pankov type for problem (1.1).

Lemma 2.14. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then there exist a constant $c_* \in [\kappa, m]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.10}$$

Proof. Choose $v_k \in \mathcal{N}^-$ such that

$$m \le \Phi(v_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.11}$$

By Lemma 2.9, $\|v_k^+\| \ge \sqrt{2m} > 0$. Since $v_k \in H^1(\mathbb{R}^N)$, $\text{meas}\{x \in \mathbb{R}^N : |v_k(x)| \le \alpha_0\} = \infty$. It follows from (F2′) that

$$\int_{\mathbb{R}^N} \frac{f(x, v_k) f_\infty(x, v_k)}{V_\infty(x)}\, dx < 0. \tag{2.12}$$

Set $e_k = v_k^+/\|v_k^+\|$. Then $e_k \in E^+$ and $\|e_k\| = 1$. By virtue of Corollary 2.8 and (2.12),

$$\begin{aligned}
&\|e_k\|^2 - \|w\|^2 - \int_{\mathbb{R}^N} V_\infty(x)(e_k + w)^2\, dx \\
&= \frac{\|v_k^+\|^2}{\|v_k^+\|^2} - \|w\|^2 - \int_{\mathbb{R}^N} V_\infty(x)\Big(\frac{v_k}{\|v_k^+\|} + w - \frac{v_k^-}{\|v_k^+\|}\Big)^2 dx \\
&\le -\Big\|w - \frac{v_k^-}{\|v_k^+\|}\Big\|^2 - \frac{1}{\|v_k^+\|^2} \int_{\mathbb{R}^N} \frac{v_k f(x, v_k) V_\infty(x) - [f(x, v_k)]^2}{V_\infty(x)}\, dx \\
&= -\Big\|w - \frac{v_k^-}{\|v_k^+\|}\Big\|^2 + \frac{1}{\|v_k^+\|^2} \int_{\mathbb{R}^N} \frac{f(x, v_k) f_\infty(x, v_k)}{V_\infty(x)}\, dx < 0 \quad \forall w \in E^-.
\end{aligned}$$

This shows that $e_k \in E_0^+$. In view of Corollary 2.11, there exists $r_k > \max\{\rho, \|v_k\|\}$ such that $\sup \Phi(\partial Q_k) \le 0$, where

$$Q_k = \{w + s e_k : w \in E^-, s \ge 0, \|w + s e_k\| \le r_k\}, \quad k \in \mathbb{N}. \tag{2.13}$$

Hence, applying Lemma 2.13 to the above set Q_k, there exist a positive constant $c_k \in [\kappa, \sup \Phi(Q_k)]$ and a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$\Phi(u_{k,n}) \to c_k, \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{2.14}$$

By virtue of Corollary 2.5,

$$\Phi(v_k) \ge \Phi(w + t v_k) \quad \forall t \ge 0, w \in E^-. \tag{2.15}$$

Since $v_k \in Q_k$, it follows from (2.13) and (2.15) that $\Phi(v_k) = \sup \Phi(Q_k)$. Hence, by (2.11) and (2.14),

$$\Phi(u_{k,n}) \to c_k < m + \frac{1}{k}, \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}.$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\kappa - \frac{1}{k} < \Phi(u_{k,n_k}) < m + \frac{1}{k}, \quad \|\Phi'(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence,

$$\Phi(u_n) \to c_* \in [\kappa, m], \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \qquad \square$$

Lemma 2.15. *Suppose that (V), (F1), (F2′) and (WN) are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying (2.10) is bounded in E.*

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$; then $\|v_n\| = 1$. By the Sobolev embedding theorem, there exists a constant $C_1 > 0$ such that $\|v_n\|_2 \le C_1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n^+|^2\,dx = 0,$$

then, by Lions' concentration compactness principle [8] or [16, Lemma 1.21], $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Fix $R > [2(1 + c^*)]^{1/2}$ and $p \in (2, 2^*)$. By virtue of (F1) and (F2′), for $\varepsilon = 1/4(RC_1)^2 > 0$, there exists $C_\varepsilon > 0$ such that $|F(x,t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^p$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Hence,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n^+)\,dx \le \varepsilon(RC_1)^2 + R^pC_\varepsilon \lim_{n\to\infty} \|v_n^+\|_p^p = \frac{1}{4}. \tag{2.16}$$

Let $t_n = R/\|u_n\|$. Hence, by virtue of (2.10), (2.16) and Corollary 2.5,

$$\begin{aligned}
c^* + o(1) = \Phi(u_n) &\ge \frac{t_n^2}{2}(\|u_n^+\|^2 + \|u_n^-\|^2) - \int_{\mathbb{R}^N} F(x, t_nu_n^+)\,dx + \frac{1-t_n^2}{2}\langle\Phi'(u_n), u_n\rangle \\
&\quad + t_n^2\langle\Phi'(u_n), u_n^-\rangle \\
&= \frac{R^2}{2}(\|v_n^+\|^2 + \|v_n^-\|^2) - \int_{\mathbb{R}^N} F(x, Rv_n^+)\,dx + \Big(\frac{1}{2} - \frac{R^2}{2\|u_n\|^2}\Big)\langle\Phi'(u_n), u_n\rangle \\
&\quad + \frac{R^2}{\|u_n\|^2}\langle\Phi'(u_n), u_n^-\rangle \\
&= \frac{R^2}{2} - \int_{\mathbb{R}^N} F(x, Rv_n^+)\,dx + o(1) \ge \frac{R^2}{2} - \frac{1}{4} + o(1) > c^* + \frac{3}{4} + o(1),
\end{aligned}$$

which implies that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |v_n^+|^2\,dx > \delta/2$. Let $\tilde{v}_n(x) = v_n(x + k_n)$. Then $\|\tilde{v}_n\| = \|v_n\| = 1$ and

$$\int_{B_{1+\sqrt{N}}(0)} |\tilde{v}_n^+|^2\,dx > \frac{\delta}{2}. \tag{2.17}$$

Passing to a subsequence, we have $\tilde{v}_n \rightharpoonup \tilde{v}$ in E, $\tilde{v}_n \to \tilde{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $\tilde{v}_n \to \tilde{v}$ almost everywhere on $\mathbb{R}^N$. Thus, (2.17) implies that $\tilde{v}^+ \ne 0$ and so $\tilde{v} \ne 0$.

Now we define $\tilde{u}_n(x) = u_n(x + k_n)$; then $\tilde{u}_n/\|u_n\| = \tilde{v}_n \to \tilde{v}$ almost everywhere on $\mathbb{R}^N$, $\tilde{v} \ne 0$. For $x \in \Omega := \{y \in \mathbb{R}^N : \tilde{v}(y) \ne 0\}$, we have $\lim_{n\to\infty} |\tilde{u}_n(x)| = \infty$. For any $\phi \in C_0^\infty(\mathbb{R}^N)$, setting $\phi_n(x) = \phi(x - k_n)$,

$$\begin{aligned}
\langle\Phi'(u_n), \phi_n\rangle &= (u_n^+ - u_n^-, \phi_n) - (V_\infty u_n, \phi_n)_{L^2} - \int_{\mathbb{R}^N} f_\infty(x, u_n)\phi_n\,dx \\
&= \|u_n\|\Big[(v_n^+ - v_n^-, \phi_n) - (V_\infty v_n, \phi_n)_{L^2} - \int_{\mathbb{R}^N} \frac{f_\infty(x, u_n)}{|u_n|}|v_n|\phi_n\,dx\Big] \\
&= \|u_n\|\Big[(\tilde{v}_n^+ - \tilde{v}_n^-, \phi) - (V_\infty\tilde{v}_n, \phi)_{L^2} - \int_{\mathbb{R}^N} \frac{f_\infty(x, \tilde{u}_n)}{|\tilde{u}_n|}|\tilde{v}_n|\phi\,dx\Big],
\end{aligned}$$

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which, together with (2.10), yields

$$(\tilde v_n^+ - \tilde v_n^-, \phi) - (V_\infty \tilde v_n, \phi)_{L^2} - \int_{\mathbb{R}^N} \frac{f_\infty(x, \tilde u_n)}{|\tilde u_n|} |\tilde v_n| \phi \, dx = o(1).$$

Note that

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} \frac{f_\infty(x, \tilde u_n)}{|\tilde u_n|} |\tilde v_n| \phi \, dx\Big| &\le \int_{\mathbb{R}^N} \Big|\frac{f_\infty(x, \tilde u_n)}{\tilde u_n}\Big| |\tilde v_n - \tilde v| \, |\phi| \, dx + \int_{\mathbb{R}^N} \Big|\frac{f_\infty(x, \tilde u_n)}{\tilde u_n}\Big| |\tilde v| \, |\phi| \, dx \\ &\le C_2 \int_{\operatorname{supp}\phi} |\tilde v_n - \tilde v| \, |\phi| \, dx + \int_{\Omega} \Big|\frac{f_\infty(x, \tilde u_n)}{\tilde u_n}\Big| |\tilde v| |\phi| \, dx = o(1).\end{aligned}$$

Hence,

$$(\tilde v^+ - \tilde v^-, \phi) - (V_\infty \tilde v, \phi)_{L^2} = 0.$$

Thus, $\tilde v$ is an eigenfunction of the operator $\mathcal{B} := -\triangle + (V - V_\infty)$, contradicting the fact that $\mathcal{B}$ has only a continuous spectrum. This contradiction shows that $\{u_n\}$ is bounded. □

Proof of Theorem 1.1. Applying Lemmas 2.14 and 2.15, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (2.10). A standard argument shows that $\{u_n\}$ is a nonvanishing sequence. Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^2 \, dx > \delta/2$ for some $\delta > 0$. Let $v_n(x) = u_n(x + k_n)$. Then

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^2 \, dx > \frac{\delta}{2}. \tag{2.18}$$

Since $V(x)$ and $f(x, u)$ are periodic in x, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c_*, \quad \|\Phi'(v_n)\|(1 + \|v_n\|) \to 0. \tag{2.19}$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\text{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $v_n \to v$ almost everywhere on $\mathbb{R}^N$. Obviously, (2.18) and (2.19) imply that $v \ne 0$ and $\Phi'(v) = 0$. This shows that $v \in \mathcal{N}^-$ and so $\Phi(v) \ge m$. On the other hand, by using (2.19), (WN) and Fatou's lemma,

$$\begin{aligned} m &\ge c_* \\ &= \lim_{n\to\infty}\Big[\Phi(v_n) - \frac{1}{2}\langle \Phi'(v_n), v_n\rangle\Big] = \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{2} f(x, v_n) v_n - F(x, v_n)\Big] dx \\ &\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \Big[\frac{1}{2} f(x, v_n) v_n - F(x, v_n)\Big] dx = \int_{\mathbb{R}^N} \Big[\frac{1}{2} f(x, v) v - F(x, v)\Big] dx \\ &= \Phi(v) - \frac{1}{2}\langle \Phi'(v), v\rangle = \Phi(v). \end{aligned}$$

This shows that $\Phi(v) \le m$ and so $\Phi(v) = m = \inf_{\mathcal{N}^-} \Phi > 0$. □

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References

[1] Y. Ding, *Variational Methods for Strongly Indefinite Problems* (World Scientific, Singapore, 2007).

[2] Y. Ding and C. Lee, 'Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms', *J. Differential Equations* **222** (2006), 137–163.

[3] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators* (Clarendon Press, Oxford, 1987).

[4] Y. Egorov and V. Kondratiev, *On Spectral Theory of Elliptic Operators* (Birkhäuser, Basel, 1996).

[5] J. Jeanjean, 'On the existence of bounded Palais–Smale sequence and application to a Landesman–Lazer-type problem on $\mathbb{R}^N$', *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), 787–809.

[6] W. Kryszewski and A. Szulkin, 'Generalized linking theorem with an application to a semilinear Schrödinger equation', *Adv. Differential Equations* **3** (1998), 441–472.

[7] G. B. Li and A. Szulkin, 'An asymptotically periodic Schrödinger equation with indefinite linear part', *Commun. Contemp. Math.* **4** (2002), 763–776.

[8] P. L. Lions, 'The concentration–compactness principle in the calculus of variations. The locally compact case, part 2', *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), 223–283.

[9] A. Pankov, 'Periodic nonlinear Schrödinger equation with application to photonic crystals', *Milan J. Math.* **73** (2005), 259–287.

[10] A. Szulkin and T. Weth, 'Ground state solutions for some indefinite variational problems', *J. Funct. Anal.* **257** (2009), 3802–3822.

[11] A. Szulkin and W. M. Zou, 'Homoclinic orbits for asymptotically linear Hamiltonian systems', *J. Funct. Anal.* **187** (2001), 25–41.

[12] X. H. Tang, 'New conditions on nonlinearity for a periodic Schrödinger equation having zero as spectrum', *J. Math. Anal. Appl.* **413** (2014), 392–410.

[13] X. H. Tang, 'New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation', *Adv. Nonlinear Stud.* **14** (2014), 361–374.

[14] X. H. Tang, 'Non-Nehari manifold method for superlinear Schrödinger equation', *Taiwanese J. Math.* **18** (2014), 1950–1972.

[15] X. H. Tang, Ground state solutions of Nehari–Pankov type for asymptotically periodic Schrödinger equation, Preprint, arXiv:1405.2607v1 [math.AP], 12 May 2014.

[16] M. Willem, *Minimax Theorems* (Birkhäuser, Boston, MA, 1996).

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