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EXPONENTIAL SUMS FOR $O(2n+1, q)$ AND THEIR APPLICATIONS

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Abstract. For a nontrivial additive character λ and a multiplicative character χ of the finite field with q elements (q a power of an odd prime), and for each positive integer r, the exponential sums $\sum \lambda((\operatorname{tr} w)^r)$ over $w \in SO(2n+1, q)$ and $\sum \chi(\det w)\lambda((\operatorname{tr} w)^r)$ over $O(2n+1, q)$ are considered. We show that both of them can be expressed as polynomials in q involving certain exponential sums. Also, from these expressions we derive the formulas for the number of elements w in $SO(2n+1, q)$ and $O(2n+1, q)$ with $(\operatorname{tr} w)^r = \beta$, for each β in the finite field with q elements.

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1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbb{F}_q$, χ a multiplicative character of $\mathbb{F}_q$, and let r be a positive integer. Throughout this paper, we assume that q is a power of an odd prime. Then we consider the exponential sum

$$\sum_{w \in SO(2n+1,q)} \lambda((\operatorname{tr} w)^r), \tag{1.1}$$

where $SO(2n+1, q)$ is a special orthogonal group over $\mathbb{F}_q$ (cf. (2.3)) and $\operatorname{tr} w$ is the trace of w. Also, we consider

$$\sum_{w \in O(2n+1,q)} \chi(\det w)\lambda((\operatorname{tr} w)^r), \tag{1.2}$$

where $O(2n+1, q)$ is an orthogonal group over $\mathbb{F}_q$ (cf. (2.2)) and $\det w$ is the determinant of w.

The main purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). We will show that (1.1) is a polynomial in q times

$$\sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \tag{1.3}$$

plus another polynomial in q involving certain exponential sums (cf. (2.14) (2.15)), of which O-estimates were given in [**14**]. On the other hand, the expression for (1.2)

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is that for (1.1) plus $\chi(-1)$ times a similar one corresponding to the subsum of (1.2) over $O(2n+1,q) - SO(2n+1,q) = \rho SO(2n+1,q)$ (cf. (2.12)).

In [**10**], the sums in (1.1) and (1.2) were studied for $r = 1$ and the connection of the sum in (1.2) for χ trivial with Hodges' signed generalized Kloosterman sum over nonsingular symmetric matrices was also investigated ([**5**], [**6**]). Since the sum in (1.3) vanishes for $r = 1$, the polynomials involving (1.3) do not appear in that case. For $r = 1$, similar sums for other classical groups over a finite field had been considered ([**7**]–[**12**], [**16**], [**17**]).

The sums in (1.1) and (1.2) may be viewed as generalizations to $SO(2n+1,q)$ and $O(2n+1,q)$, respectively, of the sum in (1.3) which was studied by several authors ([**1**]–[**3**]).

Another purpose of this paper is to find formulas for the number of elements w in $O(2n+1,q)$ and $SO(2n+1,q)$ with $(\operatorname{tr} w)^r = \beta$, for each $\beta \in \mathbb{F}_q$. We derive them from (5.2) based on well-known principles, though they could be also obtained from the expressions for (1.1) and (1.2) by specializing them to $r = q-1$ and $r = 1$ cases.

Finally, we state the main results of this paper. The reader is referred to the next section for some notations here.

THEOREM A. *For any nontrivial additive character λ of $\mathbb{F}_q$ and any positive integer r, the exponential sum over $SO(2n+1,q)$*

$$\sum_{w \in SO(2n+1,q)} \lambda((\operatorname{tr} w)^r)$$

is given by

$$\begin{aligned}
&q^{n^2-1}\Bigg\{\prod_{j=1}^{n}(q^{2j}-1) - \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \\
&\times \sum_{l=1}^{[(n-2b+2)/2]} q^{l-1}(q-1)^{n-2b+2-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Bigg\} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \\
&+ q^{n^2-1} \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \\
&\times \sum_{l=1}^{[(n-2b+2)/2]} q^{l} MK_{n-2b+2-2l}(\lambda^r; 1, 1; 1) \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),
\end{aligned}$$

where both of the unspecified sums run respectively over the set of integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \cdots \le j_1 \le n-2b+1$ and they are 1 for $l = 1$, and $MK_m(\lambda^r; a, b; c)$ is the exponential sum defined in (2.14) and (2.15).

THEOREM B. *With λ and r as above, let χ be a multiplicative character of $\mathbb{F}_q$. Then the exponential sum over $O(2n+1,q)$*

$$\sum_{w \in O(2n+1,q)} \chi(\det w)\lambda((\operatorname{tr} w)^r) \tag{0.6}$$

is given by

$$
\begin{aligned}
&(1+\chi(-1))q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^{2j}-1)-\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q \\
&\times\prod_{j=1}^{b}(q^{2j-1}-1)\sum_{l=1}^{[(n-2b+2)/2]}q^{l-1}(q-1)^{n-2b+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) \\
&+q^{n^2-1}\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1) \\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l}\big(MK_{n-2b+2-2l}(\lambda^r;1,1;1)+\chi(-1)MK_{n-2b+2-2l}(\lambda^r;1,1;-1)\big) \\
&\times\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),
\end{aligned}
$$

where both of the unspecified sums are as in Theorem A, and one is referred to (2.14) and (2.15) for $MK_m(\lambda^r;a,b;c)$.

THEOREM C. *For each $\beta\in\mathbb{F}_q$ and each positive integer r, the number of elements $N_{SO(2n+1,q)}(\beta;r)$ of all $w\in SO(2n+1,q)$ with $(\operatorname{tr} w)^r=\beta$ is given by*

$$
\begin{aligned}
&N(y^r=\beta)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1)+q^{n^2-1}\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1) \\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l-1}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1) \\
&\times\Big\{q\sum_{y^r=\beta}\delta(n-2b+2-2l,q;y-1)-N(y^r=\beta)(q-1)^{n-2b+2-2l}\Big\},
\end{aligned}
$$

where $N(y^r=\beta)$ denotes the number of y in $\mathbb{F}_q$ with $y^r=\beta$, $\delta(m,q;\gamma)$ is as in (4.27) and (4.28), and the sum in the curly bracket is over all $y\in\mathbb{F}_q$ with $y^r=\beta$.

The above Theorems A, B, and C are respectively stated below as Theorems 4.1, 4.2, and 5.2.

2. Preliminaries. In this section, we will fix some notations and gather some facts that will be needed in the sequel. For some elementary facts of the following, one is referred to [**4**] and [**16**].

Let $\mathbb{F}_q$ denote the finite field with q elements, $q=p^d$ (p an odd prime, d a positive integer).

In the following, $\operatorname{tr} A$ and $\det A$ denote respectively the trace of A and the determinant of A for a square matrix A, and ${}^tB$ denotes the transpose of B for any matrix B.

Let $GL(n,q)$ denote the group of all $n\times n$ invertible matrices with entries in $\mathbb{F}_q$. The order of $GL(n,q)$ equals

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$$g_n = \prod_{j=0}^{n-1}(q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1). \tag{2.1}$$

$O(2n+1, q)$ denotes the orthogonal group defined by

$$O(2n+1, q) = \left\{ w \in GL(2n+1, q) \mid {}^t wJw = J \right\}, \tag{2.2}$$

where

$$J = \begin{bmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also,

$$SO(2n+1, q) = \left\{ w \in O(2n+1, q) \mid \det w = 1 \right\} \tag{2.3}$$

is a subgroup of index 2 in $O(2n+1, q)$. It is well-known that

$$|O(2n+1, q)| = 2q^{n^2} \prod_{j=1}^{n}(q^{2j} - 1), \tag{2.4}$$

$$|SO(2n+1, q)| = q^{n^2} \prod_{j=1}^{n}(q^{2j} - 1). \tag{2.5}$$

Put

$$\begin{aligned} P &= P(2n+1, q) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^th \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{c} A \in GL(n, q), i = \pm 1, \\ B + {}^tB + {}^thh = 0 \end{array} \right\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} Q &= Q(2n+1, q) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^th \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{c} A \in GL(n, q), \\ B + {}^tB + {}^thh = 0 \end{array} \right\}. \end{aligned} \tag{2.7}$$

Here $Q(2n+1, q) = P(2n+1, q) \cap SO(2n+1, q)$ is a subgroup of index 2 in $P(2n+1, q)$.

It was noted in [**10**] that, starting from the Bruhat decomposition

$$O(2n+1, q) = \coprod_{b=0}^{n} P\sigma_b P,$$

one can get the following decompositions.

$$SO(2n+1,q) = \Big(\coprod_{\substack{0\le b\le n \\ b \text{ even}}} Q\sigma_b(B_b\backslash Q)\Big) \coprod \Big(\coprod_{\substack{0\le b\le n \\ b \text{ odd}}} (\rho Q)\sigma_b(B_b\backslash Q)\Big), \tag{2.8}$$

$$\begin{aligned} O(2n+1,q) = &\Big(\coprod_{\substack{0\le b\le n \\ b \text{ even}}} Q\sigma_b(B_b\backslash Q)\Big) \\ &\coprod \Big(\coprod_{\substack{0\le b\le n \\ b \text{ odd}}} (\rho Q)\sigma_b(B_b\backslash Q)\Big) \\ &\coprod \Big(\coprod_{\substack{0\le b\le n \\ b \text{ odd}}} Q\sigma_b(B_b\backslash Q)\Big) \\ &\coprod \Big(\coprod_{\substack{0\le b\le n \\ b \text{ even}}} (\rho Q)\sigma_b(B_b\backslash Q)\Big), \end{aligned} \tag{2.9}$$

where

$$B_b = B_b(q) = \{w \in Q(2n+1,q) \mid \sigma_b w \sigma_b^{-1} \in P(2n+1,q)\}, \tag{2.10}$$

$$\sigma_b = \begin{bmatrix} 0 & 0 & 1_b & 0 & 0 \\ 0 & 1_{n-b} & 0 & 0 & 0 \\ 1_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-b} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.11}$$

$$\rho = \begin{bmatrix} 1_n & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.12}$$

From (3.20) and the line just below (5.10) of [**10**] (cf. (2.18)), we have

$$|B_b(q)\backslash Q(2n+1,q)| = q^{\binom{b+1}{2}} \begin{bmatrix} n \\ b \end{bmatrix}_q. \tag{2.13}$$

Let λ be a nontrivial additive character of $\mathbb{F}_q$, $a, b, c \in \mathbb{F}_q$, and let r be a positive integer. Then we define the exponential sum $MK_m(\lambda^r; a, b; c)$ as

$$MK_m(\lambda^r; a, b; c) = \sum_{\gamma_1,\ldots,\gamma_m \in \mathbb{F}_q^\times} \lambda((a\gamma_1 + b\gamma_1^{-1} + \cdots + a\gamma_m + b\gamma_m^{-1} + c)^r) \tag{2.14}$$

for $m \ge 1$, and

$$MK_0(\lambda^r; a, b; c) = \lambda(c^r). \tag{2.15}$$

Note that, for $r = 1$,

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$$MK_m(\lambda; a, b; c) = \lambda(c)K(\lambda; a, b)^m, \tag{2.16}$$

where $K(\lambda; a, b)$ is the usual Kloosterman sum

$$K(\lambda; a, b) = \sum_{\gamma \in \mathbb{F}_q^\times} \lambda(a\gamma + b\gamma^{-1}). \tag{2.17}$$

For integers n, b with $0 \leq b \leq n$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ b \end{bmatrix}_q = \prod_{j=0}^{b-1} (q^{n-j} - 1)/(q^{b-j} - 1), \tag{2.18}$$

Then the q-binomial theorem says

$$\sum_{b=0}^{n} \begin{bmatrix} n \\ b \end{bmatrix}_q (-1)^b q^{\binom{b}{2}} x^b = (x; q)_n, \tag{2.19}$$

where

$$(x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}),$$

for x an indeterminate and n a nonnegative integer.

Finally, $[y]$ denotes the greatest integer $\leq y$, for a real number y.

3. A proposition. In this section, we will prove a proposition which is a generalization of Proposition 4.2 in [**10**] and is of use in the next section.

PROPOSITION 3.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$, $c \in \mathbb{F}_q$, r, b positive integers, and let Ω_b be the set of all $b \times b$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then*

$$a_b(\lambda; r; c) := \sum_{B \in \Omega_b} \sum_{h \in \mathbb{F}_q^{1 \times b}} \lambda((hB\,{}^t h + c)^r) \tag{3.1}$$

$$= q^{b-1} s_b \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) + q^{-1} a_b \sum_{\beta \in \mathbb{F}_q^\times} \lambda(c\beta) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r - \beta\gamma) \tag{3.2}$$

$$= (q^{b-1} s_b - q^{-1} a_b) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) + a_b \lambda(c^r), \tag{3.3}$$

where s_b is the number of all $b \times b$ nonsingular symmetric matrices over $\mathbb{F}_q$, and

$$a_b := a_b(\lambda; 1; 0) = \sum_{B \in \Omega_b} \sum_{h \in \mathbb{F}_q^{1 \times b}} \lambda(hB\,{}^t h)$$
$$= \begin{cases} q^{b(b+2)/4} \prod_{j=1}^{b/2} (q^{2j-1} - 1) & \text{for } b \text{ even}, \\ 0 & \text{for } b \text{ odd}, \end{cases} \tag{3.4}$$

(cf. [**10**, (4.10)]).

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REMARK. The independence of a_b from λ is clear from its definition or the expression of it in (3.4) above.

Proof. Put, for each $\gamma \in \mathbb{F}_q$,

$$N_\gamma = |\Big\{(B, h) \in \Omega_b \times \mathbb{F}_q^{1\times b} \mid hB\,{}^t h + c = \gamma\Big\}|.$$

Then $a_b(\lambda; r; c) = \sum_{\gamma\in\mathbb{F}_q} N_\gamma \lambda(\gamma^r)$, with

$$N_\gamma = q^{-1}\Big\{s_b q^b + \sum_{\beta\in\mathbb{F}_q^\times} \lambda(-\gamma\beta) \sum_{B,h} \lambda((hB\,{}^t h + c)\beta)\Big\}.$$

Hence

$$\begin{aligned} a_b(\lambda; r; c) &= q^{b-1} s_b \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) + q^{-1} \sum_{\beta\in\mathbb{F}_q^\times} \lambda(c\beta) \sum_{B,h} \lambda((hB\,{}^t h)\beta) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r - \beta\gamma) \\ &= q^{b-1} s_b \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) + q^{-1} a_b \sum_{\beta\in\mathbb{F}_q^\times} \lambda(c\beta) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r - \beta\gamma), \end{aligned}$$

since, as was noted in the above Remark, the sum over B, h is independent of $\beta \in \mathbb{F}_q^\times$. This shows (3.2). Now, (3.3) follows from (3.2) by interchanging the order of summation in the second term of (3.2). □

4. Main theorems. Let λ be a nontrivial additive character of $\mathbb{F}_q$, and let r be any positive integer. Then we will consider first the sum in (1.1)

$$\sum_{w\in SO(2n+1,q)} \lambda((\operatorname{tr} w)^r),$$

and find an explicit expression for this by using the decomposition in (2.8).

The sum in (1.1) can be written, using (2.8), as

$$\sum_{\substack{0\le b\le n \\ b \text{ even}}} |B_b \backslash Q| \sum_{w\in Q} \lambda((\operatorname{tr} w\sigma_b)^r) + \sum_{\substack{0\le b\le n \\ b \text{ odd}}} |B_b \backslash Q| \sum_{w\in Q} \lambda((\operatorname{tr} \rho w\sigma_b)^r), \tag{4.1}$$

where $B_b = B_b(q), Q = Q(2n+1, q), \sigma_b, \rho$ are respectively as in (2.10), (2.7), (2.11), (2.12).

Here one has to observe that, for each $q \in Q$,

$$\begin{aligned} \sum_{w\in Q} \lambda((\operatorname{tr} w\sigma_b q)^r) &= \sum_{w\in Q} \lambda((\operatorname{tr} qw\sigma_b)^r) \\ &= \sum_{w\in Q} \lambda((\operatorname{tr} w\sigma_b)^r) \end{aligned}$$

and $\rho^{-1} q\rho \in Q$. Write $w \in Q$ (cf. (2.7)) as

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$$w=\begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1_n & B & -{}^th \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix},$$

with

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad {}^tA^{-1}=\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},\quad B=\begin{bmatrix} B_{11} & -{}^tB_{21}-{}^th_1h_2 \\ B_{21} & B_{22} \end{bmatrix},$$

$$h=[h_1,h_2],$$

$$B_{11}+{}^tB_{11}+{}^th_1h_1=0,\quad B_{22}+{}^tB_{22}+{}^th_2h_2=0, \tag{4.2}$$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are respectively of sizes $b\times b$, $b\times(n-b)$, $(n-b)\times b$, $(n-b)\times(n-b)$, similarly for ${}^tA^{-1}$, B, and h_1 is of size $1\times b$.

Note here that, if we denote the upper right block of B by B_{12}, then $B_{12}+{}^tB_{21}+{}^th_1h_2=0$, and the conditions in (4.2) together with this are equivalent to $B+{}^tB+{}^thh=0$. For any b $(0\leq b\leq n)$,

$$\sum_{w\in Q}\lambda((\operatorname{tr} w\sigma_b)^r) \tag{4.3}$$

$$=\sum\lambda((\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{12}B_{21}+\operatorname{tr} A_{22}+\operatorname{tr} E_{22}+1)^r) \tag{4.4}$$

and

$$\sum_{w\in Q}\lambda((\operatorname{tr} \rho w\sigma_b)^r) \tag{4.5}$$

$$=\sum\lambda((\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{12}B_{21}+\operatorname{tr} A_{22}+\operatorname{tr} E_{22}-1)^r), \tag{4.6}$$

where both of the sums in (4.4) and (4.6) are respectively over $A, B_{11}, B_{21}, B_{22}, h$ subject to the conditions in (4.2).

Consider the sum in (4.4) first for the case $1\leq b\leq n-1$ so that A_{12} does appear. We separate the sum into two subsums, with $A_{12}\neq 0$ and with $A_{12}=0$; the latter is further divided into two subsums, with A_{11} symmetric or not. Namely, we write the sum in (4.4) as

$$\sum_{A_{12}\neq 0}\cdots+\sum_{\substack{A_{12}=0\\ A_{11}\text{ not symmetric}}}\cdots+\sum_{\substack{A_{12}=0\\ A_{11}\text{ symmetric}}}\cdots. \tag{4.7}$$

Let $A_{11}=(\alpha_{ij})$, $B_{11}=(\beta_{ij})$, $h_1=[h_{11}\ldots h_{1b}]$. Then the first condition in (4.2) is equivalent to

$$\beta_{ii}=-\frac{1}{2}h_{1i}^2\quad\text{for } 1\leq i\leq b, \tag{4.8}$$

$$\beta_{ij}+\beta_{ji}=-h_{1i}h_{1j}\quad\text{for } 1\leq i<j\leq b.$$

In particular, for each given h_1,

$$|\{\,B_{11}\mid B_{11}+{}^tB_{11}+{}^th_1h_1=0\,\}|=q^{\binom{b}{2}}. \tag{4.9}$$

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Similarly, for each given h_2,

$$|\{ B_{22} \mid B_{22} + {}^tB_{22} + {}^th_2h_2 = 0 \}| = q^{\binom{n-b}{2}}. \tag{4.10}$$

Also, as was noted in [**10**], one shows, using the relations in (4.8), that

$$\begin{aligned}\operatorname{tr} A_{11}B_{11} = -\frac{1}{2}h_1A_{11}{}^th_1 + \sum_{1\le i<j\le b} (\alpha_{ji} - \alpha_{ij}) \\ \times (\beta_{ij} + \frac{1}{2}h_{1i}h_{1j}).\end{aligned} \tag{4.11}$$

Using (4.10), the first sum in (4.7) is

$$q^{\binom{n+1-b}{2}} \sum_{\substack{A \text{ with } A_{12}\neq 0\\ B_{11},h_1}} \sum_{B_{21}} \lambda((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{12}B_{21} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22} + 1)^r). \tag{4.12}$$

Fix A with $A_{12} \neq 0$, B_{11}, h_1. Write $A_{12} = (\mu_{ij})$, $B_{21} = (\nu_{ij})$. Then $\mu_{kl} \neq 0$ for some k, l $(1 \le k \le b,\ 1 \le l \le n-b)$. For $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$, we have

$$\sum_{\gamma\in\mathbb{F}_q} \lambda((a\gamma+b)^r) = \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r),$$

and hence the inner sum of (4.12) equals

$$\sum_{\substack{\text{all } \nu_{ji}\\ \text{with } (j,i)\neq(l,k)}} \sum_{\nu_{lk}} \lambda((\mu_{kl}\nu_{lk} + \cdots)^r) = q^{b(n-b)-1} \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r). \tag{4.13}$$

Combining (4.12) and (4.13), and using (4.9), the first sum in (4.7) equals

$$q^{(n^2+n-2)/2}(g_n - g_bg_{n-b}q^{n(n-b)}) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r). \tag{4.14}$$

The subsum of the sum in (4.4) with $A_{12} = 0$ is

$$\begin{aligned}&\sum_{A_{21},B_{21},B_{22},h_2} \sum_{A_{11},A_{22},B_{11},h_1} \lambda((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r)\\ &= q^{\binom{n-b}{2}+(2b+1)(n-b)} \times \sum_{A_{11},A_{22},B_{11},h_1} \lambda((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r).\end{aligned} \tag{4.15}$$

The subsum of the sum in (4.15) with A_{11} not symmetric is

$$\sum_{\substack{A_{11} \text{ not symmetric}\\ A_{22},h_1}} \sum_{B_{11}} \lambda((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r). \tag{4.16}$$

Since $A_{11} = (\alpha_{ij})$ is not symmetric, $\alpha_{ts} - \alpha_{st} \neq 0$, for some s, t with $1 \le s < t \le b$. By the same argument as in the case of (4.12) and in view of (4.9) and (4.11), we see that the inner sum in (4.16) is

$$q^{\binom{b}{2}-1} \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) \tag{4.17}$$

Combining (4.15)–(4.17), we see that the middle sum in (4.7) is given by

$$q^{(n^2+n-2)/2+b(n-b)} g_{n-b}(g_b - s_b) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r), \tag{4.18}$$

where s_b denotes the number of $b \times b$ nonsingular symmetric matrices over $\mathbb{F}_q$ for each positive integer b.

The subsum of the sum in (4.15) with A_{11} symmetric, using (4.11), is

$$\begin{aligned} &\sum_{h_1} \sum_{B_{11}} \sum_{\substack{A_{22} \\ A_{11}\ \text{symmetric}}} \lambda((-\frac{1}{2} h_1 A_{11} {}^t h_1 + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r) \\ &= q^{\binom{b}{2}} \sum_{\substack{A_{22}, h_1 \\ A_{11}\ \text{symmetric}}} \lambda((h_1 A_{11} {}^t h_1 + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r). \end{aligned} \tag{4.19}$$

From (4.15) and (4.19), we see that the last sum in (4.7) is

$$q^{n(n+1)/2+b(n-b-1)} \times \sum_{A_{22}} \sum_{\substack{h_1 \\ A_{11}\ \text{symmetric}}} \lambda((h_1 A_{11} {}^t h_1 + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r). \tag{4.20}$$

For each fixed A_{22}, from (3.1) and (3.2) the inner sum of (4.20) is given by

$$\begin{aligned} &\sum_{\substack{h_1 \\ A_{11}\ \text{symmetric}}} \lambda((h_1 A_{11} {}^t h_1 + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)^r) \\ &= a_b(\lambda; r; \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1) \\ &= q^{b-1} s_b \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) + q^{-1} a_b \sum_{\beta \in \mathbb{F}_q^\times} \lambda(\beta(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + 1)) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r - \beta\gamma), \end{aligned} \tag{4.21}$$

where a_b is as in (3.4).

Summing (4.21) over A_{22}, we see that the double sum in (4.20) is

$$q^{b-1} g_{n-b} s_b \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) + q^{-1} a_b \sum_{\beta \in \mathbb{F}_q^\times} \lambda(\beta) K_{GL(n-b,q)}(\lambda; \beta, \beta) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r - \beta\gamma), \tag{4.22}$$

where, for $a, b \in \mathbb{F}_q$,

$$K_{GL(t,q)}(\lambda; a, b) = \sum_{w \in GL(t,q)} \lambda(a \operatorname{tr} w + b \operatorname{tr} w^{-1}). \tag{4.23}$$

From the explicit expression of (4.23) in [**8**, (4.19)], (4.22) can be written as

$$\begin{aligned} &q^{b-1} g_{n-b} s_b \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \\ &+ q^{(n-b-2)(n-b+1)/2-1} a_b \sum_{l=1}^{[(n-b+2)/2]} q^l \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \\ &\times \sum_{\gamma \in \mathbb{F}_q} \Big(\sum_{\beta \in \mathbb{F}_q^\times} K(\lambda; \beta, \beta)^{n-b+2-2l} \lambda((1-\gamma)\beta) \Big) \lambda(\gamma^r), \end{aligned} \tag{4.24}$$

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where the unspecified sum runs over all integers

$$j_1, \ldots, j_{l-1} \text{ satisfying } 2l-1 \leq j_{l-1} \leq j_{l-2} \leq \cdots \leq j_1 \leq n-b+1 \tag{4.25}$$

and it is 1 for $l = 1$ by our convention.

Recall from [**13**, (5.3)] that, for $\gamma \in \mathbb{F}_q$ and m nonnegative integer, we have

$$\sum_{\beta \in \mathbb{F}_q^\times} \lambda(-\gamma\beta) K(\lambda; \beta, \beta)^m = q\delta(m, q; \gamma) - (q-1)^m, \tag{4.26}$$

where, for $m \geq 1$,

$$\delta(m, q; \gamma) = |\left\{ (\alpha_1, , \alpha_m) \in (\mathbb{F}_q^\times)^m \mid \alpha_1 + \alpha_1^{-1} + \cdots + \alpha_m + \alpha_m^{-1} = \gamma \right\}| \tag{4.27}$$

and

$$\delta(0, q; \gamma) = \begin{cases} 1 & \text{if } \gamma = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

It is immediately seen from (4.26) that

$$\begin{aligned} &\sum_{\gamma \in \mathbb{F}_q} \Big(\sum_{\beta \in \mathbb{F}_q^\times} K(\lambda; \beta, \beta)^m \lambda((1-\gamma)\beta) \Big) \lambda(\gamma^r) \\ &= qMK_m(\lambda^r; 1, 1; 1) - (q-1)^m \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r), \end{aligned} \tag{4.29}$$

where $MK_m(\lambda^r; a, b; c)$ is as in (2.14) and (2.15).

Substituting the expression in (4.29) into (4.24), the double sum in (4.20) equals

$$\begin{aligned} &\Big\{ q^{b-1} g_{n-b} s_b - q^{(n-b-2)(n-b+1)/2} a_b \\ &\times \sum_{l=1}^{[(n-b+2)/2]} q^{l-1} (q-1)^{n-b+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \Big\} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \\ &+ q^{(n-b-2)(n-b+1)/2} a_b \sum_{l=1}^{[(n-b+2)/2]} q^l MK_{n-b+2-2l}(\lambda^r; 1, 1; 1) \\ &\times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \end{aligned} \tag{4.30}$$

where both of the unspecified sums run over the same set of integers as in (4.25) and they are 1 for $l = 1$.

For $1 \leq b \leq n-1$, (4.14), (4.18), and (4.20) with the expression of the double sum there in (4.30) add up to give

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$$\sum_{w\in Q}\lambda((\operatorname{tr} w\sigma_b)^r)$$
$$= q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^j-1) - q^{-\binom{b+1}{2}}a_b$$
$$\times \sum_{l=1}^{[(n-b+2/2]} q^{l-1}(q-1)^{n-b+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) \tag{4.31}$$
$$+ q^{n^2-1}q^{-\binom{b+1}{2}}a_b$$
$$\times \sum_{l=1}^{[(n-b+2)/2]} q^{l}MK_{n-b+2-2l}(\lambda^r;1,1;1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where both of the unspecified sums are as in (4.30).

It is an easy matter to check that, even for $b=0$ and $b=n$, the sum in (4.3) is given by the same expression as in (4.31) with the convention $a_0=1$. The details are left to the reader. Also, we observe that $a_0=1$ is natural in view of the formula in (3.4).

Glancing through the above argument for the sum in (4.3), we see that the sum in (4.5) is given by the same expression as in (4.31), except that $MK_{n-b+2-2l}(\lambda^r;1,1;1)$ is now replaced by $MK_{n-b+2-2l}(\lambda^r;1,1;-1)$. Namely, for any b with $0\le b\le n$, the sum in (4.5) equals

$$\sum_{w\in Q}\lambda((\operatorname{tr} \rho w\sigma_b)^r)$$
$$= q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^j-1) - q^{-\binom{b+1}{2}}a_b$$
$$\times \sum_{l=1}^{[(n-b+2)/2]} q^{l-1}(q-1)^{n-b+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) \tag{4.32}$$
$$+ q^{n^2-1}q^{-\binom{b+1}{2}}a_b$$
$$\times \sum_{l=1}^{[(n-b+2)/2]} q^{l}MK_{n-b+2-2l}(\lambda^r;1,1;-1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where both of the unspecified sums are as in (4.30).

Now the next theorem follows from (2.13), (3.4), (4.1), (4.31), (4.32).

THEOREM 4.1. *For any nontrivial additive character λ of $\mathbb{F}_q$ and any positive integer r, the exponential sum over $SO(2n+1,q)$*

$$\sum_{w\in SO(2n+1,q)}\lambda((\operatorname{tr} w)^r)$$

is given by

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$$
\begin{aligned}
&q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^{2j}-1)-\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l-1}(q-1)^{n-2b+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)\\
&+q^{n^2-1}\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l}MK_{n-2b+2-2l}(\lambda^r;1,1;1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),
\end{aligned}
\tag{4.33}
$$

where both of the unspecified sums run respectively over the set of integers $j_1, \dots, j_{l-1}$ *satisfying* $2l-1 \leq j_{l-1} \leq \cdots \leq j_1 \leq n-2b+1$ *and they are 1 for* $l = 1$.

With λ, r as before, let χ be a multiplicative character of $\mathbb{F}_q$. Then we next want to consider the sum in (1.2)

$$
\sum_{w\in O(2n+1,q)}\chi(\det w)\lambda((\operatorname{tr} w)^r)
$$

and to find an explicit expression for it.

From the decompositions in (2.8) and (2.9), we see that the above sum is $\sum_{w\in SO(2n+1,q)}\lambda((\operatorname{tr} w)^r)$ plus

$$
\begin{aligned}
&\chi(-1)\Big\{\sum_{\substack{0\leq b\leq n\\ \text{b odd}}}|B_q\backslash Q|\sum_{w\in Q}\lambda((\operatorname{tr} w\sigma_b)^r)\\
&+\sum_{\substack{0\leq b\leq n\\ \text{b even}}}|B_b\backslash Q|\sum_{w\in Q}\lambda((\operatorname{tr}\rho w\sigma_b)^r)\Big\}.
\end{aligned}
\tag{4.34}
$$

The following expression of (4.34) is obtained from (2.13), (3.4), (4.31), (4.32).

$$
\begin{aligned}
&\chi(-1)q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^{2j}-1)-\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l-1}(q-1)^{n-2b+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)\\
&+\chi(-1)q^{n^2-1}\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l}MK_{n-2b+2-2l}(\lambda^r;1,1;-1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),
\end{aligned}
\tag{4.35}
$$

where both of the unspecified sums are as in (4.33).

Adding up (4.33) and (4.35), we get the following result.

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THEOREM 4.2. *Let λ, χ be respectively a nontrivial additive and a multiplicative character of $\mathbb{F}_q$, and let r be a positive integer. Then the exponential sum of $O(2n+1, q)$*

$$\sum_{w \in O(2n+1,q)} \chi(\det w)\lambda((\operatorname{tr} w)^r)$$

is given by

$$\begin{aligned}
&(1+\chi(-1))q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^{2j}-1) - \sum_{b=0}^{[n/2]} q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \\
&\times \sum_{l=1}^{[(n-2b+2)/2]} q^{l-1}(q-1)^{n-2b+2-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu - 2\nu}-1)\Big\} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \\
&+ q^{n^2-1}\sum_{b=0}^{[n/2]} q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \qquad (4.36)\\
&\times \sum_{l=1}^{[(n-2b+2)/2]} q^{l}(MK_{n-2b+2-2l}(\lambda^r; 1, 1; 1) + \chi(-1)MK_{n-2b+2-2l}(\lambda^r; 1, 1; -1)) \\
&\times \sum \prod_{\nu=1}^{l-1}(q^{j_\nu - 2\nu}-1),
\end{aligned}$$

where both of the unspecified sums run over the same set of integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-2b+1$ and they are 1 for $l = 1$.

REMARKS (1) It is well-known [**18**, 5.30] that

$$\sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) = \sum_{j=1}^{e-1} G(\psi^j, \lambda),$$

where ψ is a multiplicative character of $\mathbb{F}_q$ of order $e = (r, q-1)$ and $G(\psi^j, \lambda)$ is the usual Gauss sum given by

$$G(\psi^j, \lambda) = \sum_{\gamma \in \mathbb{F}_q^\times} \psi^j(\gamma)\lambda(\gamma).$$

(2) From the expression of $\sum_{w \in Sp(2n,q)} \lambda((\operatorname{tr} w)^r)$ in [**13**, (4.22)], we see that our expression of $\sum_{w \in SO(2n+1,q)} \lambda((\operatorname{tr} w)^r)$ in (4.33) is the same as that, except that each $MK_{n-2b+2-2l}(\lambda^r; 1, 1)$ there is now replaced by $MK_{n-2b+2-2l}(\lambda^r; 1, 1; 1)$.

In the special case of $r = 1$, as was noted in [**10**, (5.12), (6.2)] or can be seen from the expressions in (4.33) and (4.36) (cf. [**8**, Theorem 5.4]), the sums in (1.1) and (1.2) are just constant multiples of the similar sum over the symplectic group $Sp(2n, q)$.

Namely, we have the following identities:

$$\sum_{w \in SO(2n+1,q)} \lambda(\operatorname{tr} w) = \lambda(1) \sum_{w \in Sp(2n,q)} \lambda(\operatorname{tr} w),$$

$$\sum_{w \in O(2n+1,q)} \chi(\det w)\lambda(\operatorname{tr} w) = (\lambda(1) + \chi(-1)\lambda(-1)) \sum_{w \in Sp(2n,q)} \lambda(\operatorname{tr} w).$$

5. Applications to certain countings. Let $G(q)$ be one of finite classical groups over $\mathbb{F}_q$. For each $\beta \in \mathbb{F}_q$ and each positive integer r, we put

$$N_{G(q)}(\beta; r) = |\{ w \in G(q) \mid (\operatorname{tr} w)^r = \beta \}|. \tag{5.1}$$

To derive formulas for (5.1) with $G(q) = SO(2n+1, q)$ and $O(2n+1, q)$, we will apply the results in Section 4. First, we collect some results from **[14]** with one modification.

For a nontrivial additive character λ of $\mathbb{F}_q$, a nonnegative integer m, and with β, r as above, we have

$$N_{G(q)}(\beta; r) = q^{-1}|G(q)| + q^{-1} \sum_{\alpha \in \mathbb{F}_q^\times} \lambda(-\beta\alpha) \sum_{w \in G(q)} \lambda(\alpha(\operatorname{tr} w)^r), \tag{5.2}$$

$$\sum_{\alpha \in \mathbb{F}_q^\times} \lambda(-\beta\alpha) \sum_{\gamma \in \mathbb{F}_q} \lambda(\alpha\gamma^r) = q\{N(y^r = \beta) - 1\}, \tag{5.3}$$

$$\sum_{\alpha \in \mathbb{F}_q^\times} \lambda(-\beta\alpha)\Big\{ \sum_{\gamma_1, \ldots, \gamma_m \in \mathbb{F}_q^\times} \lambda(\alpha(\gamma_1 + \gamma_1^{-1} + +\gamma_m + \gamma_m^{-1} \pm 1)^r)\Big\} \tag{5.4}$$

$$= q \sum_{y^r = \beta} \delta(m, q; y \mp 1) - (q-1)^m, \tag{5.5}$$

where

$$N(y^r = \beta) = |\{ y \in \mathbb{F}_q \mid y^r = \beta \}|, \tag{5.6}$$

$\delta(m, q; r)$ is as in (4.27) and (4.28), and the sum in (5.5) is over all $y \in \mathbb{F}_q$ with $y^r = \beta$. Note that, for $r = 1$, (5.5) follows from (4.26) and vice versa.

For each $\alpha \in \mathbb{F}_q^\times$, $\tilde{\lambda}(u) = \lambda(\alpha u)$ is a nontrivial additive character of $\mathbb{F}_q$. So the explicit expression of $\sum_{w \in O(2n+1,q)} \lambda(\alpha(\operatorname{tr} w)^r)$ is given by (4.36) with λ replaced by $\tilde{\lambda}$ and with χ trivial, i.e., with $\sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r)$ replaced by $\sum_{\gamma \in \mathbb{F}_q} \lambda(\alpha\gamma^r)$ and $MK_m(\lambda^r; 1, 1; 1) + MK_m(\lambda^r; 1, 1; -1)$ for various values of m replaced by the sum of the sums in the curly bracket of (5.4) for the same corresponding values of m; that of $\sum_{w \in SO(2n+1,q)} \lambda(\alpha(\operatorname{tr} w)^r)$ is given by (4.33) with the same replacement for $\sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r)$ and with $MK_m(\lambda^r; 1, 1; 1)$ replaced by the sum corresponding to '$+1$' in the curly bracket of (5.4).

The following theorems now follow immediately from these observations together with (2.4), (2.5), (4.33), (4.36), (5.2)–(5.5).

THEOREM 5.1. *For each $\beta \in \mathbb{F}_q$ and each positive integer r, $N_{O(2n+1,q)}(\beta; r)$ defined by (5.1), with $G(q) = O(2n+1, q)$, is given by*

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$$
\begin{aligned}
&2N(y^r=\beta)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1)\\
&+q^{n^2-1}\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l-1}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\\
&\times\Big\{q\Big(\sum_{y^r=\beta}\delta(n-2b+2-2l,q;y+1)\\
&+\sum_{y^r=\beta}\delta(n-2b+2-2l,q;y-1)\Big)-2N(y^r=\beta)(q-1)^{n-2b+2-2l}\Big\},
\end{aligned}
\tag{5.7}
$$

where $N(y^r=\beta)$, $\delta(m,q;\gamma)$ are respectively as in (5.6), (4.27)–(4.28)*, and the unspecified sum is over the set of integers $j_1,\dots,j_{l-1}$ satisfying $2l-1\le j_{l-1}\le\cdots\le j_1\le n-2b+1$ and it is 1 for $l=1$.*

THEOREM 5.2. *For each $\beta\in\mathbb{F}_q$ and each positive integer r, $N_{SO(2n+1,q)}(\beta;r)$ defined by (5.1), with $G(q)=SO(2n+1,q)$, is given by*

$$
\begin{aligned}
&N(y^r=\beta)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1)\\
&+q^{n^2-1}\sum_{b=0}^{[n/2]}q^{b(b+1)}\begin{bmatrix} n \\ 2b \end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2b+2)/2]}q^{l-1}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\\
&\times\Big\{q\sum_{y^r=\beta}\delta(n-2b+2-2l,q;y-1)-N(y^r=\beta)(q-1)^{n-2b+2-2l}\Big\},
\end{aligned}
\tag{5.8}
$$

where $N(y^r=\beta)$, $\delta(m,q;\gamma)$, and the unspecified sum is as in Theorem 5.1.

REMARKS. (1) For a finite classical group $G(q)$ over $\mathbb{F}_q$, we write, for brevity,

$$
\begin{aligned}
N_{G(q)}(\beta) &:= N_{G(q)}(\beta;1)\\
&=|\{\,w\in G(q)\mid \operatorname{tr} w=\beta\,\}|.
\end{aligned}
$$

Formulas for $N_{O(2n+1,q)}(\beta)$ and $N_{SO(2n+1,q)}(\beta)$ can now be obtained respectively from (5.7) and (5.8) by setting $r=1$, which amounts to replacing $N(y^r=\beta)$ by 1 and $\sum_{y^r=\beta}\delta(m,q;y\pm1)$ for various m by $\delta(m,q;\beta\pm1)$.

The reversed ways are also possible by noting that

$$N_{G(q)}(\beta;r)=\sum_{y^r=\beta}N_{G(q)}(y).$$

(2) $N(y^r=\beta)$ appearing in the above theorems can be expressed [**18**, (5.70)] as

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$$N(y^r = \beta) = \sum_{j=0}^{e-1} \psi^j(\beta),$$

where ψ is any multiplicative character of $\mathbb{F}_q$ of order $e = (r, q-1)$.

REFERENCES

1. B. C. Berndt and R. J. Evans, Sums of Gauss, Jacobi, and Jacobsthal, *J. Number Theory* **11** (1979), 349–398.

2. B. C. Berndt and R. J. Evans, Sums of Gauss, Eisenstein, Jacobi, Jacobsthal, and Brewer, *Illinois J. Math.* **23** (1979), 374–437.

3. B. C. Berndt and R. J. Evans, The determination of Gauss sums, *Bull. Amer. Math. Soc. (N.S.)* **5** (1981), 107–129.

4. B. C. Berndt, R. J. Evans, and K. S. Williams, Gauss and Jacobi sums, Canad. Math. Soc. Series of Monographs and Advanced Texts, Vol. 21 (Wiley-Interscience, 1998).

5. J. H. Hodges, Exponential sums for symmetric matrices in a finite field, *Math. Nachr.* **14** (1955), 331–339.

6. J. H. Hodges, Weighted partitions for symmetric matrices in a finite field, *Math. Z.* **66** (1956), 13–24.

7. D. S. Kim, Gauss sums for general and special linear groups over a finite field, *Arch. Math. (Basel)* **69** (1997), 297–304.

8. D. S. Kim, Gauss sums for symplectic groups over a finite field, *Monatsh. Math.* **126** (1998), 55–71.

9. D. S. Kim, Gauss sums for $O^-(2n, q)$, *Acta Arith.* **80** (1997), 343–365.

10. D. S. Kim, Gauss sums for $O(2n+1, q)$, *Finite Fields Appl.* **4** (1998), 62–86.

11. D. S. Kim, Gauss sums for $U(2n, q^2)$, *Glasgow Math. J.* **40** (1998), 79–95.

12. D. S. Kim, Gauss sums for $U(2n+1, q^2)$, *J. Korean Math. Soc.* **34** (1997), 871–894.

13. D. S. Kim, Exponential sums for symplectic groups and their applications, *Acta Arith.* **88** (1999), 151–171.

14. D. S. Kim, Exponential sums for $O^+(2n, q)$ and their applications, *Acta Math. Hungarica*, to appear.

15. D. S. Kim, Exponential sums for $O^-(2n, q)$ and their applications, *Acta Arith.* **97** (2001), 67–86.

16. D. S. Kim and I.-S. Lee, Gauss sums for $O^+(2n, q)$, *Acta Arith.* **78** (1996), 75–89.

17. D. S. Kim and Y. H. Park, Gauss sums for orthogonal groups over a finite field of characteristic two, *Acta Arith.* **82** (1997), 331–357.

18. R. Lidl and H. Niederreiter, *Finite Fields*, Encyclopedia of Mathematics and Its Applications, Vol. 20 (Cambridge University Press, 1987).