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The Chern–Ricci Flow on Oeljeklaus–Toma Manifolds

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Abstract. We study the Chern–Ricci flow, an evolution equation of Hermitian metrics, on a family of Oeljeklaus–Toma (OT-) manifolds that are non-Kähler compact complex manifolds with negative Kodaira dimension. We prove that after an initial conformal change, the flow converges in the Gromov–Hausdorff sense to a torus with a flat Riemannian metric determined by the OT-manifolds themselves.

1 Introduction

The Chern–Ricci flow is an evolution equation for Hermitian metrics by their Chern–Ricci forms on complex manifolds, which coincides exactly with the Kähler–Ricci flow when the initial metric is Kählerian. It was introduced by Gill [8] in the setting of complex manifolds with vanishing first Bott–Chern class. Tosatti and Weinkove [30, 31] investigated the flow on more general complex manifolds and proposed a program to study its behavior on all compact surfaces. The results in [10,11,15,23,30–32] are very similar to those for the Kähler–Ricci flow, and provide affirmative evidence that the Chern–Ricci flow is a natural geometric flow on complex surfaces whose properties reflect the underlying geometry of these manifolds.

Class VII surfaces are by definition non-Kähler compact complex surfaces with negative Kodaira dimension and first Betti number one. This class of surfaces are of especial interest, because there exists a well-known problem to complete their classification. Naturally, we will try to understand the properties of the Chern–Ricci flow on these surfaces, with the long-term aim of obtaining more topological or complex-geometric properties (*cf.* [28], where a different flow is considered). In this direction, in [5], the authors consider a family of Class VII surfaces, known as Inoue surfaces (see [12]) and proved that for a large class of Hermitian metrics, the Chern–Ricci flow always collapses the Inoue surface to a circle at infinite time, in the Gromov-Hausdorff sense. Also, the authors [5] posed some conjectures and open problems concerning the Chern–Ricci flow.

In this paper, we will concentrate on [5, Problem 3]; that is, we will study the behavior of the Chern–Ricci flow on a family of well-understood Oeljeklaus–Toma (OT-)

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manifolds, analogous to the Inoue–Bombieri surfaces S_M (see [12]) in high dimensions. OT-manifolds, first constructed by Oeljeklaus and Toma [16] from the view of algebraic number theory, are non-Kähler compact complex solvmanifolds with negative Kodaira dimension and without Vaisman metrics (see [16, Proposition 2.5] and [13, Section 6]). Battisti and Oeljeklaus [1, Theorem 3.5] states that OT-manifolds admit no analytic hypersurfaces and their algebraic dimension is zero. Verbitsky [34,35] also proved that OT-manifolds carry no closed 1-dimensional analytic subspaces and that OT-manifolds cannot contain any nontrivial compact complex 2-dimensional submanifolds except the Inoue surfaces. More recent progress and open problems about OT-manifolds can be found in [18] and references therein.

We investigate a class of complex m-dimensional OT-manifolds denoted by M_K with universal cover $\mathbb{H}^{m-1}\times\mathbb{C}$ and whose quotient covering map is denoted by $\pi:\mathbb{H}^{m-1}\times\mathbb{C}\to M_K$ (see Section 2), where $\mathbb{H}$ is the upper half plane. This class of OT-manifolds have locally conformally Kähler metric structure and admit no nontrivial complex subvariety (see [16, 17] and [18, Theorem 4.5]). In particular, the OT-manifolds with universal cover $\mathbb{H}^2\times\mathbb{C}$ give counterexamples to a conjecture of Vaisman [4, p. 8] (see also [18, Section 4.2]). Denote the standard coordinates on $\mathbb{H}^{m-1}\times\mathbb{C}$ by $(z_1,\dots,z_m)$. On OT-manifolds M_K a constant multiple of the product of standard Poincaré metric $\alpha=\sqrt{-1}\sum_{i=1}^{m-1}\frac{dz_i\wedge d\overline{z_i}}{4(\mathfrak{Im} z_i)^2}$ on $\mathbb{H}^{m-1}$ descends to a closed semipositive real $(1,1)$ form on M_K denoted by ω_∞ (also denoted by α itself) with

$$0\le\omega_\infty\in -c_1^{\mathrm{BC}}(M_K),$$

where $c_1^{\mathrm{BC}}(M_K)$ is the first Bott–Chern class of M_K. The $(1,1)$ form ω_∞ will play a key role in our results.

We consider the normalized Chern–Ricci flow

$$\frac{\partial}{\partial t}\omega=-\operatorname{Ric}(\omega)-\omega,\quad \omega|_{t=0}=\omega_0 \tag{1.1}$$

on M_K, with an initial Hermitian metric ω_0. Here $\operatorname{Ric}(\omega)$ is the Chern–Ricci form of the Hermitian metric $\omega=\sqrt{-1}g_{i\overline{j}}dz_i\wedge d\overline{z_j}$ defined by

$$\operatorname{Ric}(\omega)=-\sqrt{-1}\partial\overline{\partial}\log\det g.$$

Since the canonical bundle of M_K is nef, the results of [30–32] imply that there exists a unique solution to (1.1) for all time. We are concerned with the behavior of the normalized Chern–Ricci flow as $t\to\infty$.

Theorem 1.1 *Let M_K be an OT-manifold and let ω be any Hermitian metric on M_K. Then there exists a Hermitian metric $\omega_{\mathrm{LF}}=e^\sigma\omega$ in the conformal class of ω such that the following holds.*

Let $\omega(t)$ be the solution of the normalized Chern–Ricci flow (1.1) with the initial Hermitian metric of the form

$$\omega_0=\omega_{\mathrm{LF}}+\sqrt{-1}\partial\overline{\partial}\rho>0.$$

Then as $t\to\infty$, $\omega(t)\to\omega_\infty$ uniformly on M_K and exponentially fast, where ω_∞ is the $(1,1)$ form defined above. Furthermore,

$$\big(M_K,\omega(t)\big)\longrightarrow(\mathbb{T}^{m-1},g)$$

in the Gromov-Hausdorff sense, where g is defined as in (2.8), is the flat Riemannian metric on torus $\mathbb{T}^{m-1}$ determined by the OT-manifold M_K.

Therefore, we prove that the (normalized) Chern–Ricci flow collapses a Hermitian metric ω on the OT-manifold M_K to a torus, modulo an initial conformal change to ω. Indeed, we prove more than this, since our initial Hermitian metric can be any one in the $\partial\overline{\partial}$-class of $e^{\sigma}\omega$. Note that this collapsing to $\mathbb{T}^{m-1}$ is in stark contrast with the properties of the Kähler–Ricci flow that always collapses to even-dimensional manifolds (*cf.* [6, 9, 24–27, 33]).

Our conformal change is relative to a holomorphic foliation structure without singularity $\mathcal{F}$ defined by ω_∞ on the OT-manifold M_K. Now we give an outline of the explanation of this holomorphic foliation structure (more details can be found in Section 2). Note that $\mathcal{F}$ can be induced by the holomorphic foliation $\widetilde{\mathcal{F}}$ generated by ∂_{z_m} on the universal covering manifold $\mathbb{H}^{m-1} \times \mathbb{C}$ and every leaf of $\widetilde{\mathcal{F}}$ is of form $\{z'\} \times \mathbb{C}$, where $z' \in \mathbb{H}^{m-1}$. Motivated by [5], we give some definitions and deduce a useful proposition as follows.

Definition 1.1 A Hermitian metric ω on M_K is called *flat along the leaves* if the restriction of $\pi^*\omega$ to every leaf of $\widetilde{\mathcal{F}}$ is a flat Kähler metric on $\mathbb{C}$, and called *strongly flat along the leaves* if this restriction of $\pi^*\omega$ to every leaf of $\widetilde{\mathcal{F}}$ equals to

$$c\big((\Im z_1) \cdots (\Im z_{m-1}) \big) \sqrt{-1} \mathrm{d} z_m \wedge \mathrm{d}\overline{z_m},$$

where $c > 0$ is a constant independent of the leaf.

The Hermitian metric ω_{LF} we need in the statement of Theorem 1.1 is exactly strongly flat along the leaves. The following proposition shows that the assumption of being strongly flat along the leaves is not in fact restrictive, because it can always be obtained from any Hermitian metric ω by a conformal change (see also Lemma 2.2).

Proposition 1.2 *For any Hermitian metric ω on the OT-manifold M_K, there exists a smooth function $\sigma \in C^\infty(M_K, \mathbb{R})$ such that $\omega_{\mathrm{LF}} := e^{\sigma}\omega$ is strongly flat along the leaves.*

We remark that in the case of Inoue surfaces S_M, Definition 1.1 and Proposition 1.2 specialize to the corresponding ones in [5].

Another interesting question is whether we can get the smooth (C^∞) convergence of $\omega(t)$ to ω_∞ instead of the uniform (C^0) convergence in Theorem 1.1. In this direction, if the initial Hermitian metric is of a more restricted type, then we can get C^α convergence for $0 < \alpha < 1$. More precisely, Oeljeklaus and Toma [16] and Ornea and Verbitsky [17] constructed an explicit Hermitian metric ω_{OT} defined in (2.2) on the exact OT-manifold M_K we consider, which is strongly flat along the leaves. For the initial Hermitian metrics in the $\partial\overline{\partial}$-class of ω_{OT} and $0 < \alpha < 1$, we prove the C^α convergence as follows.

Theorem 1.3 *Let $\omega(t)$ be the solution of the normalized Chern–Ricci flow (1.1) on an OT-manifold M_K with an initial Hermitian metric of the form*

$$\omega_0 = \omega_{\mathrm{OT}} + \sqrt{-1}\partial\overline{\partial}\rho > 0.$$

Then the solution metric $\omega(t)$ is uniformly bounded in the C^1 topology, and for any $0 < \alpha < 1$, there holds $\omega(t) \to \omega_\infty$, as $t \to \infty$ in the C^α topology.

We note that while the strategy of the proofs is the same as in [5], new difficulties arise due to the fact that these manifolds have dimension greater than 2. This is the first general result where collapsing of the Chern–Ricci flow in dimensions greater than 2 is established for a large class of manifolds and initial metrics.

2 Oeljeklaus–Toma Manifolds

Let $\mathbb{Q}$ be the field consisting of rational numbers and let K be a finite extension field of $\mathbb{Q}$ with degree $[K:\mathbb{Q}] = n$. Then the field K admits precisely $n = s + 2t$ distinct embeddings $\sigma_1, \dots, \sigma_n$ into the field $\mathbb{C}$ consisting of complex numbers, where $\sigma_1, \dots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \dots, \sigma_n$ are complex embeddings. Without loss of generation, assume that $\sigma_{s+i} = \overline{\sigma_{s+t+i}}$ for $1 \le i \le t$, because the complex embeddings of K into $\mathbb{C}$ occur in pairs of complex conjugate embeddings. Also assume that both s and t are positive. Let

$$\sigma\colon K \longrightarrow \mathbb{C}^m, \quad \sigma(a) := \big(\sigma_1(a), \dots, \sigma_{s+t}(a)\big)$$

be the geometric representation of K.

Let O_K be the ring of algebraic integers of K and let O_K^* be the multiplicative group of units of O_K, that is,

$$O_K^* := \big\{ a \in O_K : \sigma_1(a)\cdots\sigma_s(a)|\sigma_{s+1}(a)|^2\cdots|\sigma_{s+t}(a)|^2 = \pm 1 \big\}.$$

Also, let

$$O_K^{*,+} := \big\{ a \in O_K^* : \sigma_1(a) > 0, \dots, \sigma_s(a) > 0 \big\}.$$

It is well known that the image $\sigma(O_K)$ is a lattice of rank n in $\mathbb{C}^m$, where $m := s+t$ (see, for example, [2, Theorem 1 in Section 3 of Chapter 2]). Therefore, we get a properly discontinuous action of O_K on $\mathbb{C}^m$ by translations.

Consider the multiplicative action of O_K on $\mathbb{C}^m$ given by

$$az := \big(\sigma_1(a)z_1, \dots, \sigma_m(a)z_m\big).$$

Denote by $\mathbb{H}$ the upper complex half-plane, that is, $\mathbb{H} = \{z \in \mathbb{C} : \Im\mathrm{m}\, z > 0\}$. Since for $a \in O_K$, $a\sigma(O_K) \subset \sigma(O_K)$, combining the additive action of O_K and the multiplicative action of $O_K^{*,+}$, Oeljeklaus and Toma [16] (see also [19]) obtained a free action of $O_K^{*,+} \ltimes O_K$ on $\mathbb{H}^s \times \mathbb{C}^t$. Now consider the logarithmic representation of units

$$\begin{aligned} &\mathrm{L}\colon O_K^{*,+} \longrightarrow \mathbb{R}^m, \\ &\mathrm{L}(a) := \big(\log\sigma_1(a), \dots, \log\sigma_s(a), 2\log|\sigma_{s+1}(a)|, \dots, 2\log|\sigma_{s+t}(a)|\big). \end{aligned} \tag{2.1}$$

It follows from the Dirichlet's Units Theorem (see, for example, [2]) that $\mathrm{L}(O_K^{*,+})$ is a full lattice in the subspace H of $\mathbb{R}^m$, where

$$H := \Big\{ x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 0 \Big\}.$$

For $t > 0$, the projection $\mathrm{Pr}\colon H \to \mathbb{R}^s$ given by the first s coordinate functions is surjective. So there exists subgroups G of rank s of $O_K^{*,+}$ such that $\mathrm{Pr}\circ\mathrm{L}(G)$ is a full lattice Λ in $\mathbb{R}^s$. Such a subgroup is called *admissible* for the field K by Oeljeklaus and Toma [16].

Take G admissible for K. The quotient manifold $(\mathbb{H}^s\times\mathbb{C}^t)/\sigma(O_K)$ is diffeomorphic to a trivial torus bundle $(\mathbb{R}_+)^s \times (\mathbb{S}^1)^n$, and G acts properly discontinuously on it because it induces a properly discontinuous action on $(\mathbb{R}_+)^s$. Therefore, we get an m-dimensional compact complex manifold

$$M_{K,G} := (\mathbb{H}^s \times \mathbb{C}^t)/(G \ltimes O_K)$$

which is a fiber bundle over

$$\mathbb{T}^s := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{s-\text{times}} \quad \text{with} \quad \mathbb{T}^n := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n-\text{times}}$$

as fiber. Such a manifold is called an *Oeljeklaus–Toma (OT-) manifold*.

For $s = t = 1$, $G = O_K^{*,+}$, $M_{K,G}$ is an Inoue-Bombieri surface S_M (see [12]).

In this paper, we will consider the OT-manifold in the case of $s > 0$, $t = 1$ (to prove similar results in the case when $t > 1$, it seems that new ideas will be required (see [18, 36])), that is, $M_K := M_{K,G} = (\mathbb{H}^{m-1} \times \mathbb{C})/\Gamma$, where $\Gamma := (G \ltimes O_K)$ and denote the quotient covering map by $\pi\colon \mathbb{H}^{m-1} \times \mathbb{C} \to M_K$ and the fiber projection by $p\colon M_K \to \mathbb{T}^{m-1}$.

Let

$$z_i = x_i + \sqrt{-1}y_i, \quad i = 1, \dots, m,$$

where $z_1, \dots, z_{m-1}$ is the standard coordinates of $\mathbb{H}^{m-1}$ and z_m is the standard coordinate of $\mathbb{C}$. Then we have some Γ-invariant forms on $\mathbb{H}^{m-1} \times \mathbb{C}$ that can be induced on M_K and denoted by the same symbols, defined as follows (see [16, 17]):

$$\alpha := \sqrt{-1}\sum_{i=1}^{m-1} \frac{dz_i \wedge d\overline{z_i}}{4y_i^2}, \ \beta := \sqrt{-1}(y_1 \cdots y_{m-1}) dz_m \wedge d\overline{z_m}, \ \gamma := \sqrt{-1}\sum_{k,\ell=1}^{m-1} \frac{dz_k \wedge d\overline{z_\ell}}{4y_k y_\ell}.$$

In addition, α is d-closed and also denoted by ω_∞ when it descends to M_K in Section 1. Therefore, we can construct a Hermitian metric ω_{OT} by

$$\omega_{\mathrm{OT}} = \alpha + \beta + \gamma \tag{2.2}$$

with Ricci form

$$\mathrm{Ric}(\omega_{\mathrm{OT}}) = -\alpha \in c_1^{\mathrm{BC}}(M_K).$$

If we define a function $\psi(z) = (y_1 \cdots y_{m-1})^{-1}$ on $\mathbb{H}^{m-1} \times \mathbb{C}$, then ω_{OT} was defined in [16, 17] to be

$$\frac{\sqrt{-1}\partial\overline{\partial}(\psi(z) + |z_m|^2)}{\psi(z)},$$

and a simple calculation shows that this is equal to $\alpha + \beta + \gamma$. In the case of Inoue–Bombieri surfaces, $\omega_{\mathrm{T}} = 4\alpha + \beta$ is called the Tricerri metric [29].

Now we give more details about the holomorphic foliation $\mathcal{F}$ mentioned in Section 1. We begin with the holomorphic foliation $\widetilde{\mathcal{F}}$ without singularity on $\mathbb{H}^{m-1} \times \mathbb{C}$ generated by vector field ∂_{z_m}. The foliation $\widetilde{\mathcal{F}}$ is Γ-invariant and is also the kernel of the Γ-invariant form α. Therefore, it induces a holomorphic foliation $\mathcal{F}$ without singularity on M_K with the kernel $\alpha = \omega_\infty$ (see [17]). A leaf of the foliation $\widetilde{\mathcal{F}}$ including the point $(t_1,\dots,t_m)\in\mathbb{H}^{m-1}\times\mathbb{C}$ is given as

$$\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}} := \big\{(z',z_m)\in\mathbb{H}^{m-1}\times\mathbb{C} : z'=(t_1,\dots,t_{m-1})\in\mathbb{H}^{m-1}\big\}.$$

Since the isotropy group of the leaf denoted by

$$G_{\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}}} := \big\{g\in\Gamma : g\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}} = \widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}}\big\}.$$

is trivial, we get a leaf $\mathcal{L}$ of $\mathcal{F}$ via the natural immersion of $\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}}/G_{\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}}}$ diffeomorphic to $\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}}$ into M_K. All the leaves of $\mathcal{F}$ can be obtained in this way (see [14]). For the closure Z of a leaf $\mathcal{L}=\pi(\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}})$ of $\mathcal{F}$, Ornea and Verbitsky [17, Proposition 3.2] proved that

$$\pi^{-1}(Z)\supseteq Z_{\alpha_1,\dots,\alpha_{m-1}} := \{(z_1,\dots,z_m)\in\mathbb{H}^{m-1}\times\mathbb{C} : \alpha_i=\Im\mathrm{m} z_i, 1\le i\le m-1\},$$

where $\alpha_i=\Im\mathrm{m} t_i, 1\le i\le m-1$ and $Z_{\alpha_1,\dots,\alpha_{m-1}}$ is the closure of $O_K(\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}})$. Therefore, we can deduce the following lemma.

Lemma 2.1 *For any point $a\in M_K$, the leaf $\mathcal{L}_a$ of the foliation $\mathcal{F}$ through this point is dense in the $\mathbb{T}^{m+1}$-fiber of the point $p(a)\in\mathbb{T}^{m-1}$; that is, for any point $t=(t_1,\dots,t_m)\in\mathbb{H}^{m-1}\times\mathbb{C}$, $\pi(\widetilde{\mathcal{L}}_{t_1,\dots,t_{m-1}})$ is dense in the $\mathbb{T}^{m+1}$-fiber of the point $p\circ\pi(t)\in\mathbb{T}^{m-1}$.*

The following lemma shows that every Hermitian metric ω on M_K is conformal to a Hermitian one that is strongly flat along the leaves.

Lemma 2.2 *A Hermitian metric ω_{LF} on m-dimensional M_K is flat along the leaves if and only if*

$$\alpha^{m-1}\wedge\omega_{\mathrm{LF}} = (p^*\eta)\alpha^{m-1}\wedge\beta, \tag{2.3}$$

where

$$\eta\colon\mathbb{T}^{m-1} = \underbrace{\mathbb{S}^1\times\cdots\times\mathbb{S}^1}_{(m-1)-\text{times}} \longrightarrow \mathbb{R}$$

is a smooth positive function. And it is strongly flat along the leaves if and only if

$$\alpha^{m-1}\wedge\omega_{\mathrm{LF}} = c\alpha^{m-1}\wedge\beta, \tag{2.4}$$

where $c>0$ is a constant. For any Hermitian metric ω on M_K, define $\sigma\in C^\infty(M_K,\mathbb{R})$ by

$$e^\sigma = \frac{\alpha^{m-1}\wedge\beta}{\alpha^{m-1}\wedge\omega}.$$

Then $\omega_{\mathrm{LF}}=e^\sigma\omega$ satisfies (2.4) with $c=1$ and hence is strongly flat along the leaves.

Proof Write the pullback of the Hermitian ω_{LF} as

$$\pi^*\omega_{\mathrm{LF}} = \sum_{i,j=1}^m g_{i\bar{j}}\mathrm{d}z_i\wedge\mathrm{d}\overline{z_j},$$

and we have

$$\frac{\alpha^{m-1}\wedge\pi^*\omega_{\mathrm{LF}}}{\alpha^{m-1}\wedge\beta} = \frac{g_{m\overline{m}}}{y_1\cdots y_{m-1}}. \tag{2.5}$$

So (2.3) is equivalent to

$$\frac{g_{m\overline{m}}}{y_1\cdots y_{m-1}} = \pi^* p^* \eta.$$

Notice that the function $\pi^* p^* \eta$ depends only on $(y_1, \dots, y_{m-1})$. Since the restriction of $\pi^* \omega_{\mathrm{LF}}$ to a leaf $\{z'\} \times \mathbb{C}$ equals $\sqrt{-1} g_{m\overline{m}} \mathrm{d}z_m \wedge \mathrm{d}\overline{z_m}$, and its Ricci curvature equals $-\partial_m \partial_{\overline{m}} \log g_{m\overline{m}}$, we can deduce that if (2.3) holds then ω_{LF} is flat along the leaves.

Conversely, if ω_{LF} is flat along the leaves, then for each fixed $z' \in \mathbb{H}^{m-1}$ we have that

$$\partial_m \partial_{\overline{m}} \log \frac{g_{m\overline{m}}}{y_1\cdots y_{m-1}} = \partial_m \partial_{\overline{m}} \log g_{m\overline{m}} = 0.$$

Thanks to (2.5) we get that the function $\log(g_{m\overline{m}}/y_1\cdots y_{m-1})$ on $\mathbb{H}^{m-1} \times \mathbb{C}$ is Γ-invariant, hence bounded (because it is the pullback of a function from M_K). Therefore, $\log(g_{m\overline{m}}/y_1\cdots y_{m-1})$ for $(z_1, \dots, z_{m-1}) \in \mathbb{H}^{m-1}$ fixed is a bounded harmonic function on $\mathbb{C}$, and so it must be constant. In other words, the ratio $(\alpha^{m-1} \wedge \omega_{\mathrm{LF}})/(\alpha^{m-1} \wedge \beta)$ is constant along each leaf of $\mathcal{F}$. Since every leaf is dense in the $\mathbb{T}^{m+1}$ fiber that contains it, we obtain that $(\alpha^{m-1} \wedge \omega_{\mathrm{LF}})/(\alpha^{m-1} \wedge \beta)$ equals the pullback of a function from $\mathbb{T}^{m-1}$.

On the other hand, it is now clear that ω is strongly flat along the leaves if and only if (2.4) holds, or equivalently,

$$\frac{g_{m\overline{m}}}{y_1\cdots y_{m-1}} = c,$$

where $c > 0$ is a constant. The last assertion of the lemma is immediate. ■

To end this section, we give some details about the Riemannian metric on $\mathbb{T}^{m-1}$ induced from α. On $\mathbb{H}^{m-1}$, α corresponds to the Riemannian metric

$$\sum_{i=1}^{m-1} \frac{\mathrm{d}x_i \otimes \mathrm{d}x_i + \mathrm{d}y_i \otimes \mathrm{d}y_i}{2y_i^2},$$

which is restricted on $(\mathbb{R}_+)^{m-1}$,

$$\sum_{i=1}^{m-1} \frac{\mathrm{d}y_i \otimes \mathrm{d}y_i}{2y_i^2}. \tag{2.6}$$

Under the local coordinate

$$f\colon (\mathbb{R}_+)^{m-1} \longrightarrow \mathbb{R}^{m-1}, \quad (y_1, \dots, y_{m-1}) \longmapsto (\log y_1, \dots, \log y_{m-1}),$$

the metric (2.6) can be expressed as

$$\frac{1}{2} \sum_{i=1}^{m-1} \mathrm{d}x_i \otimes \mathrm{d}x_i. \tag{2.7}$$

Now let $a_1, \dots, a_{m-1}$ be the generators of the admissible group G. Then under the logarithmic representation (2.1),

$$\big(\log \sigma_1(a_i), \dots, \log \sigma_{m-1}(a_i)\big) =: (v_{i1}, \dots, v_{i,m-1}), \quad i = 1, \dots, m-1$$

is the basis of the full lattice Λ in $\mathbb{R}^{m-1}$ and $\mathbb{T}^{m-1} = \mathbb{R}^{m-1}/\Lambda$, where $\mathbb{R}^{m-1}$ is equipped with the metric (2.7). So the metric on $\mathbb{T}^{m-1}$ is

$$\frac{1}{2}\sum_{k,\ell=1}^{m-1}\Big(\sum_{i=1}^{m-1} v_{ki}v_{\ell i}\Big)\mathrm{d}x^k \otimes \mathrm{d}x^\ell, \tag{2.8}$$

and the radius of the k-th ($k = 1, \dots, m-1$) factor $\mathbb{S}^1$ of $\mathbb{T}^{m-1}$ is

$$\frac{1}{2\sqrt{2}\pi}\Big(\sum_{i=1}^{m-1} v_{ki}v_{ki}\Big)^{1/2}.$$

Obviously, the metric on $\mathbb{T}^{m-1}$ depends on the lattice Λ. In fact, the metrics g_Λ and $g_{\Lambda'}$ defined on $\mathbb{T}^{m-1}$ are isometric if and only if there exists an isometry of $\mathbb{R}^{m-1}$ that sends the lattice Λ on the lattice Λ' (see [7, Theorem 2.23]).

3 The Chern–Ricci flow on OT-manifolds

We will write the normalized Chern–Ricci flow as a parabolic complex Monge–Ampère equation. Let $\omega_{\mathrm{LF}} = \sqrt{-1}\sum_{i,j=1}^m (g_{\mathrm{LF}})_{i\bar{j}}\mathrm{d}z_i \wedge \mathrm{d}\overline{z_j}$ be the Hermitian metric that is strongly flat along the leaves, as in the setup of Theorem 1.1. First, we define

$$\widetilde{\omega} = \widetilde{\omega}(t) = e^{-t}\omega_{\mathrm{LF}} + (1-e^{-t})\alpha > 0, \tag{3.1}$$

and denote by $\widetilde{g}$ the Hermitian metric associated with $\widetilde{\omega}$. We define a volume form Ω by

$$\Omega = m\alpha^{m-1} \wedge \omega_{\mathrm{LF}} = mc\alpha^{m-1} \wedge \beta, \tag{3.2}$$

with the constant c defined by (2.4). Direct calculation using (3.2) implies

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = \alpha.$$

It follows that the normalized Chern–Ricci flow (1.1) is equivalent to the parabolic complex Monge–Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log\frac{e^t(\widetilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^m}{\Omega} - \varphi, \quad \widetilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \quad \varphi(0) = \rho. \tag{3.3}$$

Namely, if φ solves equation (3.3), then $\omega(t) = \widetilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ solves the normalized Chern–Ricci flow (1.1), as is readily checked. Conversely, given a solution $\omega(t)$ of (1.1), we can find a solution (see [30]) $\varphi = \varphi(t)$ of (3.3) such that $\omega(t) = \widetilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$.

Let $\varphi = \varphi(t)$ be the solution to (3.3) and write $\omega = \omega(t) = \widetilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ for the corresponding Hermitian metrics along the normalized Chern–Ricci flow (1.1). We first prove uniform estimates on the potential φ and its time derivative $\dot{\varphi}$. Given the choice of $\widetilde{\omega}$ and Ω, the proof is very similar to the one in [27, Lemmas 3.6.3 and 3.6.7] (see also [5, 6, 9, 24, 32]).

Lemma 3.1 *There exists a uniform positive constant C such that on $M_K \times [0, \infty)$,*

(i) $|\varphi| \le C(1+t)e^{-t}$,
(ii) $|\dot{\varphi}| \le C$,
(iii) $C^{-1}\widetilde{\omega}^m \le \omega^m \le C\widetilde{\omega}^m$.

Proof Since the discussion is very similar to those in [9, 27, 32], we will be brief. For part (i), first we claim that, by the choice of $\widetilde{\omega}$ and Ω, there holds

$$\Big|e^t \log \frac{e^t\widetilde{\omega}^m}{\Omega}\Big| \le C', \tag{3.4}$$

for uniform C'. Indeed, from (3.1) and (3.2), we have

$$\begin{aligned}\frac{e^t\widetilde{\omega}^m}{\Omega} &= \frac{e^t\sum_{k=0}^m \binom{m}{k}(1-e^{-t})^k\alpha^k e^{-(m-k)t}\omega_{\mathrm{LF}}^{m-k}}{\Omega}\\ &= \frac{m(1-e^{-t})^{m-1}\alpha^{m-1}\wedge\omega_{\mathrm{LF}} + e^{-t}\sum_{k=0}^{m-2}\binom{m}{k}(1-e^{-t})^k e^{-(m-2-k)t}\alpha^k\wedge\omega_{\mathrm{LF}}^{m-k}}{\Omega}\\ &= 1+O(e^{-t}),\end{aligned} \tag{3.5}$$

which implies (3.4). From now on, $O(f(t))$ will mean $\le Cf(t)$ for a uniform constant C, where $f(t)$ is a positive function of t (*e.g.*, e^{-t}, 1, e^t). Now consider the quantity

$$P = e^t\varphi - (C'+1)t.$$

If $\sup_{M_K\times[0,\,t_0]} P = P(x_0,\,t_0)$ for some $x_0 \in M_K$ and $t_0 > 0$, we have at this point,

$$0 \le \frac{\partial P}{\partial t} \le e^t \log\frac{e^t\widetilde{\omega}^m}{\Omega} - C' - 1 \le -1,$$

which is absurd. Therefore, $\sup_{M_K} P$ is bounded from above by its initial value, which implies $\varphi \le C(1+t)e^{-t}$. The lower bound is similar.

To prove (ii), choose a constant C_0 satisfying $C_0\widetilde{\omega} > \alpha$ for all $t \ge 0$. Then compute, for the Laplacian $\Delta = g^{\bar{j}i}\partial_i\partial_{\bar{j}}$,

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)\big(\dot{\varphi} - (C_0-1)\varphi\big) &= 1 + \mathrm{tr}_\omega(\alpha - \widetilde{\omega}) - C_0\dot{\varphi} + (C_0-1)\,\mathrm{tr}_\omega(\omega - \widetilde{\omega})\\ &< 1 - C_0\dot{\varphi} + m(C_0-1).\end{aligned}$$

The maximum principle implies that $\dot{\varphi}$ is bounded from above. For the lower bound of $\dot{\varphi}$,

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)(\dot{\varphi} + 2\varphi) &= \mathrm{tr}_\omega(\alpha - \widetilde{\omega}) + 1 + \dot{\varphi} - 2\,\mathrm{tr}_\omega(\omega - \widetilde{\omega})\\ &\ge \mathrm{tr}_\omega\,\widetilde{\omega} + \dot{\varphi} - (2m-1).\end{aligned} \tag{3.6}$$

By the geometric-arithmetic means inequality, we have

$$e^{-\frac{\dot{\varphi}+\varphi}{m}} = \Big(\frac{\Omega}{e^t\omega^m}\Big)^{\frac{1}{m}} \le C\Big(\frac{\widetilde{\omega}^m}{\omega^m}\Big)^{\frac{1}{m}} \le \frac{C}{m}\,\mathrm{tr}_\omega\,\widetilde{\omega}, \tag{3.7}$$

where we use (3.5). Combining (3.6), (3.7), and the maximum principle indicates that $\dot{\varphi}$ is bounded from below.

Finally, (iii) follows from (i), (ii), and equation (3.3). ∎

Next, we bound the torsion and curvature of the reference metrics $\widetilde{g}$. We will denote the Chern connection, torsion, and curvature of $\widetilde{g}$ by $\widetilde{\nabla}$, $\widetilde{T}$, and $\widetilde{\mathrm{Rm}}$, respectively, and also write

$$\widetilde{T}_{ij\bar{\ell}} = \widetilde{T}^k_{ij}\widetilde{g}_{k\bar{\ell}} = \partial_i \widetilde{g}_{j\bar{\ell}} - \partial_j \widetilde{g}_{i\bar{\ell}}.$$

Since α is a closed form, we have

$$\widetilde{T}_{ij\bar{\ell}} = e^{-t}(T_{\mathrm{LF}})_{ij\bar{\ell}}, \tag{3.8}$$

where T_{LF} is the torsion of the metric g_{LF}. We can deduce the following bounds on torsion and curvature of $\widetilde{g}$, which are analogous to those in [32, Lemma 4.1].

Lemma 3.2 *There exists a uniform constant C such that*

(i) $|\widetilde{T}|_{\widetilde{g}} \le C$,

(ii) $|\overline{\partial}\widetilde{T}|_{\widetilde{g}} + |\widetilde{\nabla}\widetilde{T}|_{\widetilde{g}} + |\widetilde{\mathrm{Rm}}|_{\widetilde{g}} \le Ce^{t/2}$.

Proof Denote by $\widetilde{g}_{k\bar{\ell}}$ the component of metric matrix $(\widetilde{g}_{p\bar{q}})$ in k-th row and ℓ-th column. We have

$$\begin{aligned} \widetilde{g}_{k\bar{k}} &= e^{-t}(g_{\mathrm{LF}})_{k\bar{k}} + (1-e^{-t})\alpha_{k\bar{k}} = O(1), \quad 1 \le k \le m-1, \\ \widetilde{g}_{k\bar{\ell}} &= e^{-t}(g_{\mathrm{LF}})_{k\bar{\ell}} = O(e^{-t}), \quad \text{otherwise}. \end{aligned}$$

Denote by $\widetilde{g}^{\bar{\ell}k}$ the component of the inverse matrix of $(\widetilde{g}_{p\bar{q}})$ in ℓ-th row and k-th column, and by $G_{k\ell}$ the algebraic cofactor of the component $\widetilde{g}_{k\bar{\ell}}$. Note that

$$\begin{aligned} G_{mm} &= \widetilde{g}_{1\bar{1}} \cdots \widetilde{g}_{m-1\overline{m-1}} + \sum \text{terms with factor } e^{-t} \\ &= \alpha_{1\bar{1}} \cdots \alpha_{m-1\overline{m-1}} + \sum \text{terms with factor } e^{-t} \end{aligned}$$

and

$$G_{k\ell} = \sum \text{terms with factor } e^{-t}, \quad (k,\ell) \ne (m,m).$$

A preliminary analysis implies that there exists a uniform constant c_0 independent of t such that

$$\det(\widetilde{g}_{p\bar{q}}) = \sum_{\ell=1}^{m} \widetilde{g}_{m\bar{\ell}} G_{m\ell} = e^{-t}\sum_{\ell=1}^{m} (g_{\mathrm{LF}})_{m\bar{\ell}} G_{m\ell} \ge c_0 e^{-t},$$

where we also use the fact that $(g_{\mathrm{LF}})_{m\overline{m}} > 0$. Therefore, we can deduce that all the components of the inverse metric matrix $(\widetilde{g}^{\bar{q}p})$ are bounded by $O(1)$ except that $\widetilde{g}^{\overline{m}m}$ is bounded by $O(e^t)$, where we use the formula $\widetilde{g}^{\bar{\ell}k} = \frac{G_{k\ell}}{\det(\widetilde{g}_{p\bar{q}})}$.

From (3.8), we have

$$|\widetilde{T}|^2_{\widetilde{g}} = e^{-2t}(T_{\mathrm{LF}})_{ik\bar{q}}\overline{(T_{\mathrm{LF}})_{j\ell\bar{p}}}\widetilde{g}^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{g}^{\bar{q}p} \le C,$$

since the only term involving the cube of $\widetilde{g}^{\overline{m}m}$ vanishes by the skew-symmetry of $(T_{\mathrm{LF}})_{ik\bar{q}}$ in i and k, and by the bounds of other components of $(\widetilde{g}^{\bar{\ell}k})$ all other terms are bounded.

Since if one of the indexes $1 \le i,\ k,\ \ell \le m$ equals m, we have

$$\partial_i \widetilde{g}_{k\bar{\ell}} = \partial_{\bar{i}} \widetilde{g}_{k\bar{\ell}} = O(e^{-t}), \tag{3.9}$$

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and

$$\partial_m \widetilde{g}_{m\overline{m}} = \partial_{\overline{m}} \widetilde{g}_{m\overline{m}} = 0, \tag{3.10}$$

where for (3.10) we use the fact that $\widetilde{g}_{m\overline{m}} = ce^{-t}(y_1 \cdots y_{m-1})$, we can bound on the $\widetilde{g}$ norm of the Christoffel symbols $\widetilde{\Gamma}^p_{ik}$ of the Chern connection of $\widetilde{g}$ by

$$|\widetilde{\Gamma}^p_{ik}|^2_{\widetilde{g}} = \widetilde{\Gamma}^p_{ik}\overline{\widetilde{\Gamma}^q_{j\ell}}\widetilde{g}^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{g}_{p\bar{q}} = \widetilde{g}^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{g}^{\bar{q}p}\partial_i\widetilde{g}_{k\bar{q}}\partial_{\bar{j}}\widetilde{g}_{p\bar{\ell}} \le C. \tag{3.11}$$

Note that the quantity $|\widetilde{\Gamma}^p_{ik}|^2_{\widetilde{g}}$ is only locally defined.

Since

$$(T_{\mathrm{LF}})_{im\overline{m}} = \partial_i (g_{\mathrm{LF}})_{m\overline{m}} - \partial_m (g_{\mathrm{LF}})_{i\overline{m}} = O(1),$$
$$\partial_{\overline{m}}(T_{\mathrm{LF}})_{im\overline{m}} = \partial_i\partial_{\overline{m}}(g_{\mathrm{LF}})_{m\overline{m}} - \partial_m\partial_{\overline{m}}(g_{\mathrm{LF}})_{i\overline{m}} = -\partial_m\partial_{\overline{m}}(g_{\mathrm{LF}})_{i\overline{m}} = O(1),$$

we have, using the skew-symmetry of $(T_{\mathrm{LF}})_{ik\bar{q}}$ in i and k,

$$|(T_{\mathrm{LF}})_{ij\bar{r}}|^2_{\widetilde{g}} \le Ce^{2t}, \quad |\partial_{\bar{\ell}}(T_{\mathrm{LF}})_{ij\bar{k}}|^2_{\widetilde{g}} \le Ce^{3t}.$$

Therefore, from (3.8) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\overline{\partial}\widetilde{T}|^2_{\widetilde{g}} = |\overline{\nabla}\widetilde{T}|^2_{\widetilde{g}} &= e^{-2t}|\partial_{\bar{\ell}}(T_{\mathrm{LF}})_{ij\bar{k}} - \overline{\widetilde{\Gamma}^r_{\ell k}}(T_{\mathrm{LF}})_{ij\bar{r}}|^2_{\widetilde{g}}\\ &\le 2e^{-2t}|\partial_{\bar{\ell}}(T_{\mathrm{LF}})_{ij\bar{k}}|^2_{\widetilde{g}} + 2e^{-2t}|\overline{\widetilde{\Gamma}^r_{\ell k}}|^2_{\widetilde{g}}|(T_{\mathrm{LF}})_{ij\bar{r}}|^2_{\widetilde{g}}\\ &\le 2e^{-2t}|\partial_{\bar{\ell}}(T_{\mathrm{LF}})_{ij\bar{k}}|^2_{\widetilde{g}} + Ce^{-2t}|(T_{\mathrm{LF}})_{ij\bar{r}}|^2_{\widetilde{g}} \le Ce^t.\end{aligned}$$

Similarly, we can deduce $|\nabla\widetilde{T}|^2_{\widetilde{g}} \le Ce^t$.

Recall that the curvature of the Chern connection of $\widetilde{g}$ is given by

$$\widetilde{R}_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}} + \widetilde{g}^{\bar{q}p}\partial_i\widetilde{g}_{k\bar{q}}\partial_{\bar{j}}\widetilde{g}_{p\bar{\ell}}.$$

For the bound of $|\widetilde{R}_{i\bar{j}k\bar{\ell}}|^2_{\widetilde{g}}$, using (3.11), we just need to bound $|\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}}|^2_{\widetilde{g}}$. Thanks to (3.9) and (3.10), we can obtain

$$\begin{aligned}|\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}}|^2_{\widetilde{g}} = \sum_{k,\ell=1}^{m-1}(\partial_m\partial_{\overline{m}}\widetilde{g}_{k\overline{m}})(\partial_m\partial_{\overline{m}}\widetilde{g}_{m\bar{\ell}})\widetilde{g}^{\overline{m}m}\widetilde{g}^{\overline{m}m}\widetilde{g}^{\overline{m}m}\widetilde{g}^{\bar{\ell}k}&\\ + \text{ terms bounded by constant } \le Ce^t.&\end{aligned}$$

So $|\widetilde{\mathrm{Rm}}|^2_{\widetilde{g}} \le Ce^t$, as required. ∎

Now we can apply the arguments of [32, 33] to establish the estimates of the solution $\omega(t)$ to the normalized Chern–Ricci flow (1.1) and also the solution $\varphi(t)$ to the parabolic complex Monge–Ampère equation (3.3).

Theorem 3.3 *For $\varphi = \varphi(t)$ solving (3.3) on M_K, the following estimates hold:*

(i) *There exists a uniform constant C such that $C^{-1}\widetilde{\omega} \le \omega(t) \le C\widetilde{\omega}$.*
(ii) *The Chern scalar curvature R satisfies the bound $-C \le R \le Ce^{t/2}$, where C is uniform constant.*

(iii) *For any $\eta \in (0, 1/2)$ and $\sigma \in (0, 1/4)$, there exists a constant $C_{\eta,\sigma}$ such that*

$$-C_{\eta,\sigma} e^{-\eta t} \le \dot{\varphi} \le C_{\eta,\sigma} e^{-\sigma t}.$$

Therefore, combining Lemma 3.1(i) *and this bound gives (taking $\eta = \sigma$)*

$$|\varphi + \dot{\varphi}| \le C_\sigma e^{-\sigma t}.$$

(iv) *For any $\varepsilon \in (0, 1/8)$, there exists a constant C_ε satisfying*

$$\|\omega - \widetilde{\omega}\|_{C^0(M_K,\, \omega_0)} \le C_\varepsilon e^{-\varepsilon t}.$$

Proof Given Lemmas 3.1 and 3.2, the proof is almost identical to the discussion in [32, 33]. Therefore, we give only a brief outline and point out the main differences.

For part (i), we claim that

$$(3.12)\quad \Big(\frac{\partial}{\partial t} - \Delta\Big) \log \operatorname{tr}_{\widetilde{\omega}} \omega \le \frac{2}{(\operatorname{tr}_{\widetilde{\omega}} \omega)^2} \operatorname{Re}\Big(\widetilde{g}^{\bar{\ell} i} g^{\bar{q} k} \widetilde{T}_{k i \bar{\ell}} \partial_{\bar{q}} \operatorname{tr}_{\widetilde{\omega}} \omega\Big) + C e^{t/2} \operatorname{tr}_\omega \widetilde{\omega}, \quad t \ge 0.$$

Indeed, this inequality can be obtained by an argument that is almost identical to the one in [32, Lemma 5.2]. From [30, Proposition 3.1] we have

$$(3.13)\quad \Big(\frac{\partial}{\partial t} - \Delta\Big) \log \operatorname{tr}_{\widetilde{\omega}} \omega = (I) + (II) + (III) - \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega} \widetilde{g}^{\bar{\ell} p} \widetilde{g}^{\bar{q} k} \alpha_{p \bar{q}} g_{k \bar{\ell}},$$

where

$$\begin{aligned}
(I) = \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\Big[&- g^{\bar{j} p} g^{\bar{q} i} \widetilde{g}^{\bar{\ell} k} \widetilde{\nabla}_k g_{i \bar{j}} \widetilde{\nabla}_{\bar{\ell}} g_{p \bar{q}} + \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega} g^{\bar{\ell} k} \partial_k (\operatorname{tr}_{\widetilde{\omega}} \omega) \partial_{\bar{\ell}} (\operatorname{tr}_{\widetilde{\omega}} \omega) \\
&- 2 \operatorname{Re}\Big(g^{\bar{j} i} \widetilde{g}^{\bar{\ell} k} \widetilde{T}^p_{k i} \widetilde{\nabla}_{\bar{\ell}} g_{p \bar{j}}\Big) - g^{\bar{j} i} \widetilde{g}^{\bar{\ell} k} \widetilde{T}^p_{i k} \overline{\widetilde{T}^q_{j \ell}} g_{p \bar{q}} \Big] \\
(II) = \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\Big[& g^{\bar{j} i} \widetilde{g}^{\bar{\ell} k} \Big(\widetilde{\nabla}_i \overline{\widetilde{T}^q_{j \ell}} - \widetilde{R}_{i \bar{\ell} p \bar{j}} \widetilde{g}^{\bar{q} p}\Big) g_{k \bar{q}} \Big] \\
(III) = -\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\Big[& g^{\bar{j} i} \Big(\widetilde{\nabla}_i \overline{\widetilde{T}^\ell_{j \ell}} \Big) + \Big(\widetilde{\nabla}_{\bar{\ell}} \widetilde{T}^p_{i k}\Big) g^{\bar{j} i} \widetilde{g}^{\bar{\ell} k} \widetilde{g}_{p \bar{j}} - \overline{\widetilde{T}^q_{j \ell}} \widetilde{T}^p_{i k} g^{\bar{j} i} \widetilde{g}^{\bar{\ell} k} \widetilde{g}_{p \bar{q}} \Big].
\end{aligned}$$

Let us point out some differences from the calculation in [30]. Here, ω is evolved by normalized Chern–Ricci flow (1.1), and our reference metric $\widetilde{\omega}$ also depends on time. In particular, in our case we have $T_{i j \bar{\ell}} = \widetilde{T}_{i j \bar{\ell}}$ (instead of $T_{i j \bar{\ell}} = (T_0)_{i j \bar{\ell}}$ in [30]) and the metrics $\widehat{g}$ and g_0 in [30] are replaced by $\widetilde{\omega}$. The last term in (3.13) comes from the $-\omega$ term on the right side of (1.1) and the time derivative of $\widetilde{\omega}$. Fortunately, we have

$$-\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega} \widetilde{g}^{\bar{\ell} p} \widetilde{g}^{\bar{q} k} \alpha_{p \bar{q}} g_{k \bar{\ell}} \le 0.$$

Proposition 3.1 in [30] gives us

$$(I) \le \frac{2}{(\operatorname{tr}_{\widetilde{\omega}} \omega)^2} \operatorname{Re}\Big(\widetilde{g}^{\bar{\ell} i} g^{\bar{q} k} \widetilde{T}_{k i \bar{\ell}} \partial_{\bar{q}} (\operatorname{tr}_{\widetilde{\omega}} \omega)\Big).$$

Therefore, to complete the proof of the lemma, we only need to show that

$$(II) + (III) \le C e^{t/2} \operatorname{tr}_\omega \widetilde{\omega}.$$

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To see this, from Lemma 3.1(iii) and the geometric-arithmetic means inequality, we can deduce $\operatorname{tr}_{\widetilde{\omega}} \omega \geq C^{-1}$ for a uniform constant C, which, combining with $|g|_{\widetilde{g}} \leq \operatorname{tr}_{\widetilde{\omega}} \omega$ and $|g^{-1}|_{\widetilde{g}} \leq \operatorname{tr}_{\omega} \widetilde{\omega}$, implies

$$\begin{aligned}
\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\big| g^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{\nabla}_i \overline{\widetilde{T}^q_{j\ell}} g_{k\bar{q}}\big| &\leq \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}|g^{-1}|_{\widetilde{g}}|\widetilde{g}^{-1}|_{\widetilde{g}}|\overline{\widetilde{\nabla}}\widetilde{T}|_{\widetilde{g}}|g|_{\widetilde{g}} \leq C(\operatorname{tr}_{\omega}\widetilde{\omega})e^{t/2},\\
\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\big| g^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{g}^{\bar{q}p} g_{k\bar{q}}\widetilde{R}_{i\bar{\ell}p\bar{j}}\big| &\leq \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}|g^{-1}|_{\widetilde{g}}|\widetilde{g}^{-1}|^2_{\widetilde{g}}|g|_{\widetilde{g}}|\widetilde{\mathrm{Rm}}|_{\widetilde{g}} \leq C(\operatorname{tr}_{\omega}\widetilde{\omega})e^{t/2},\\
\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\big| g^{\bar{j}i}\widetilde{\nabla}_i \overline{\widetilde{T}^\ell_{j\ell}}\big| &\leq \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}|g^{-1}|_{\widetilde{g}}|\overline{\widetilde{\nabla}}\widetilde{T}|_{\widetilde{g}} \leq C(\operatorname{tr}_{\omega}\widetilde{\omega})e^{t/2},\\
\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\big| g^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{g}_{p\bar{j}}\widetilde{\nabla}_{\bar{\ell}}\widetilde{T}^p_{ik}\big| &\leq \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}|g^{-1}|_{\widetilde{g}}|\widetilde{g}^{-1}|_{\widetilde{g}}|\widetilde{g}|_{\widetilde{g}}|\overline{\widetilde{\nabla}}\widetilde{T}|_{\widetilde{g}} \leq C(\operatorname{tr}_{\omega}\widetilde{\omega})e^{t/2},\\
\frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}\big| g^{\bar{j}i}\widetilde{g}^{\bar{\ell}k}\widetilde{T}^p_{ik}\overline{\widetilde{T}^q_{j\ell}}\widetilde{g}_{p\bar{q}}\big| &\leq \frac{1}{\operatorname{tr}_{\widetilde{\omega}} \omega}|g^{-1}|_{\widetilde{g}}|\widetilde{g}^{-1}|_{\widetilde{g}}|\widetilde{g}|_{\widetilde{g}}|\widetilde{T}|^2_{\widetilde{g}} \leq C(\operatorname{tr}_{\omega}\widetilde{\omega}),
\end{aligned}$$

which completes the proof of the claim.

To prove part (i), first note that Lemma 3.1(i) implies that $e^{t/2}\varphi$ is uniformly bound. Using the idea from Phong–Sturm [21], we consider the quantity

$$Q = \log \operatorname{tr}_{\widetilde{\omega}} \omega - Ae^{t/2}\varphi + \frac{1}{e^{t/2}\varphi + \widetilde{C}},$$

where $\widetilde{C}$ is a constant satisfying $e^{t/2}\varphi + \widetilde{C} \geq 1$ and A is a large constant that will be determined later. Notice that

$$0 \leq \frac{1}{e^{t/2}\varphi + \widetilde{C}} \leq 1.$$

Since $\Delta\varphi = m - \operatorname{tr}_{\omega}\widetilde{\omega}$, using the bounds for φ and $\dot{\varphi}$ in Lemma 3.1, we have

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t} - \Delta\Big)\Big(-Ae^{t/2}\varphi + \frac{1}{e^{t/2}\varphi + \widetilde{C}}\Big)\\
&\quad = -\Big(A + \frac{1}{(e^{t/2}\varphi + \widetilde{C})^2}\Big)(e^{t/2}\dot{\varphi} + \frac{1}{2}e^{t/2}\varphi)\\
&\qquad + \Big(A + \frac{1}{(e^{t/2}\varphi + \widetilde{C})^2}\Big)\Delta(e^{t/2}\varphi) - \frac{2|\partial(e^{t/2}\varphi)|^2_g}{(e^{t/2}\varphi + \widetilde{C})^3}\\
&\quad \leq CAe^{t/2} - Ae^{t/2}\operatorname{tr}_{\omega}\widetilde{\omega} - \frac{2|\partial(e^{t/2}\varphi)|^2_g}{(e^{t/2}\varphi + \widetilde{C})^3}.
\end{aligned} \tag{3.14}$$

At the point (x_0, t_0) with $t_0 > 0$ where Q attains a maximum, we have $\partial_{\bar{q}}Q = 0$, implying

$$\frac{\partial_{\bar{q}}\operatorname{tr}_{\widetilde{\omega}}\omega}{\operatorname{tr}_{\widetilde{\omega}}\omega} = \Big(A + \frac{1}{(e^{t_0/2}\varphi + \widetilde{C})^2}\Big)e^{t_0/2}\partial_{\bar{q}}\varphi.$$

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At this point,

$$
\begin{aligned}
&\frac{2}{(\operatorname{tr}_{\widetilde{\omega}}\omega)^2}\operatorname{Re}\big(\widetilde{g}^{\bar{\ell}i}g^{\bar{q}k}\widetilde{T}^{p}_{ki}\widetilde{g}_{p\bar{\ell}}\partial_{\bar{q}}\operatorname{tr}_{\widetilde{\omega}}\omega\big)\\
&\quad=\frac{2}{\operatorname{tr}_{\widetilde{\omega}}\omega}\operatorname{Re}\Bigg(\widetilde{g}^{\bar{\ell}i}g^{\bar{q}k}\widetilde{T}^{p}_{ki}\widetilde{g}_{p\bar{\ell}}\Bigg(A+\frac{1}{(e^{t_0/2}\varphi+\widetilde{C})^2}\Bigg)e^{t_0/2}\partial_{\bar{q}}\varphi\Bigg)\\
&\quad\le\frac{CA^2}{(\operatorname{tr}_{\widetilde{\omega}}\omega)^2}(e^{t_0/2}\varphi+\widetilde{C})^3g^{\bar{q}k}\widetilde{T}^{i}_{ki}\overline{\widetilde{T}^{r}_{qr}}+\frac{|\partial(e^{t_0/2}\varphi)|^2_g}{(e^{t_0/2}\varphi+\widetilde{C})^3}\\
&\quad\le\frac{CA^2}{(\operatorname{tr}_{\widetilde{\omega}}\omega)^2}\operatorname{tr}_{\omega}\widetilde{\omega}+\frac{|\partial(e^{t_0/2}\varphi)|^2_g}{(e^{t_0/2}\varphi+\widetilde{C})^3},
\end{aligned}
\tag{3.15}
$$

where for the last step we used Lemmas 3.1 and 3.2 and the Cauchy–Schwarz inequality for the quantity $g^{\bar{q}k}\widetilde{T}^{i}_{ki}\overline{\widetilde{T}^{r}_{qr}}$. Combining (3.12), (3.14), and (3.15), we have, at a point (x_0, t_0), for a uniform $C>0$,

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)Q&\le CA^2\operatorname{tr}_{\omega}\widetilde{\omega}+Ce^{t_0/2}\operatorname{tr}_{\omega}\widetilde{\omega}+CAe^{t_0/2}-Ae^{t_0/2}\operatorname{tr}_{\omega}\widetilde{\omega}\\
&\le CA^2\operatorname{tr}_{\omega}\widetilde{\omega}+Ce^{t_0/2}\operatorname{tr}_{\omega}\widetilde{\omega}+CAe^{t_0/2}-Ae^{t_0/2}\operatorname{tr}_{\omega}\widetilde{\omega},
\end{aligned}
$$

where we are assuming, without loss of generality, that at this maximum point of Q we have $\operatorname{tr}_{\widetilde{\omega}}\omega\ge 1$. Choose a uniform A satisfying $A\ge C+1$. We can also assume $t_0>T_0$, where T_0 satisfies

$$
CA^2-e^{t/2}\le -1,\quad \forall\ t\ge T_0.
$$

Then we can deduce

$$
[\operatorname{tr}_{\omega}\widetilde{\omega}](x_0, t_0)\le\frac{CAe^{t_0/2}}{e^{t_0/2}-CA^2}\in(CA, CA+C^2A^3),
$$

at the maximum of Q, implying that $[\operatorname{tr}_{\widetilde{\omega}}\omega](x_0, t_0)$ is bounded from above, by using Lemma 3.1(iii) and the fact (see, for example, [27, Corollary 3.5])

$$
\operatorname{tr}_{\widetilde{\omega}}\omega\le\frac{1}{(m-1)!}\frac{\omega^m}{\widetilde{\omega}^m}(\operatorname{tr}_{\omega}\widetilde{\omega})^{m-1}.
\tag{3.16}
$$

Combining the upper bound of $[\operatorname{tr}_{\widetilde{\omega}}\omega](x_0, t_0)$, the definition of Q, and Lemma 3.1(i) gives the uniform estimate

$$
\operatorname{tr}_{\widetilde{\omega}}\omega\le C
\tag{3.17}
$$

Again using (3.16) and Lemma 3.1(iii), the uniform upper bound (3.17) shows that ω is equivalent to $\widetilde{\omega}$.

As for part (ii), we claim that there exists a uniform constant $C>0$ satisfying

$$
\Big(\frac{\partial}{\partial t}-\Delta\Big)\operatorname{tr}_{\widetilde{\omega}}\omega\le -C^{-1}|\widetilde{\nabla}g|^2_g+Ce^{t/2},
\tag{3.18}
$$

$$
\Big(\frac{\partial}{\partial t}-\Delta\Big)\operatorname{tr}_{\omega}\alpha\le|\widetilde{\nabla}g|^2_g-C^{-1}|\nabla\operatorname{tr}_{\omega}\alpha|^2_g+Ce^{t/2}
\tag{3.19}
$$

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along the normalized Chern–Ricci flow (1.1). As a result, uniform positive constants C_0 and C_1 exist such that for all $t \geq 0$

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\operatorname{tr}_\omega \alpha + C_0 \operatorname{tr}_{\widetilde{\omega}} \omega) \leq -|\widetilde{\nabla} g|_g^2 - C_1^{-1}|\nabla \operatorname{tr}_\omega \alpha|_g^2 + C_1 e^{t/2}. \tag{3.20}$$

Indeed, the claim follows from an argument similar to the one in [32, Lemma 6.2] (see also [5]). From [30, Proposition 3.1], as the argument in the proof of inequality (3.13), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_{\widetilde{\omega}} \omega = J_1 + J_2 + J_3 - \widetilde{g}^{\overline{\ell}p}\widetilde{g}^{\overline{q}k}\alpha_{p\overline{q}}g_{k\overline{\ell}},$$

where

$$\begin{aligned}
J_1 &= -g^{\overline{j}p}g^{\overline{q}i}\widetilde{g}^{\overline{\ell}k}\widetilde{\nabla}_k g_{i\overline{j}}\widetilde{\nabla}_{\overline{\ell}}g_{p\overline{q}} - 2\operatorname{Re}\left(g^{\overline{j}i}\widetilde{g}^{\overline{\ell}k}\widetilde{T}^p_{ki}\widetilde{\nabla}_{\overline{\ell}}g_{p\overline{j}}\right) - g^{\overline{j}i}\widetilde{g}^{\overline{\ell}k}\widetilde{T}^p_{ik}\overline{\widetilde{T}^q_{j\ell}}g_{p\overline{q}},\\
J_2 &= g^{\overline{j}i}\widetilde{g}^{\overline{\ell}k}\left(\widetilde{\nabla}_i\overline{\widetilde{T}^q_{j\ell}} - \widetilde{R}_{i\overline{\ell}p\overline{j}}\widetilde{g}^{\overline{q}p}\right)g_{k\overline{q}},\\
J_3 &= -\left[g^{\overline{j}i}\left(\widetilde{\nabla}_i\overline{\widetilde{T}^\ell_{j\ell}}\right) + \left(\widetilde{\nabla}_{\overline{\ell}}\widetilde{T}^p_{ik}\right)g^{\overline{j}i}\widetilde{g}^{\overline{\ell}k}\widetilde{g}_{p\overline{j}} - \overline{\widetilde{T}^q_{j\ell}}\widetilde{T}^p_{ik}g^{\overline{j}i}\widetilde{g}^{\overline{\ell}k}\widetilde{g}_{p\overline{q}}\right].
\end{aligned}$$

Using Lemma 3.2(i) and the Cauchy–Schwarz inequality, we get (3.18).

Inequality (3.19) comes from a parabolic Schwarz Lemma argument as in [5,24,37]. A key difference is that we do not have a global holomorphic map from M_K to a lower dimensional complex manifold. Fortunately, we have a locally defined holomorphic map f from a holomorphic chart in M_K to the cross product $\mathbb{H}^{m-1}$ of $m-1$ upper half planes with the property that $\alpha = f^*\omega_{\mathbb{H}}$, where $\omega_{\mathbb{H}}$ is the multiple of the product of Poincaré metrics

$$\omega_{\mathbb{H}} = \sqrt{-1}\sum_{i=1}^{m-1}\frac{\mathrm{d}z_i \wedge \mathrm{d}\overline{z_i}}{4y_i^2}$$

on $\mathbb{H}^{m-1}$. Since the parabolic Schwarz Lemma calculation is completely local, we can deduce inequality (3.19) exactly as in [32].

Now we turn to the estimate of the bound of Chern scalar curvature R. First note that the minimum principle and the evolution equation of R $\frac{\partial R}{\partial t} = \Delta R + |\operatorname{Ric}|^2 + R$ imply the lower bound $R \geq -C$ directly. For the upper bound of R, we consider the quantity $u := \varphi + \dot{\varphi}$ with the property that $-\Delta u = R + \operatorname{tr}_\omega \alpha \geq R$. We want to bound $-\Delta u$ from above by $Ce^{t/2}$. Using a Cheng–Yau type argument in [3] (*cf.* [22,24]) and applying the maximum principle to

$$Q_1 := \frac{|\nabla u|^2}{A-u} + C_1(\operatorname{tr}_\omega \alpha + C_0 \operatorname{tr}_{\widetilde{\omega}} \omega)$$

for A and C_1 chosen sufficiently large, we can deduce the estimate $|\nabla u|_g^2 \leq Ce^{t/2}$, exactly as in [32, Proposition 6.3] (replacing ω_S with α wherever it occurs). A direct calculation gives

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|_g^2 \leq -\frac{1}{2}|\nabla\overline{\nabla}u|_g^2 - |\nabla\nabla u|_g^2 + |\nabla \operatorname{tr}_\omega \alpha|_g^2 + |\widetilde{\nabla}g|_g^2 + Ce^t. \tag{3.21}$$

On the other hand, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(-\Delta u) \leq 2|\nabla\overline{\nabla}u|_g^2 - \Delta u + Ce^{t/2} + |\widehat{\nabla}g|_g^2 - C^{-1}|\nabla \operatorname{tr}_\omega \alpha|_g^2. \tag{3.22}$$

Combining (3.20), (3.21), and (3.22) implies, for C_1 large,

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\big(-\Delta u+6|\nabla u|_g^2+C_1(\operatorname{tr}_\omega\alpha+C_0\operatorname{tr}_{\widetilde{\omega}}\omega)\big)\le -|\nabla\overline{\nabla}u|_g^2-\Delta u+Ce^t,$$

and it follows from the maximum principle that $-\Delta u\le Ce^{t/2}$, giving the upper bounded of the Chern scalar curvature $R\le Ce^{t/2}$. Next, using the discussion in [32, Lemma 6.4] (*cf.* [27]), we can deduce the bound (iii) on $\dot{\varphi}$ follows from the bounds on R, Lemma 3.1, and the evolution equation

$$\frac{\partial}{\partial t}\dot{\varphi}=-R-(m-1)-\dot{\varphi}.$$

For (iv), we first obtain, as in [32, Lemma 7.3] (replacing ω_S with α, whenever it occurs),

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\operatorname{tr}_\omega\widetilde{\omega}\le Ce^{t/2}-C^{-1}|\widetilde{\nabla}g|_g^2.$$

Combining this and the bounds of φ and $\dot{\varphi}$, we consider the quantity

$$e^{\varepsilon t}(\operatorname{tr}_\omega\widetilde{\omega}-m)-e^{\delta t}\varphi$$

for $0<\varepsilon<1/4$ and $\delta,\delta'>0$ chosen carefully. The maximum principle discussion of [32, Proposition 7.3] (replacing $\operatorname{tr}_\omega\widetilde{\omega}-2$ with $\operatorname{tr}_\omega\widetilde{\omega}-m$, whenever it occurs) gives

$$\operatorname{tr}_\omega\widetilde{\omega}-m\le Ce^{-\varepsilon t}. \tag{3.23}$$

On the other hand, there exists a uniform $T_I>0$ depending only on the initial data of M_K such that the following holds, for $t\ge T_I$:

$$\begin{aligned}\frac{\widetilde{\omega}^m}{\omega^m}&=\frac{e^t\widetilde{\omega}^m}{\Omega}\frac{\Omega}{e^t\omega^m}=\frac{e^t\widetilde{\omega}^m}{\Omega}e^{-\varphi-\dot{\varphi}}=e^{-\varphi-\dot{\varphi}}(1+O(e^{-t}))\\&\ge e^{-\varphi-\dot{\varphi}}-Ce^{-t}\ge 1-C'e^{-\sigma t},\end{aligned}\tag{3.24}$$

where we use (3.5), Lemma 3.1, and the bound of $\varphi+\dot{\varphi}$ in part (iii). From (3.23) and (3.24) we can apply [33, Lemma 2.6] (choose coordinates such that ω is the identity and $\widetilde{\omega}$ is given by matrix A and take $\varepsilon=\sigma$ in (3.23) and (3.24)) to get

$$\|\omega-\widetilde{\omega}\|_{C^0(M_K,\omega)}\le C_\sigma e^{-\sigma t/2}.$$

Noting $\omega\le C\omega_0$, we can deduce part (iv) for $t\ge T_I$. Then we can modify the uniform constant such that part (iv) holds for all $t\ge 0$. ■

Using the estimate in Theorem 3.3(i), we can get the Gromov–Hausdorff convergence of $\omega(t)$ in Theorem 1.1.

Proof of Theorem 1.1 Note that $\omega_\infty=\alpha$ represents $-c_1^{\mathrm{BC}}(M_K)$, and by definition of the reference metric, $\widetilde{\omega}(t)\to\alpha$ uniformly and exponentially fast as $t\to\infty$. Theorem 3.3(iv) implies that the same convergence holds for $\omega(t)$.

Now we turn to determining the Gromov–Hausdorff limit of $(M_K,\omega(t))$. Call

$$F=p\colon M_K\longrightarrow\mathbb{T}^{m-1}$$

the projection map and denote by $T_a=F^{-1}(a)$ the $\mathbb{T}^{m+1}$-fiber over $a\in\mathbb{T}^{m-1}$. Fix $\varepsilon>0$ and let L_t be the length of a curve in M_K measured with respect to metric $\omega(t)$. Let d_t be the induced distance function on M_K. Also denote by L_∞ and d_∞ the length and distance functions of the degenerate metric α on M_K and by L and d the length and distance functions of the flat Riemannian metric g defined by (2.8) on $\mathbb{T}^{m-1}$. Define

$$G: \mathbb{T}^{m-1} \longrightarrow M_K$$

by mapping every point $a \in \mathbb{T}^{m-1}$ to some chosen point in M_K on the fiber T_a. Note that the map G will in general be discontinuous.

Clearly, we have $F \circ G = \mathrm{Id}$, while $G \circ F$ is a fiber-preserving discontinuous map of M_K. In particular, for any $a \in \mathbb{T}^{m-1}$ we have trivially

$$\mathrm{d}(a, F \circ G(a)) = 0. \tag{3.25}$$

Since $\mathcal{F}$ is the kernel of α and each leaf of $\mathcal{F}$ is dense in a $\mathbb{T}^{m-1}$-fiber, we conclude that $\mathrm{d}_\infty(x, y) = 0$ for all $x, y \in M_K$ with $F(x) = F(y)$. Therefore, combining Theorem 3.3(i) and uniform convergence of $\widetilde{\omega}(t)$ to α as $t \to \infty$ implies that for any $x \in M_K$ and for all t large enough, we have

$$\mathrm{d}_t(x, G \circ F(x)) \leq \varepsilon. \tag{3.26}$$

Now take any two points $x, y \in M_K$ and let γ be a curve joining x to y with $L_t(\gamma) = \mathrm{d}_t(x, y)$. Then $F(\gamma)$ is a path in $\mathbb{T}^{m-1}$ between $F(x)$ and $F(y)$. We claim that

$$L\big(F(\gamma)\big) \leq L_\infty(\gamma).$$

Indeed, for any tangent vector V on M_K, we can write it locally as

$$V = \sum_{i=1}^{m} (X^i \partial_{x_i} + Y^i \partial_{y_i}),$$

and from the definitions of ω_∞ and g we see that

$$|F_* V|_g^2 = \sum_{i=1}^{m-1} \frac{(Y^i)^2}{2y_i^2} \leq \sum_{i=1}^{m-1} \frac{(X^i)^2 + (Y^i)^2}{2y_i^2} = |V|_{\omega_\infty}^2,$$

where for convenience we use the local coordinates $y_1, \dots, y_{m-1}$ on $\mathbb{T}^{m-1}$ and write the flat Riemannian metric g on $\mathbb{T}^{m-1}$ using the form of (2.6). Choosing $V = \dot{\gamma}$ implies the claim. Therefore, given two points $x, y \in M_K$, letting γ be a minimizing geodesic for the metric $\omega(t)$ joining them, we can get

$$\mathrm{d}\big(F(x), F(y)\big) \leq L\big(F(\gamma)\big) \leq L_\infty(\gamma) \leq L_t(\gamma) + \varepsilon = \mathrm{d}_t(x, y) + \varepsilon, \tag{3.27}$$

for all t large. Obviously this also implies that

$$\mathrm{d}(a, b) = \mathrm{d}\big(F \circ G(a), F \circ G(b)\big) \leq \mathrm{d}_t\big(G(a), G(b)\big) + \varepsilon, \tag{3.28}$$

for all $a, b \in \mathbb{T}^{m-1}$ and all t large.

Lastly, given $x, y \in M_K$, let γ be a minimizing geodesic in $\mathbb{T}^{m-1}$ connecting $F(x)$ and $F(y)$, and let $\widetilde{\gamma}$ be a lift of the curve γ starting at x; that is, $\widetilde{\gamma}$ is a curve in M_K with $F(\widetilde{\gamma}) = \gamma$ and initial point x. This lift can always be constructed because $F = p$ is the bundle projection map. We then concatenate $\widetilde{\gamma}$ with a curve $\widetilde{\gamma}_1$ contained in the fiber $T_{F(y)}$ joining the endpoint of $\widetilde{\gamma}$ with y, and obtain a curve $\widehat{\gamma}$ in M_K joining x and y. We claim that $L_\infty(\widetilde{\gamma}) = L(\gamma) = \mathrm{d}(F(x), F(y))$. In fact we can construct a lift $\widetilde{\gamma}$ such that in local coordinates we have $\dot{\widetilde{\gamma}} = \sum_{i=1}^{m-1} Y^i(t)\partial_{y_i}$, as in [31]. So we can obtain

$$|\dot{\gamma}|_g^2 = |F_* \dot{\widetilde{\gamma}}|_g^2 = \sum_{i=1}^{m-1} \frac{(Y^i(t))^2}{2y_i^2} = |\dot{\widetilde{\gamma}}|_{\omega_\infty}^2,$$

as required. Therefore, we can conclude that

(3.29) $$d_t(x, y) \le L_t(\widetilde{\gamma}) = L_t(\widetilde{\gamma}) + L_t(\widetilde{\gamma}_1) \le L_\infty(\widetilde{\gamma}) + 2\varepsilon = d\big(F(x), F(y)\big) + 2\varepsilon,$$

for all large t. This also implies

(3.30) $$d_t\big(G(a), G(b)\big) \le d\big(F \circ G(a), F \circ G(b)\big) + 2\varepsilon = d(a, b) + 2\varepsilon$$

for all $a, b \in \mathbb{T}^{m-1}$. Combining (3.25)–(3.30) implies that $(M_K, \omega(t))$ converges to $(\mathbb{T}^{m-1}, g)$ in the Gromov–Hausdorff sense. ∎

Now we turn to Theorem 1.3. With the special reference metric

$$\widetilde{\omega} = e^{-t}\omega_{\mathrm{OT}} + (1 - e^{-t})\alpha,$$

we can get better estimates than the ones in Lemma 3.2 as follows.

Lemma 3.4 *There exists a uniform constant C such that*

(i) $|\widetilde{T}|_{\widetilde{g}} \le C$,

(ii) $|\overline{\partial}\widetilde{T}|_{\widetilde{g}} + |\widetilde{\nabla}\widetilde{T}|_{\widetilde{g}} + |\widetilde{\mathrm{Rm}}|_{\widetilde{g}} \le C$,

(iii) $|\overline{\widetilde{\nabla}}\widetilde{\nabla}\widetilde{T}|_{\widetilde{g}} + |\widetilde{\nabla}\overline{\widetilde{\nabla}}\widetilde{T}|_{\widetilde{g}} + |\widetilde{\nabla}\widetilde{\mathrm{Rm}}|_{\widetilde{g}} \le C$.

Proof For this new reference metric $\widetilde{g}$, we also have

$$\widetilde{g}_{k\bar{k}} = e^{-t}(g_{\mathrm{OT}})_{k\bar{k}} + (1 - e^{-t})\alpha_{k\bar{k}} = O(1), \quad 1 \le k \le m-1,$$
$$\widetilde{g}_{k\bar{\ell}} = e^{-t}(g_{\mathrm{OT}})_{k\bar{\ell}} = O(e^{-t}), \quad \text{otherwise},$$

and that all the components of the inverse metric matrix $(\widetilde{g}^{\bar{\ell}k})$ are bounded by $O(1)$ except that $\widetilde{g}^{\overline{m}m}$ is bounded by $O(e^t)$. Part (i) was proved in Lemma 3.2.

Since $|\widetilde{\Gamma}^p_{ik}|^2_{\widetilde{g}} \le C$, that is, the $\widetilde{g}$ norm of the first order derivatives of $\widetilde{g}$ is bounded, using the Cauchy–Schwarz inequality, to complete the proof of the rest of the lemma, we just need to bound the $\widetilde{g}$ norm of the second and third order derivatives of the new reference metric $\widetilde{g}$. Since

(3.30) $$\partial_m\widetilde{g}_{k\bar{\ell}} = \partial_{\overline{m}}\widetilde{g}_{k\bar{\ell}} = 0, \quad 1 \le k, \ell \le m$$

and

(3.31) $$\partial_i\widetilde{g}_{m\overline{m}} = \partial_{\bar{i}}\widetilde{g}_{m\overline{m}} = O(e^{-t}), \quad \widetilde{g}_{i\overline{m}} = \widetilde{g}_{m\bar{i}} = 0, \quad 1 \le i \le m-1,$$

we can deduce

$$|\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}}|^2_{\widetilde{g}} = \sum_{i,j,k,\ell,r,s,p,q=1}^{m-1} (\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}})(\partial_s\partial_{\bar{r}}\widetilde{g}_{q\bar{p}})\widetilde{g}^{\bar{r}i}\widetilde{g}^{\bar{j}s}\widetilde{g}^{\bar{p}k}\widetilde{g}^{\bar{\ell}q}$$
$$+ \sum_{i,j,r,s=1}^{m-1} (\partial_i\partial_{\bar{j}}\widetilde{g}_{m\overline{m}})(\partial_s\partial_{\bar{r}}\widetilde{g}_{m\overline{m}})\widetilde{g}^{\bar{r}i}\widetilde{g}^{\bar{j}s}\widetilde{g}^{\overline{m}m}\widetilde{g}^{\overline{m}m} \le C$$

and

$$|\partial_a\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}}|^2_{\widetilde{g}} = \sum_{a,b,i,j,k,\ell,r,s,p,q=1}^{m-1} (\partial_a\partial_i\partial_{\bar{j}}\widetilde{g}_{k\bar{\ell}})(\partial_{\bar{b}}\partial_s\partial_{\bar{r}}\widetilde{g}_{q\bar{p}})\widetilde{g}^{\bar{b}a}\widetilde{g}^{\bar{r}i}\widetilde{g}^{\bar{j}s}\widetilde{g}^{\bar{p}k}\widetilde{g}^{\bar{\ell}q}$$
$$+ \sum_{a,b,i,j,r,s=1}^{m-1} (\partial_a\partial_i\partial_{\bar{j}}\widetilde{g}_{m\overline{m}})(\partial_{\bar{b}}\partial_s\partial_{\bar{r}}\widetilde{g}_{m\overline{m}})\widetilde{g}^{\bar{b}a}\widetilde{g}^{\bar{r}i}\widetilde{g}^{\bar{j}s}\widetilde{g}^{\overline{m}m}\widetilde{g}^{\overline{m}m} \le C,$$

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as required. ■

The estimates in Theorem 3.3 imply that the special reference metric $\widetilde{g}$ is equivalent to the solution metric g uniformly. The other key ingredient of the proof of Theorem 1.3 is the following Calabi-type "third order" estimate.

Proposition 3.5 *For the normalized Chern–Ricci flow* (1.1) *with the reference metric*

$$\widetilde{\omega} = e^{-t}\omega_{\mathrm{OT}} + (1-e^{-t})\alpha,$$

we have

$$|\widetilde{\nabla} g|_{\widetilde{g}} \le C. \tag{3.32}$$

Proof This Calabi-type estimate is very similar to the ones established in [23] and [32, Section 8] (see also [20]), so we give only a brief outline, pointing out the main differences. We consider the quantity $S := |\widetilde{\nabla} g|_g^2 = |\Psi|_g^2$, where $\Psi_{ij}^k = \Gamma_{ij}^k - \widetilde{\Gamma}_{ij}^k$ as in [23]. The quantity S is equivalent to $|\widetilde{\nabla} g|_{\widetilde{g}}^2$ because g is equivalent to $\widetilde{g}$.

Compared to the setup in [23], here we consider the normalized Chern–Ricci flow and the reference metric $\widetilde{g}$ now depends on time t, while the reference metric $\widehat{g}$ in [23] is fixed. Combining these differences and the calculation in [23], we observe that there is one new term in the evolution of the quantity S of the form

$$-2\,\mathrm{Re}\Big(g_{i\overline{r}} g^{\overline{u}j} g^{\overline{v}k} \widetilde{g}^{\overline{q}i} \widetilde{\nabla}_j \alpha_{k\overline{q}} \overline{\Psi_{uv}^r}\Big). \tag{3.33}$$

We claim that

$$|\widetilde{\nabla}\alpha|_{\widetilde{g}} \le C. \tag{3.34}$$

Indeed, since $|\widetilde{\Gamma}_{ij}^k|_{\widetilde{g}}^2 \le C$ and

$$|\alpha|_{\widetilde{g}}^2 = \sum_{i,j,k,\ell=1}^{m-1} \widetilde{g}^{\overline{j}i}\widetilde{g}^{\overline{\ell}k}\alpha_{i\overline{\ell}}\alpha_{k\overline{j}} = \sum_{i,j=1}^{m-1} \widetilde{g}^{\overline{j}i}\widetilde{g}^{\overline{i}j}\alpha_{i\overline{i}}\alpha_{j\overline{j}} \le C,$$

from the Cauchy–Schwarz inequality, to prove (3.34), it is enough to bound $\partial_i \alpha_{j\overline{\ell}}$. Noting that

$$\partial_m \alpha_{i\overline{j}} = \partial_{\overline{m}}\alpha_{i\overline{j}} = \alpha_{m\overline{\ell}} = \alpha_{\ell\overline{m}} = 0, \quad 1 \le i,\ j \le m-1,\ 1 \le \ell \le m,$$

we have

$$\begin{aligned} |\partial_i \alpha_{j\overline{\ell}}|_{\widetilde{g}}^2 &= \sum_{i,j,k,\ell,p,q=1}^{m-1} \widetilde{g}^{\overline{j}i}\widetilde{g}^{\overline{q}p}\widetilde{g}^{\overline{\ell}k}(\widetilde{\partial}_i\alpha_{k\overline{q}})(\widetilde{\partial}_{\overline{j}}\alpha_{p\overline{\ell}}) \\ &= \sum_{i,j,k,\ell=1}^{m-1} \widetilde{g}^{\overline{j}i}\widetilde{g}^{\overline{k}\ell}\widetilde{g}^{\overline{\ell}k}(\widetilde{\partial}_i\alpha_{k\overline{k}})(\widetilde{\partial}_{\overline{j}}\alpha_{\ell\overline{\ell}}) \le C, \end{aligned}$$

as required. Therefore, the new term (3.33) is of the order $O(\sqrt{S})$ and harmless. Combining [23, Remark 3.1] and the estimates in Lemma 3.4 gives the bound S, and hence we can obtain (3.32). ■

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Proof of Theorem 1.3 We claim that

$$|\widetilde{\Gamma} - \Gamma_{\mathrm{OT}}|_{g_{\mathrm{OT}}} \leq C, \tag{3.35}$$

where Γ_{OT} is the Christoffel symbols of g_{OT}. Indeed, we just need to bound the g_{OT} norm of $\widetilde{\Gamma}$. Thanks to (3.30) and (3.31), we have

$$\begin{aligned}
|\widetilde{\Gamma}|^2_{g_{\mathrm{OT}}} &= (g_{\mathrm{OT}})_{k\overline{q}}(g_{\mathrm{OT}})^{\overline{r}i}(g_{\mathrm{OT}})^{\overline{s}j}\widetilde{g}^{\overline{\ell}k}\widetilde{g}^{\overline{q}p}(\partial_i\widetilde{g}_{j\overline{\ell}})(\partial_{\overline{r}}\widetilde{g}_{p\overline{s}}) \\
&= (g_{\mathrm{OT}})_{m\overline{m}}(g_{\mathrm{OT}})^{\overline{r}i}(g_{\mathrm{OT}})^{\overline{s}j}\widetilde{g}^{\overline{m}m}\widetilde{g}^{\overline{m}m}(\partial_i\widetilde{g}_{j\overline{m}})(\partial_{\overline{r}}\widetilde{g}_{m\overline{s}}) \\
&\quad + \text{other terms bounded by constant} \\
&= (g_{\mathrm{OT}})_{m\overline{m}}(g_{\mathrm{OT}})^{\overline{r}i}(g_{\mathrm{OT}})^{\overline{m}m}\widetilde{g}^{\overline{m}m}\widetilde{g}^{\overline{m}m}0(\partial_i\widetilde{g}_{m\overline{m}})(\partial_{\overline{r}}\widetilde{g}_{m\overline{m}}) \\
&\quad + \text{other terms bounded by constant} \leq C,
\end{aligned}$$

as required. Now we use (3.32), (3.35), and the fact $\widetilde{g} \leq C g_{\mathrm{OT}}$ to deduce

$$|\nabla_{\mathrm{OT}} g|_{g_{\mathrm{OT}}} \leq |\widetilde{\nabla} g|_{g_{\mathrm{OT}}} + C \leq |\widetilde{\nabla} g|_{\widetilde{g}} + C \leq C,$$

which completes the proof of Theorem 1.3. ■

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