

ON A SHEAF REPRESENTATION OF A CLASS OF NEAR-RINGS

GEORGE SZETO

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Abstract

It is proved that R is a near-ring with identity in which every element is a power of itself if and only if it is isomorphic with a near-ring of sections of a sheaf of near-fields in which every element is a power of itself. We also obtain that the Boolean spectrum is homeomorphic with the space of all completely prime ideals of R with the Zariski topology.

1. Introduction

Grothendieck (1960) proved that a commutative ring is isomorphic with the ring of sections of a sheaf of local rings. Then Dauns and Hofmann (1966), Koh (1971), Lambek (1971) and Pierce (1967) obtained sheaf representations for different classes of rings (commutative and noncommutative) and modules. A lot of applications of sheaf representations of an algebraic structure have been found (Bergman (1971), DeMeyer (1972), Magid (1971), Pierce (1967), and Villamayor and Zelinsky (1969)). The purpose of the present paper is to show a sheaf representation for a class of near-rings in which every element is a power of itself. The main result is the following: R is a near-ring with identity in which every element is a power of itself if and only if it is isomorphic with a near-ring of sections of a sheaf of near-fields in which every element is a power of itself.

2. Preliminaries

Throughout, we assume that R is a near-ring with identity 1 such that $0r = 0$ for all r in R , and that every nonzero element is a power of itself where the power is greater than 1. For such an R , Bell (1970) has found the following useful properties:

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- (1) $(R, +)$ is a commutative group (Bell (1970), Theorem 1).
- (2) All idempotents are central (Bell (1970), Lemma 2).
- (3) R contains a family of completely prime ideals with trivial intersection (Bell (1970), Lemma 3).

3. Ideals and topological spaces

Let Π be the set of completely prime ideals of R (it is not a void set from (3) in Section 2), and let $\Gamma(I) = \{p \in \Pi \mid I \not\subseteq p\}$. We shall show that Π is a topological space with the basic open set, $\{\Gamma(I) \text{ for all ideals } I\}$. From (1) and (2) in Section 2, the set of idempotents $B(R)$ of R are central in R , so $B(R)$ is a commutative ring with identity 1. Hence $B(R)$ is a Boolean ring under the multiplication $e \cdot e' = ee'$ and addition $e + e' = e + e' - ee'$, and, as usual, the set of maximal ideals of $B(R)$ is a topological space $\text{Spec } B(R)$ with kernel-hull topology. It is well known that $\text{Spec } B(R)$ is totally disconnected, compact and Hausdorff. In this section, we shall show that the two topological spaces Π and $\text{Spec } B(R)$ are homeomorphic. We begin with some properties of ideals of R . Recall that I is an ideal of the near-ring R if (1) $(I, +)$ is a normal subgroup of $(R, +)$, (2) $RI \subset I$, and (3) $(x + a)y - xy$ is in I for all x, y in R and a in I .

LEMMA 3.1. (1) *If I and I' are ideals of R then II' and $I \cap I'$ are ideals of R .*

(2) *If I_α are ideals for α in an index set J , then $\sum I_\alpha$ for all α in J is an ideal of R .*

PROOF. Clearly, $I \cap I'$ is an ideal. For II' , let $\sum a_i b_i$ and $\sum a'_i b'_i$ be elements in II' with a_i, a'_i in I and b_i, b'_i in I' , we have $\sum a_i b_i - \sum a'_i b'_i = \sum a_i b_i + \sum a'_i (-b'_i)$, which is in II' (for $(R, +)$ is commutative under addition). For an r in R , $r(\sum a_i b_i) = \sum (ra_i) b_i$, an element in II' . Moreover, for $z = \sum a_i b_i$ in II' , there exists an integer n such that $z^n = z$, so z^{n-1} is in $B(R) \cap II'$. Hence the Pierce decomposition theorem implies that $R \cong z^{n-1}R + (1 - z^{n-1})R$. Denote z^{n-1} by e . Then for all x, y in R ,

$$\begin{aligned} (1 - e)((x + z)y - xy) &= (1 - e)((x + ez)y - xy) = (1 - e)xy + (1 - e)(-xy) \\ &= (1 - e)(xy + (-xy)) = 0. \end{aligned}$$

This implies that $((x + z)y - xy)$ is in (eR) , which is in II' .

Part (2) can be proved by the similar argument to part (1).

Following Pierce (1967), an ideal I is called regular if $I = (I \cap B(R))R$.

LEMMA 3.2. *Every ideal I of R is regular.*

PROOF. For $a \neq 0$ in I , there exists an integer $n > 1$ such that $a^n = a$, so a^{n-1} is an idempotent. Hence it is central. In this way, each element r in R induces an element e_r in $B(R)$. Clearly, $I \cap B(R) = \{e_r / r \text{ in } I\}$. Now, for an $r \neq 0$ in I with $r^n = r$, $r = e_r r$ which is in $(I \cap B(R))R$. Hence $I \subset (I \cap B(R))R$. The other inclusion is immediate.

Now we show that a topology can be imposed on the set of completely prime ideals.

THEOREM 3.3. *Let Π be the set of completely prime ideals of R and $\Gamma(I) = \{p \in \Pi / I \text{ an ideal of } R \text{ with } I \not\subset p\}$. Then Π is a topological space with a basic open set $\Gamma(I)$ for all ideals I of R .*

PROOF. Since each p in $\Gamma(I)$ is completely prime, it is easy to see that $\Gamma(I) \cap \Gamma(I') = \Gamma(II')$, and that $\cup \Gamma(I_\alpha) = \Gamma(\sum I_\alpha)$, where I_α are ideals of R with α in some index set J . From Lemma 3.1, II' and $\sum I_\alpha$ are ideals, so the proof is complete.

Next we want to show that Π and $\text{Spec } B(R)$ are homeomorphic.

THEOREM 3.4. *The topological space Π is homeomorphic with the Boolean space $\text{Spec } B(R)$ of R .*

PROOF. From Lemma 3.2, we have $P = (P \cap B(R))R$ for each P in Π . Define a mapping F from Π to $\text{Spec } B(R)$ by $F(P) = P \cap B(R)$. Since P is completely prime, $P \cap B(R)$ is prime in $B(R)$, and so it is a point in $\text{Spec } B(R)$. Also, Lemma 3.2 implies that F is one to one. Let x be a point in $\text{Spec } B(R)$. We claim that xR is a completely prime ideal in R . In fact, let rr' be in xR for any r, r' in R . If r and r' are not in xR , e_r and $e_{r'}$ are not in x (note that xR is an ideal). But then $e_r e_{r'}$ is not in x . Hence $(e_r e_{r'} + x) = B(R)$, so $1 = e_r e_{r'} s + s'$ for some s and s' in $B(R)$ with s' in x . Thus $e_r e_{r'} R + xR = R$ contradicting to the fact that $e_r e_{r'} R \subset rr'R \subset xR$. This proves that xR is in Π such that $F(xR) = x$ in $\text{Spec } B(R)$. So F is onto. Moreover, noting that Π has a basic open set $\Gamma(r)$ for r in R and that $\Gamma(r) = \Gamma(e_r)$, we conclude that $F(\Gamma(e_r)) = \Gamma_0(e_r)$, where $\Gamma_0(e_r) = \{x \text{ in } \text{Spec } B(R) / e_r \notin x\}$. But $\{\Gamma_0(e_r)\}$ is a basic open set for $\text{Spec } B(R)$, then F is a required homeomorphism.

4. A sheaf representation

As defined for a sheaf of rings (Pierce (1967)), a sheaf Σ of near-rings R_x for x in a topological space X is a disjoint union of R_x such that (1) for each x in X , a near-ring R_x is given with identity 1_x , (2) $R_x \cap R_y = \emptyset$, a void set for $x \neq y$ in X , (3) the projection π from Σ to X maps r in R_x to x for each r , (4) a topology is imposed on Σ such that 1) if r is in Σ , there exists an open set U in

Σ with r in U and $N \subset X$ such that π maps U homeomorphically on an open set N of X , 2) let $\Sigma + \Sigma$ denote $\{(r, s)/\pi(r) = \pi(s)\}$, with the product topology in $\Sigma \times \Sigma$, then the inverse map $r \rightarrow -r$, the addition map $(r, s) \rightarrow r + s$ and the product map $(r, s) \rightarrow rs$ are continuous, and 3) the constant map $x \rightarrow 1_x$ is continuous on X to Σ .

The near-rings R_x are called stalks of the sheaf Σ . For a subset U of X , $\lambda(U, \Sigma)$ is the collection of all continuous functions from U to Σ , called sections from U to Σ . Now we take R/p as R_p for each p in Π . That is, $R/p = R/xR$ by Lemma 3.2, where $x = p \cap B(R)$. Then we have:

THEOREM 4.1. *The disjoint union Σ of R/p is a sheaf with a basic open set $\hat{r}(\Gamma(e))$ for all e in $B(R)$ and r in R , where \hat{r} is a section such that $\hat{r}(p) = \bar{r}$ in R/p .*

PROOF. For p in $\Gamma(e') \cap \Gamma(e'')$, let $\hat{r}'(p) = \hat{r}''(p)$ with r' and r'' in R . Then $\bar{r}' = \bar{r}''$ in R/p . So, $(r' - r'') = e_{(r'-r'')}s$ for some s in R and $e_{(r'-r'')} \in p$. Hence $\bar{r}' = \bar{r}''$ in R/p' for all p' in $\Gamma(1 - e_{(r'-r'')})$. Denote $(1 - e_{(r'-r'')})$ by e . We have $\bar{r}' = \bar{r}''$ in R/p for all p in $\Gamma(e'e''e)$. Thus $\hat{r}'(e'e''e) \subset \hat{r}'\Gamma(e') \cap \hat{r}''\Gamma(e'')$. This proves that $\{\hat{r}\Gamma(e)/r$ in R and e in $B(R)\}$ is a basic open set for a topology imposed on Σ . Now let p be a point in $(\hat{r}^{-1})(\hat{r}'(\Gamma(e)))$ for r, r' in R and e in $B(R)$. Then $\hat{r}(p) = \hat{r}'(p)$ and p is in $\Gamma(e)$. The above result implies that $\hat{r}(p') = \hat{r}'(p')$ in some open set $\hat{r}(\Gamma(e'))$ contained in $\hat{r}'(\Gamma(e))$. Thus \hat{r} is a section for all r in R . Finally, it is routine to check that Σ is a sheaf of near-rings R/p with the topology having a basic open set $\hat{r}(\Gamma(e))$ for all e in $B(R)$ and r in R .

We say a near-ring has property P if every element is a power of itself.

THEOREM 4.2. *The near-ring R has property P if and only if it is isomorphic with the near-ring of sections of a sheaf of near-fields, each having property P .*

PROOF. Let R have property P . By Theorem 4.1, we know that \hat{r} is a section for each r in R . Now let $\hat{r} = 0$. Then $\bar{r} = 0$ in R/p for all p in Π . Hence $r = 0$ by (3) in Section 2; and so the map: $r \rightarrow \hat{r}$ is one-to-one. Moreover, since p is completely prime, R/p has no non-trivial zero divisors. But R has property P , so R/p is a near-field having property P . Next we claim that the map: $r \rightarrow \hat{r}$ is onto from R to Σ . Let f be a section of Σ . Then $f(p)$ is in R/p , and so $f(p) = \bar{r}$ in R/p for some r in R . Hence $f = \hat{r}$ in a basic open set $\Gamma(e)$ for some e in $B(R)$ by a standard property of sheaves. Let p vary over Π . Then Π is covered by such $\Gamma(e)$'s. Thus by the partition property of $\text{Spec } B(R)$, there are a finite cover of Π , $\Gamma(e_i)$ for $i = 1, 2, \dots, n$ such that e_i

are orthogonal idempotents summing to 1 and $f = \hat{r}_i$ at each p in $\Gamma(e_i)$ for all i , where r_i are in R (see proposition 12–13 in Pierce (1967) for the partition property of $B(R)$ and hence of Π). Therefore $f = \hat{r}$ with $r = \sum r_i e_i$.

Conversely, let r be a nonzero element in R . Then $(\bar{r})^n = \bar{r}$ in R/p for some p in Π and an integer $n > 1$ depending on p . That is, $(\bar{r}^n) = \bar{r}$. Considering $(r)^n$ and r as sections of Σ , we have a basic open set $\Gamma(e)$ containing p such that $(\hat{r})^n = \hat{r}$ over $\Gamma(e)$. Now let p vary over Π . Then Π is covered with such $\Gamma(e)$'s. By the partition property of Π , there is a finite cover $\Gamma(e_i)$ for $i = 1, 2, \dots, k$ which is a refinement of the $\Gamma(e)$'s such that e_i are orthogonal and summing to 1. Thus $(e_i r)^n = e_i r$ for each i . Observing that $(e_i r)^{n-1}$ is an idempotent for each i , we let $m = (n_1 - 1)(n_2 - 1) \cdots (n_k - 1) + 1$. Then

$$r^m = r r^{m-1} = r(r \cdot 1)^{m-1} = r(\sum(re_i))^{m-1}.$$

We next claim that $(\sum(re_i))^{m-1} = \sum(re_i)^{m-1}$. In fact, e_i are central orthogonal idempotents, so $e_i R e_j = 0$ for $i \neq j$. Hence

$$\begin{aligned} (re_i + r'e_j)^2 &= (re_i + r'e_j)re_i + (re_i + r'e_j)r'e_j \\ &= e_i(re_i + r'e_j)re_i + e_j(re_i + r'e_j)r'e_j \\ &= (re_i + 0)re_i + (0 + r'e_j)r'e_j \\ &= (re_i)^2 + (r'e_j)^2. \end{aligned}$$

By repeating this calculation, we have

$$r^m = r(r^{m-1}) = r(\sum(re_i))^{m-1} = r(\sum(re_i)^{m-1}) = r(\sum(re_i)^{n-1})$$

because $m - 1 = (n_1 - 1)(n_2 - 1) \cdots (n_k - 1)$ and $(re_i)^{n-1}$ is an idempotent for each i . But then

$$r^m = \sum r(re_i)^{n-1} = \sum(re_i)^n = \sum(re_i) = r(\sum(e_i)) = r.$$

Thus the proof is complete.

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Department of Mathematics,
Bradley University,
Peoria, Illinois 61625.

The University of Chicago,
Chicago, Illinois 60637,
U.S.A.