Can. J. Math., Vol. XXXIX, No. 6, 1987, pp. 1489-1530

HARMONIC MAPPINGS ONTO CONVEX DOMAINS

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1. Introduction. Let D be a simply-connected domain and w_0 a fixed point of D. Denote by S_D the set of all complex-valued, harmonic, orientation-preserving, univalent functions f from the open unit disk U onto D with $f(0) = w_0$. Unlike conformal mappings, harmonic mappings are not essentially determined by their image domains. So, it is natural to study the set S_D.

In Section 2, we give some mapping theorems. We prove the existence, when D is convex and unbounded, of a univalent, harmonic solution f of the differential equation

$$\overline{f_{\bar{z}}} = af_z, \quad z \in U,$$

where a is analytic and $|a| < 1$, such that $f(U) \subset D$ and

$$\hat{f}(e^{it}) = \lim_{r \to 1} f(re^{it}) \in \partial D \text{ a.e.}$$

General bounded domains with locally connected boundaries were considered earlier in [7]. We show also that if D is convex and unbounded and if $f \in S_D$, then $f + AP(\cdot, t)$ is a univalent, orientation-preserving mapping onto a convex domain for suitable constants A and t. Here, and in what follows, P denotes the Poisson kernel

$$P(z, t) = \frac{1}{2\pi} \operatorname{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right].$$

In Section 3, we choose D to be a wedge W. We determine the extreme points for the closed convex hull of S_W. As an application, we estimate the Fourier coefficients. In Section 4, D is chosen to be a half-plane, and we carry out a parallel development.

In Section 5, we give an application to nonparametric minimal surfaces over D when D is a wedge, a half-plane, or a strip. In particular, we give lower bounds for the Gaussian curvature of the surface over a point in D.

2. Mapping theorems. Recall that f belongs to the space h^1 if f is harmonic in U and

Received November 25, 1986. The work of the second author was supported in part by a grant from the National Science Foundation (USA).

$$\int_0^{2\pi} |f(re^{it})|dt$$

is bounded as $r \to 1$. Since we admit complex-valued functions, this implies (cf. [**3**]) that

$$f(z) = \int_0^{2\pi} P(z, t)d\mu(t)$$

where μ is a complex-valued measure of finite variation. Furthermore, the radial limits

$$\hat{f}(e^{it}) = \lim_{r\to 1} f(re^{it})$$

exist a.e., and $\hat{f}$ is equal a.e. to the Radon-Nikodym derivative of $d\mu$.

The following lemma asserts that, under certain hypotheses, the absolutely continuous part of a limit measure is the limit of the absolutely continuous parts, at least for a subsequence. Its proof was suggested to us by J. G. Stampfli.

LEMMA 2.1. *Let g_n and g belong to $L^1[0, 2\pi]$,*

$$\lim_{n\to\infty} g_n = g \text{ a.e.,} \quad \textit{and}$$

$$\int_0^{2\pi} |g_n(x)|dx \leqq M \quad \textit{for all } n = 1, 2, 3, \dots .$$

Then there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $g_{n_k}dx$ converges in the weak $*$ topology as $k \to \infty$ to $gdx + ds$, where ds is a singular measure on $[0, 2\pi]$ with respect to the Lebesgue measure dx.

Proof. Alaoglu's theorem implies that there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $h \in L^1[0, 2\pi]$ so that

$$g_{n_k}dx \to hdx + ds \quad \text{as } k \to \infty$$

in the weak $*$ topology, where ds is singular with respect to Lebesgue measure. For each $\epsilon > 0$ there is, by Egoroff's theorem, a compact subset A_ϵ of $[0, 2\pi]$ with measure $2\pi - \epsilon$ such that $g_{n_k} \to g$ uniformly on A_ϵ as $k \to \infty$. Since the restriction of $hdx + ds$ to A_ϵ is also the weak $*$ limit of g_{n_k} on A_ϵ, we may conclude that $h = g$ a.e. on A_ϵ. Let $\epsilon \to 0$. Then $h = g$ a.e. on $[0, 2\pi]$.

The following application of Lemma 2.1 will be useful in our mapping theorem.

LEMMA 2.2. *Assume that*

$$f_n(z) = \int_0^{2\pi} P(z, t)\hat{f}_n(e^{it})dt$$

and that

$$\lim_{n\to\infty} \hat{f}_n = g \quad \text{a.e.}$$

where $\hat{f}_n$, $g \in L^1$ *and*

$$\int_0^{2\pi} |\hat{f}_n(e^{it})|dt \leqq M \quad \textit{for all } n = 1, 2, 3, \dots .$$

If f_n *converges locally uniformly on* U *to a function* $f \in h^1$, *then* $\hat{f} = g$ a.e.

Proof. Lemma 2.1 implies that there is a subsequence f_{n_k} that converges for each $z \in U$ to

$$\int_0^{2\pi} P(z, t)[g(e^{it})dt + ds]$$

where ds is singular. On the other hand,

$$f_{n_k}(z) \to f(z) \quad \text{as } k \to \infty,$$

and so

$$f(z) = \int_0^{2\pi} P(z, t)[g(e^{it})dt + ds] \quad \text{for all } z \in U.$$

Hence $\hat{f} = g$ a.e.

The following is our mapping theorem. The linear space of analytic functions on U is denoted by $H(U)$.

THEOREM 2.3. *Let D be a convex domain. Fix $w_0 \in D$, and let $a \in H(U)$ satisfy $a(U) \subset U$. Then there exists a univalent, harmonic, orientation-preserving mapping f with the following properties.*

(a) $f(U) \subset D$, $f(0) = w_0$, *and* $f_z(0) > 0$;

(b) f *is a solution of* $\overline{f_{\bar{z}}} = af_z$;

(c) *the limits* $\lim_{r\to 1} f(re^{it})$ *exist and belong to* ∂D *for* a.e. t.

Proof. Case 1. If D is convex and bounded, then ∂D is locally connected and this theorem is a special case of [7, Theorem 4.2].

Case 2. If D is convex and unbounded, but neither a strip nor a half-plane, then there is an infinite wedge containing D. Since the mapping $w \to \alpha w + \beta$ preserves the convexity of the domain, harmonicity, and the form of the equation, we may assume without loss of generality that D is contained in the wedge bounded by the rays $L_+ : te^{i\alpha}$ and $L_- : te^{-i\alpha}$, $t \geqq 0$, where α is fixed and $0 < \alpha < \pi/2$.

Let φ be the univalent analytic mapping from U onto D with $\varphi(0) = w_0$ and $\varphi'(0) > 0$. Let

$$D_n = \varphi(\,|z| < n/(n + 1)\,) \quad \text{and}$$

$$a_n(z) = a(nz/(n + 1)).$$

Since D_n is bounded and convex, there exists by Case 1 a univalent, harmonic, and orientation-preserving mapping f_n satisfying (a), (b), and (c) with D and a replaced by D_n and a_n. Furthermore, since $\|a_n\|_\infty < 1$, the prime-end theory for quasiconformal mappings implies that f_n extends to a homeomorphism of $\bar{U}$ onto $\overline{D_n}$.

Write $f_n = h_n + \overline{g_n}$ where h_n and g_n are analytic, $h_n(0) = w_0$, and $g_n(0) = 0$. Since D_n is contained in the wedge, the analytic functions

$$F_n^{\pm} = e^{\pm i(\pi/2 - \alpha)} h_n + e^{\pm i(\alpha - \pi/2)} g_n$$

satisfy

$$\operatorname{Re} F_n^{\pm} = \operatorname{Re}\{e^{\pm i(\pi/2 - \alpha)} f_n\} > 0 \quad \text{and}$$

$$F_n^{\pm}(0) = e^{\pm i(\pi/2 - \alpha)} w_0.$$

By Montel's theorem there is a subsequence $F_{n_j}^{\pm}$ converging uniformly on compact subsets, say to $F^{\pm}$. Now

$$f_{n_j} = [e^{-i\alpha} \operatorname{Re}\{F_{n_j}^{+}\} + e^{i\alpha} \operatorname{Re}\{F_{n_j}^{-}\}]/\sin(2\alpha)$$

converges in a similar fashion to

$$f = [e^{-i\alpha} \operatorname{Re}\{F^{+}\} + e^{i\alpha} \operatorname{Re}\{F^{-}\}]/\sin(2\alpha)$$

and $f(0) = w_0$. To see that f is not constant, we use the fact from [**2**, Theorem 2.1] that

$$[f_n - w_0]/[(f_n)_z(0)]$$

omits some point on the circle $|w| = 1.72$. Hence

$$(f_n)_z(0) \geqq \frac{\delta_1}{1.72}$$

where δ_1 is the distance from w_0 to ∂D_1. Consequently,

$$f_z(0) \geqq \frac{\delta_1}{1.72} > 0,$$

and so f is not constant. This and Lemma 3.1 in [**7**] imply that f is a univalent, orientation-preserving, harmonic mapping and that f satisfies the equation

$$\overline{f_{\bar{z}}} = af_z.$$

It is clear that $f(U) \subset D$ since f is open, $D_n \subset D$, and D is convex.

It remains to show part (c). We use the Helly selection theorem as in [**7**, Theorem 3.2] to find a further subsequence $f_{n_{j_k}}$, call it f_k, such that

https://doi.org/10.4153/CJM-1987-071-4 Published online by Cambridge University Press

$\varphi^{-1} \circ f_k$ is bounded and converges a.e. on ∂U to a function η on ∂U with $|\eta| = 1$. Since φ is continuous on $\bar{U}$ with respect to the spherical metric, it follows that $f_k \to \varphi \circ \eta$ a.e. on ∂U in that metric. If we write $f_k = u_k + iv_k$, then

$$|v_k| \leqq (\tan \alpha)u_k$$

since $\overline{D_k}$ lies in the wedge. Hence

$$\int_0^{2\pi} |f_k(e^{it})|dt \leqq (1 + \tan \alpha) \int_0^{2\pi} u_k(e^{it})dt$$
$$= 2\pi(1 + \tan \alpha)\mathrm{Re}\{w_0\}.$$

This and Fatou's lemma give

$$\int_0^{2\pi} |\varphi \circ \eta(e^{it})|dt \leqq \lim_{k\to\infty} \int_0^{2\pi} |f_k(e^{it})|dt$$
$$\leqq 2\pi(1 + \tan \alpha)\mathrm{Re}\{w_0\}.$$

Thus $\varphi \circ \eta$ is finite a.e., and it follows from Lemma 2.2 that $\hat{f}$ exists and equals $\varphi \circ \eta$ a.e. Since $\varphi \circ \eta \in \partial D$, part (c) is proved.

Case 3. If D is a half-plane, we may suppose without loss of generality that D is the right half-plane. An explicit representation for the solution will be given in Remark 4.4.

Case 4. If D is a strip, then an explicit representation was given earlier in [**8**, Section 2].

We remark that Case 3 of the proof remains valid for any domain D that is contained in a convex wedge and has a locally connected boundary. Secondly, uniqueness of the mapping with the properties of Theorem 2.3 is not known in general. However, for the cases where D is a strip or half-plane the explicit representations for f show that the correspondence between f and a is one-to-one.

It is possible to represent univalent, harmonic, orientation-preserving mappings $f = h + \bar{g}$ of U onto convex domains in various ways. For example, J. Clunie and T. Sheil-Small [**2**, Lemma 5.11] showed that there are real constants λ and μ such that

$$p = (e^{-i\mu}h' + e^{i\mu}g')(e^{i\lambda} - e^{-i\lambda}z^2)$$

satisfies Re $p \geqq 0$. If $a = g'/h'$, this implies that

$$f(z) = f(0) + \int_0^z \frac{p(\zeta)d\zeta}{(e^{i\lambda} - e^{-i\lambda}\zeta^2)(e^{-i\mu} + e^{i\mu}a(\zeta))}$$
$$+ \overline{\int_0^z \frac{a(\zeta)p(\zeta)d\zeta}{(e^{i\lambda} - e^{-i\lambda}\zeta^2)(e^{-i\mu} + e^{i\mu}a(\zeta))}}.$$

This representation was sufficient for Clunie and Sheil-Small [**2**] to obtain some sharp coefficient estimates.

Let us consider another representation. If f is a univalent harmonic mapping onto a bounded convex domain, then

$$f(z) = \int_0^{2\pi} P(z, t)\hat{f}(e^{it})dt$$

and, moreover, f has unrestricted limits at the boundary except possibly at points of a countable set at which the cluster sets are straight line segments [**7**, Theorem 4.3]. The following theorem concerns the unbounded case.

THEOREM 2.4. *Let f be a univalent, harmonic, orientation-preserving mapping from U onto an unbounded convex domain D which is neither a strip nor a half-plane. Then*

(a) $f \in h^1$;

(b) *there is only one point $e^{i\lambda}$ that corresponds to ∞;*

(c) $$f(z) = \int_0^{2\pi} P(z, t)\hat{f}(e^{it})dt + AP(z, \lambda)$$

for some constant $A \in \mathbf{C}$;

(d) *there is a countable set $E \subset \partial U \backslash \{e^{i\lambda}\}$ such that*

(i) *the unrestricted limit* $\lim f(z)$ *exists as $z \to e^{i\theta}$, $z \in U$, and is continuous for all points $e^{i\theta} \in \partial U \backslash [E \cup \{e^{i\lambda}\}]$,*

(ii) $$\lim_{t \uparrow \theta} \hat{f}(e^{it}) \quad \textit{and} \quad \lim_{t \downarrow \theta} \hat{f}(e^{it})$$

exist and are different for $e^{i\theta} \in E$,

(iii) *and the cluster set of f at $e^{i\theta} \in E$ is the line segment joining*

$$\lim_{t \uparrow \theta} \hat{f}(e^{it}) \quad \textit{to} \quad \lim_{t \downarrow \theta} \hat{f}(e^{it});$$

(e) *the cluster set at $e^{i\lambda}$ is either the point at ∞, a half-line, or two parallel half-lines.*

By correspondence in (b), we mean that there is a sequence $\{z_k\} \subset U$ so that

$$\lim_{k \to \infty} z_k = e^{i\lambda} \quad \text{and} \quad \lim_{k \to \infty} f(z_k) = \infty.$$

Proof. Since D is neither a strip nor a half-plane, there is a convex wedge that contains D inside it. Use an affine transformation

$$w \to \mathscr{A}(w) = aw + b\bar{w} + c$$

to map this wedge onto the first quadrant. Then $\mathscr{A} \circ f$ is harmonic, $\operatorname{Re} \mathscr{A} \circ f \geqq 0$, $\operatorname{Im} \mathscr{A} \circ f \geqq 0$, and so $\mathscr{A} \circ f \in h^1$. Since the inverse of $\mathscr{A}$ is also affine and h^1 is closed under affine transformations, part (a) of the theorem is proved.

The existence of a point $e^{i\lambda}$ corresponding to ∞ is clear because f is open. To show uniqueness, choose two points $e^{i\alpha}$ and $e^{i\beta}$ on ∂U at which f has finite radial limits. Denote the union of the radii to $e^{i\alpha}$ and $e^{i\beta}$ by L. Then L divides U into two disjoint sectors U_1 and U_2. Suppose that $e^{i\lambda} \in \partial U_1$. Then $f(L)$ is a finite Jordan arc that divides D into two disjoint domains, one bounded and the other unbounded. Necessarily, $f(U_1)$ is unbounded and $f(U_2)$ is bounded. But f has radial limits almost everywhere, and so we can choose $e^{i\alpha}$ and $e^{i\beta}$ arbitrarily close to $e^{i\lambda}$ and on either side. Hence $e^{i\lambda}$ is the only point that corresponds to ∞.

Next, consider part (c). The function $\mathscr{A} \circ f$, defined earlier, belongs to h^1, and so it has the representation

$$\mathscr{A} \circ f(z) = \int_0^{2\pi} P(z, t)\mathscr{A} \circ \hat{f}(e^{it})dt + \int_0^{2\pi} P(z, t)ds(t)$$

where ds is singular with respect to dt. Since Re $\mathscr{A} \circ f \geqq 0$ and Im $\mathscr{A} \circ f \geqq 0$, we may write $ds = ds_1 + ids_2$ where ds_1 and ds_2 are non-negative singular measures.

Suppose that ds_j, for some j, is not zero, and let E_j denote its closed support. Then it is known [**4**, p. 77] that

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} s_j((t - \epsilon, t + \epsilon)) = +\infty$$

for s_j-almost all $t \in E_j$. This, in turn, implies [**3**, p. 4] that

$$\int_0^{2\pi} P(z, t)ds_j(t)$$

has radial limit $+\infty$ at such points. Since the real and imaginary parts of

$$\int_0^{2\pi} P(z, t)\mathscr{A} \circ \hat{f}(e^{it})dt$$

are nonnegative, it follows that

$$\lim_{r \to 1} \mathscr{A} \circ f(re^{it}) = \infty$$

for s_j-almost all $t \in E_j$. But only $e^{i\lambda}$ corresponds to ∞ by part (b). This implies that $E_j = \{e^{i\lambda}\}$. Therefore ds_j is a point mass at $e^{i\lambda}$. This proves part (c) of the theorem for $\mathscr{A} \circ f$ and, consequently, for f.

The proof of (d) is the same as in [**7**, Theorem 4.3]. The boundary of D is locally connected since D is convex, and the hypothesis in [**7**, Theorem 4.3] that f is bounded can be replaced by the properties (a), (b), and (c) above.

To prove (e) we use the representation (c). First suppose that $A = 0$. Since $P(z, t)dt$ is a probability measure for each fixed z, it follows that D is the open convex hull of the radial boundary values $\hat{f}$. If $\hat{f}$ is unbounded from both sides of $e^{i\lambda}$, then an argument as in part (b) shows that the cluster set at $e^{i\lambda}$ is only the point at ∞. Similarly, if $\hat{f}$ is bounded from one side and unbounded from the other, then the cluster set must be a half-line. In this case, $\hat{f}$ cannot be bounded from both sides of $e^{i\lambda}$, for then D would be bounded.

Finally, assume that $A \neq 0$ so that $P(z, \lambda)$ plays a role. If $\hat{f}$ is bounded, then by approaching $e^{i\lambda}$ along circular arcs tangent to ∂U at $e^{i\lambda}$, one sees that the cluster set contains two parallel half-lines. Since D is convex, that is all it contains. If $\hat{f}$ is bounded from one side of $e^{i\lambda}$ and unbounded from the other, then by approaching $e^{i\lambda}$ along the same circular arcs, but only from the bounded side, it follows that the cluster set contains one half-line. The convexity of D and an argument as in (b) implies that the cluster set is no larger. If $\hat{f}$ is unbounded from both sides, then an argument as in (b) shows that the cluster set is just the point at ∞.

Remark 2.5. It is a property of the Poisson integral representation (c) and part (d) (iii) that the radial limit of f exists for points $e^{i\theta} \in E$ and equals

$$\frac{1}{2}\left[\lim_{t\uparrow\theta} \hat{f}(e^{it}) + \lim_{t\downarrow\theta} \hat{f}(e^{it})\right].$$

Therefore the radial limit $\hat{f}$ exists at every point of $\partial U\setminus\{e^{i\lambda}\}$.

A question arises as to whether there is a univalent, harmonic, orientation-preserving mapping f onto a convex domain D so that $A \neq 0$ in part (c) of Theorem 2.4. The next theorem answers this question affirmatively.

THEOREM 2.6. *Let f be a univalent, harmonic, orientation-preserving mapping from U onto an unbounded convex domain D. Choose λ and α so that there are points $z_k \in U$ for which*

$$z_k \to e^{i\lambda}, \quad |f(z_k)| \to \infty, \quad \textit{and}$$

$$f(z_k)/|f(z_k)| \to e^{i\alpha} \quad \textit{as } k \to \infty.$$

Then for each $r > 0$

$$f + re^{i\alpha}P(\cdot, \lambda)$$

is a univalent, harmonic, orientation-preserving mapping of U onto an unbounded convex domain contained in D.

Proof. Write $f = h + \overline{g}$ where h and g are analytic, and define

$$H(z) = h(z) + \frac{r}{4\pi}e^{i\alpha}\frac{e^{i\lambda} + z}{e^{i\lambda} - z} \quad \text{and}$$

$$G(z) = g(z) + \frac{r}{4\pi}e^{-i\alpha}\frac{e^{i\lambda} + z}{e^{i\lambda} - z}.$$

Then $F = H + \bar{G}$ satisfies

$$F(z) = f(z) + re^{i\alpha}P(z, \lambda).$$

By [**2**, Theorem 5.7], the harmonic function F is univalent, orientation-preserving, and maps onto a convex domain if and only if for each $\theta \in [0, \pi)$ the function

$$\Phi = ie^{-i\theta}[H - e^{2i\theta}G]$$

is univalent and maps onto a domain that is convex in the vertical direction. This is the case [**13**, Theorem 1] if and only if Φ is nonconstant and there are parameters $\mu \in [0, 2\pi)$ and $\nu \in [0, \pi]$ such that

$$\text{(1)} \qquad \operatorname{Re}\{-ie^{i\mu}[1 - (2\cos\nu)e^{-i\mu}z + e^{-2i\mu}z^2]\Phi'(z)\} \geqq 0.$$

Choose μ and ν so that (1) is satisfied for the function

$$\varphi = ie^{-i\theta}[h - e^{2i\theta}g],$$

which does map onto a domain convex in the vertical direction because f is convex. Since

$$\Phi = \varphi - \frac{r}{2\pi}\sin(\alpha - \theta)\frac{e^{i\lambda} + z}{e^{i\lambda} - z},$$

it is sufficient to verify that

$$\text{(2)} \qquad \operatorname{Re}\left\{ie^{i\mu}[1 - (2\cos\nu)e^{-i\mu}z + e^{-2i\mu}z^2]\sin(\alpha - \theta)\frac{e^{i\lambda}}{(e^{i\lambda} - z)^2}\right\}$$

is nonnegative. It is known [**13**] that the point $e^{i(\mu+\nu)}$ corresponds to $\inf_U \operatorname{Re}\varphi$ and that $e^{i(\mu-\nu)}$ corresponds to $\sup_U \operatorname{Re}\varphi$. We consider two cases.

If

$$\operatorname{Re}\{ie^{-i\theta}e^{i\alpha}\} \geqq 0,$$

then the point $e^{i\lambda}$ corresponds to $\sup_U \operatorname{Re}\varphi$ because

$$\operatorname{Re}\varphi = \operatorname{Re}\{ie^{-i\theta}f\}.$$

In this case

$$e^{i\lambda} = e^{i(\mu-\nu)}$$

and (2) reduces to

$$\sin(\alpha - \theta)\operatorname{Re}\left\{i\frac{e^{i\mu} - e^{-i\nu}z}{e^{i(\mu-\nu)} - z}\right\}$$

$$= -\sin(\alpha - \theta)\frac{(1 - |z|^2)\sin\nu}{|e^{i(\mu-\nu)} - z|^2},$$

which is nonnegative. Similarly, if

$$\mathrm{Re}\{ie^{-i\theta}e^{i\alpha}\} < 0,$$

then the point $e^{i\lambda}$ corresponds to $\inf_U \mathrm{Re}\,\varphi$, and so

$$e^{i\lambda} = e^{i(\mu+\nu)}.$$

In this case (2) reduces to

$$\sin(\alpha - \theta)\,\mathrm{Re}\left\{i\frac{e^{i\mu} - e^{i\nu}z}{e^{i(\mu+\nu)} - z}\right\}$$

$$= \sin(\alpha - \theta)\frac{(1 - |z|^2)\sin\nu}{|e^{i(\mu+\nu)} - z|^2},$$

which is nonnegative.

To see that Φ is nonconstant, choose z_k as in the hypothesis. Then

$$F(z_k) = f(z_k) + re^{i\alpha}P(z_k, \lambda)$$

approaches infinity in the direction $e^{i\alpha}$ as $k \to \infty$. For this reason

$$\mathrm{Re}\,\Phi = \mathrm{Re}\{ie^{-i\theta}F\}$$

could be constant only if

$$\mathrm{Re}\{ie^{-i\theta}e^{i\alpha}\} = 0.$$

However, in this case

$$\mathrm{Re}\,\Phi = \mathrm{Re}\{ie^{-i\alpha}f\}$$

is not constant because f is open.

Finally, the domain $F(U)$ is contained in D since the addition of $re^{i\alpha}P(z, \lambda)$ to $f(z)$ is a translation along a ray in the convex domain D. In addition, $F(U)$ is unbounded since the points $F(z_k)$ mentioned earlier approach infinity.

Except at the point $e^{i\lambda}$ the Poisson kernel $P(z, \lambda)$ has boundary values zero. Therefore, in Theorem 2.6, the only place where the boundaries of D and $F(U)$ can differ is at the cluster sets of f and F at $e^{i\lambda}$.

3. Mappings onto a convex wedge. In this section, we consider the set S_W of all univalent, harmonic, orientation-preserving mappings f of U onto the wedge

$$W = \{w : |\arg w| < \pi/4\},$$

with normalization $f(0) = 1$. Let $\overline{S_W}$ denote the closure of S_W in the topology of locally uniform convergences. If $\mathscr{W}$ is any convex wedge, then there is an affine transformation that maps $\mathscr{W}$ onto W. So the properties of S_W will be representative.

By Theorem 2.4, every $f \in S_W$ is in h^1 and can be expressed in the form

$$(3) \quad f(z) = \int_0^{2\pi} P(z, t)\hat{f}(e^{it})dt + AP(z, \lambda)$$

where the radial limit values $\hat{f}(e^{it})$ are finite and belong to ∂W for every point $e^{it} \neq e^{i\lambda}$. The following theorem shows that this representation and other properties persist for $\overline{S_W}$, and it identifies some degeneracies.

THEOREM 3.1. *If* $f \in \overline{S_W}$, *then*

(a) $f \in h^1$, $f(U) \subset W$, *f is not constant, and* $\hat{f}(e^{it}) \in \partial W$ *for almost all* t;

(b) $\overline{f_{\bar{z}}} = af_z$ *for some analytic function a with* $a(U) \subset \bar{U}$;

(c) *if* $|a| \not\equiv 1$, *then f is univalent and orientation preserving,* $f(U)$ *is convex, and f has the representation* (3);

(d) *if* $|a| \equiv 1$, *then* $f(U)$ *is either a line segment or half-line through the point* 1 *with endpoint*(*s*) *on* ∂W.

Proof. Let $\{f_n\} \subset S_W$ converge locally uniformly to f. Since the real and imaginary parts of $(1 + i)f_n$ are positive and normalized, the same is true for $(1 + i)f$, and it follows that f belongs to h^1. Next, it is clear that $f(U) \subset \overline{W}$ and, moreover, that $f(U) \subset W$ since $f(0) = 1$ is an interior point of W. In addition, we have

$$|f_z(0)| \geqq \frac{\delta}{1.72}$$

where $\delta = 1/\sqrt{2}$ is the distance from $f_n(0) = 1$ to ∂W as in the proof of Theorem 2.3. Hence f is not constant.

In order to show that $\hat{f}(e^{it}) \in \partial W$ for almost all t, consider the functions $\varphi^{-1} \circ \hat{f}_n$ where

$$\varphi(z) = \sqrt{\frac{1 + z}{1 - z}}$$

is a conformal mapping from U onto W. Then $\varphi^{-1} \circ \hat{f}_n$ maps ∂U into ∂U, is orientation-preserving, and, by Helly's selection theorem as in the proof of Theorem 2.3, part (c), has a subsequence that converges almost everywhere to a function η with $|\eta| = 1$. Since φ is continuous from $\bar{U}$ into the Riemann sphere, a subsequence of $\{\hat{f}_n\}$ converges to $\varphi \circ \eta$ almost everywhere. Therefore, $\hat{f} = \varphi \circ \eta$ by Lemma 2.2, and so the values of $\hat{f}$ belong almost everywhere to ∂W. This completes the proof of part (a).

Since f is nonconstant [7, Lemma 3.1] implies that $\overline{f_{\bar{z}}} = af_z$ for some analytic function a with $a(U) \subset \bar{U}$, and if $|a| \not\equiv 1$ then f is univalent and orientation preserving.

First, assume that $|a| \not\equiv 1$. To show that f is convex, we write

$$f_n = h_n + \overline{g_n} \quad \text{and} \quad f = h + \overline{g}$$

with $h_n(0) = h(0) = 1$ and $g_n(0) = g(0) = 0$. Since the z and $\bar{z}$ derivatives of f_n converge to those of f, it follows by integration that $h_n \to h$ and $g_n \to g$ as $n \to \infty$. Each f_n is convex, and so the function

$$\varphi_n = ie^{-i\theta}(h_n - e^{2i\theta}g_n)$$

is convex in the vertical direction [**2**]. Thus φ_n satisfies a condition of the form (1) [**13**]. By passing through appropriate subsequences, it follows that

$$\varphi = ie^{-i\theta}(h - e^{2i\theta}g)$$

also satisfies a condition of the form (1). Moreover, φ is not constant because

$$\text{Re } \varphi = \text{Re}\{ie^{-i\theta}f\}.$$

Therefore φ is convex in the vertical position for all θ [**13**]. Hence f is convex [**2**]. Now Theorem 2.4 implies that f has the representation (3). As a result, parts (b) and (c) are proved.

If $|a| \equiv 1$, then $a(z) \equiv e^{2i\phi}$ for some constant ϕ, and the differential equation $\overline{f_{\bar{z}}} = af_z$ can be written as

$$(\text{Im}\{e^{i\phi}f\})_{\bar{z}} = 0.$$

Thus $\text{Im}\{e^{i\theta}f\}$ is constant, and the values of f lie on a straight line. Since $f(U) \subset W$, $f(0) = 1$, and the boundary values of f are almost everywhere on ∂W, only the indicated segments and half-lines are possible.

By Theorem 2.6 the functions

$$f_n(z) = \frac{1}{n}\sqrt{\frac{1+z}{1-z}} + \left(1 - \frac{1}{n}\right)2\pi P(z, 0)$$

belong to S_W. Their limit $f(z) = 2\pi P(z, 0)$ maps U onto the positive real axis. Therefore some degeneracy, as in part (d) of Theorem 3.1, can occur in $\overline{S_W}$.

For a function $f \in \overline{S_W}\backslash S_W$ the only possibilities for $f(U)$, beside segments and half-lines, are triangles, quadrilaterals, and unbounded polygons with three, four, or five sides. All cases actually occur. We shall make use of the triangles and some of the others.

Choose α, β, and γ so that $\alpha < \beta < \gamma < 2\pi + \alpha$, and let

$$I_1 = \{e^{it}:\alpha < t < \beta\} \quad \text{and} \quad I_2 = \{e^{it}:\gamma < t < 2\pi + \alpha\}.$$

The lengths of these arcs are $|I_1| = \beta - \alpha$ and $|I_2| = 2\pi + \alpha - \gamma$. Next, consider the harmonic functions

$$U_1(z) = \frac{\pi}{|I_1|} \int_\alpha^\beta P(z, t)dt = \frac{-1}{2} + \frac{1}{|I_1|} \arg \frac{e^{i\beta} - z}{e^{i\alpha} - z}$$

and

$$U_2(z) = \frac{\pi}{|I_2|} \int_\gamma^{\alpha+2\pi} P(z, t)dt = \frac{-1}{2} + \frac{1}{|I_2|} \arg \frac{e^{i\alpha} - z}{e^{i\gamma} - z}$$

where the branches of the arguments are chosen so that

$$U_1(0) = U_2(0) = \frac{1}{2}.$$

Clearly, the boundary values of U_j are $\pi/|I_j|$ on I_j and zero on $\partial U \backslash \overline{I_j}$. Finally, define

(4) $\quad T_{(\alpha,\beta,\gamma)} = (1 + i)U_1 + (1 - i)U_2.$

Then $T_{(\alpha,\beta,\gamma)}(0) = 1$ and

$$\hat{T}_{(\alpha,\beta,\gamma)}(e^{it}) = \begin{cases} \dfrac{(1 + i)\pi}{|I_1|} & \text{if } e^{it} \in I_1 \\ \dfrac{(1 - i)\pi}{|I_2|} & \text{if } e^{it} \in I_2 \\ 0 & \text{if } e^{it} \in \partial U \backslash [\overline{I_1} \cup \overline{I_2}]. \end{cases}$$

$T_{(\alpha,\beta,\gamma)}$ is a univalent, harmonic, orientation-preserving mapping of U onto the open triangle with vertices at the origin and the points

$$\frac{(1 + i)\pi}{|I_1|} \quad \text{and} \quad \frac{(1 - i)\pi}{|I_2|}.$$

The cluster sets of $T_{(\alpha,\beta,\gamma)}$ at the points $e^{i\alpha}$, $e^{i\beta}$, and $e^{i\gamma}$ are the respective sides of the triangle.

We extend the definition of $T_{(\alpha,\beta,\gamma)}$ by continuity so that

(5) $\quad T_{(\alpha,\beta,\gamma)}(z)$

$$= \begin{cases} (1 + i)U_1(z) + (1 - i)U_2(z) & \text{if } \alpha < \beta = \gamma < \alpha + 2\pi \\ (1 + i)U_1(z) + (1 - i)\pi P(z, \alpha) & \text{if } \alpha < \beta < \gamma = \alpha + 2\pi \\ (1 + i)\pi P(z, \alpha) + (1 - i)U_2(z) & \text{if } \alpha = \beta < \gamma < \alpha + 2\pi \\ \dfrac{1 + i}{2} + (1 - i)\pi P(z, \alpha) & \text{if } \alpha < \beta = \gamma = \alpha + 2\pi \\ (1 + i)\pi P(z, \alpha) + \dfrac{1 - i}{2} & \text{if } \alpha = \beta = \gamma < \alpha + 2\pi \\ 2\pi P(z, \alpha) & \text{if } \alpha = \beta < \gamma = \alpha + 2\pi \end{cases}$$

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If $\alpha < \beta = \gamma < \alpha + 2\pi$, then $T_{(\alpha,\beta,\gamma)}$ maps U onto the open segment with endpoints

$$\frac{(1+i)\pi}{|I_1|} \quad \text{and} \quad \frac{(1-i)\pi}{|I_2|}.$$

If $\alpha < \beta < \gamma = \alpha + 2\pi$ or if $\alpha = \beta < \gamma < \alpha + 2$, then $T_{(\alpha,\beta,\gamma)}$ is a univalent mapping of U onto an unbounded open triangle with vertices at the origin and either at

$$\frac{(1+i)\pi}{|I_1|} \quad \text{or at} \quad \frac{(1-i)\pi}{|I_2|}.$$

The unbounded sides are parallel, and one of them coincides with one side of ∂W. If $\alpha < \beta = \gamma = \alpha + 2\pi$ or if $\alpha = \beta = \gamma < \alpha + 2\pi$, then $T_{(\alpha,\beta,\gamma)}$ maps U onto the open half-line through 1 with initial point either $(1 + i)/2$ or $(1 - i)/2$. Finally, if $\alpha = \beta < \gamma = \alpha + 2\pi$, then $T_{(\alpha,\beta,\gamma)}(U)$ is the positive real axis.

THEOREM 3.2. *For* $\alpha \leqq \beta \leqq \gamma \leqq \alpha + 2\pi$ *the functions* $T_{(\alpha,\beta,\gamma)}$ *belong to* $\overline{S_W}$.

Proof. Since the other cases were defined by limiting processes, we may assume that $\alpha < \beta < \gamma < \alpha + 2\pi$. Set

$$\delta = \frac{\beta + \gamma}{2}.$$

Approximate the characteristic function $\chi_{(\alpha,\beta]}$ of the interval $(\alpha, \beta]$ by continuous, strictly decreasing functions σ_n on $(\alpha, \delta]$ in such a way that

$$\lim_{t\to\alpha} \sigma_n(t) = \infty, \quad \sigma_n(\delta) = 0, \quad \int_\alpha^\delta \sigma_n(t)dt = |I_1|, \quad \text{and}$$

$$\int_\alpha^\delta |\sigma_n - \chi_{(\alpha,\beta]}|dt \to 0 \quad \text{as } n \to \infty.$$

Similarly, approximate the characteristic function $\chi_{[\gamma,\alpha+2\pi)}$ of the interval $[\gamma, \alpha + 2\pi)$ by continuous, strictly increasing functions τ_n on $[\delta, \alpha + 2\pi)$ in such a way that

$$\tau_n(\delta) = 0, \quad \lim_{t\to\alpha+2\pi} \tau_n(t) = \infty, \quad \int_\delta^{\alpha+2\pi} \tau_n(t)dt = |I_2|, \quad \text{and}$$

$$\int_\delta^{\alpha+2\pi} |\tau_n - \chi_{[\gamma,\alpha+2\pi)}|dt \to 0 \quad \text{as } n \to \infty.$$

Define

$$f_n(z) = \frac{(1+i)\pi}{|I_1|} \int_\alpha^\delta P(z, t)\sigma_n(t)dt$$

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$$+ \frac{(1-i)\pi}{|I_2|} \int_\delta^{\alpha+2\pi} P(z, t)\tau_n(t)dt.$$

Then $\hat{f}_n$ is a homeomorphism of ∂U onto $\partial W \cup \{\infty\}$, and the proof of the Rado-Kneser-Choquet theorem [**10**, **1**] applies to it. Therefore f_n is a univalent, harmonic, orientation-preserving mapping of U onto W. Since $f_n(0) = 1$, the function f_n belongs to S_W. As $n \to \infty$, the functions f_n converge locally uniformly to

$$\frac{(1+i)\pi}{|I_1|} \int_\alpha^\beta P(z, t)dt + \frac{(1-i)\pi}{|I_2|} \int_\gamma^{\alpha+2\pi} P(z, t)dt$$
$$= (1 + i)U_1(z) + (1 - i)U_2(z) = T_{(\alpha,\beta,\gamma)}(z).$$

Consequently, $T_{(\alpha,\beta,\gamma)}$ belongs to $\overline{S_W}$.

Let $\mathcal{T}$ denote the set of all functions f of the form

$$f = \int_K T_{(\alpha,\beta,\gamma)} d\mu(\alpha, \beta, \gamma)$$

where μ varies over all probability measures on the compact, convex set

$$K = \{ (\alpha, \beta, \gamma) : 0 \leqq \alpha \leqq 2\pi, \alpha \leqq \beta \leqq \gamma \leqq 2\pi + \alpha\}.$$

Let HS_W denote the closed convex hull of S_W.

THEOREM 3.3. $HS_W = \mathcal{T}$.

Proof. Since $T_{(\alpha,\beta,\gamma)} \in \overline{S_W}$, it is clear that $\mathcal{T} \subset HS_W$. To verify that $HS_W \subset \mathcal{T}$, it is sufficient to show that each $f \in S_W$ belongs to $\mathcal{T}$.

Fix $f \in S_W$, and choose α and β so that the values $\hat{f}(e^{it})$ are on the ray $\arg w = \pi/4$ for $\alpha < t < \beta$ and on the ray $\arg w = -\pi/4$ for $\beta < t < \alpha + 2\pi$. Then the representation (3) becomes

$$f(z) = (1 + i)u_1(z) + (1 - i)u_2(z) + AP(z, \alpha)$$

where

$$u_1(z) = \frac{1}{\sqrt{2}} \int_\alpha^\beta P(z, t) |\hat{f}(e^{it})| dt,$$

$$u_2(z) = \frac{1}{\sqrt{2}} \int_\beta^{\alpha+2\pi} P(z, t) |\hat{f}(e^{it})| dt, \quad \text{and}$$

$$|\arg A| \leqq \pi/4.$$

If $b = u_1(0)$ and $c = u_2(0)$, then the normalization $f(0) = 1$ and restriction on $\arg A$ imply

$$\frac{A}{2\pi} = (1 - b - c) + i(c - b) \quad \text{and} \quad 0 \leqq b, c \leqq \frac{1}{2}.$$

It is possible that $\alpha = \beta$, in which case $b = 0$, or that $\beta = \alpha + 2\pi$, in which case $c = 0$.

First, we want to approximate u_1. We may assume that $b > 0$, for otherwise $u_1 \equiv 0$. Partition the interval $[\alpha, \beta]$ into n equal parts, and let

$$v_j(t) = \begin{cases} \dfrac{2\pi nb}{(\beta - \alpha)j} & \text{for } \alpha \leqq t < \alpha + \dfrac{(\beta - \alpha)j}{n} \\ 0 & \text{for } \alpha + \dfrac{(\beta - \alpha)j}{n} \leqq t \leqq \beta \end{cases}$$

for $j = 1, \ldots, n$. Then

$$\int_\alpha^\beta v_j(t)dt = 2\pi b.$$

Define also

$$d_j = \frac{n}{\sqrt{2}(\beta - \alpha)} \int_{\alpha+((\beta-\alpha)(j-1)/n)}^{\alpha+((\beta-\alpha)j/n)} |\hat{f}(e^{it})|dt$$

and $d_{n+1} = 0$. Since $|\hat{f}(e^{it})|$ is nonincreasing on (α, β), the numbers

$$s_j = \frac{(\beta - \alpha)j}{2\pi nb}(d_j - d_{j+1})$$

are nonnegative and

$$\sum_{j=1}^n s_j = \frac{\beta - \alpha}{2\pi nb} \sum_{j=1}^n j(d_j - d_{j+1})$$

$$= \frac{\beta - \alpha}{2\pi nb} \sum_{j=1}^n d_j = \frac{1}{2\sqrt{2}\pi b} \int_\alpha^\beta |\hat{f}(e^{it})|dt = 1.$$

Next, define

$$V_n = \sum_{j=1}^n s_j v_j.$$

If

$$\alpha + \frac{(\beta - \alpha)(k - 1)}{n} \leqq t < \alpha + \frac{(\beta - \alpha)k}{n},$$

then

$$V_n(t) = \sum_{j=k}^n (d_j - d_{j+1}) = d_k.$$

In addition,

$$\int_\alpha^\beta V_n(t)dt = 2\pi b.$$

Since

$$V_n(t) \to u_1(e^{it}) \quad \text{for a.e. } t,$$

it follows from Lemma 2.1 that there is a subsequence such that

$$V_{n_m}(z) \equiv \int_\alpha^\beta P(z, t)V_{n_m}(t)dt$$

converges locally uniformly to u_1.

Similarly, to approximate u_2 we may assume that $c \neq 0$ and define

$$w_k(t) = \begin{cases} 0 & \text{for } \beta \leqq t < \beta + \dfrac{(\alpha + 2\pi - \beta)(k-1)}{n} \\[2ex] \dfrac{2\pi nc}{(\alpha + 2\pi - \beta)(n - k + 1)} & \text{for } \beta + \dfrac{(\alpha + 2\pi - \beta)(k-1)}{n} \leqq t \leqq \alpha + 2\pi \end{cases}$$

for $k = 1, \ldots, n$. Then we can construct nonnegative numbers t_k with

$$\sum_{k=1}^{n} t_k = 1,$$

functions

$$W_n = \sum_{k=1}^{n} t_k w_k,$$

and their Poisson convolutions

$$W_n(z) = \int_\beta^{\alpha+2\pi} P(z, t)W_n(t)dt$$

such that

$$\int_\beta^{\alpha+2\pi} W_n(t)dt = 2\pi c$$

and a further subsequence

$$W_{n_{m_l}}(z) \to u_2(z)$$

locally uniformly.

For $1 \leqq j, k \leqq n$, let

$$T_{jk}(z) = (1 + i)v_j(z) + (1 - i)w_k(z) + AP(z, \alpha)$$

where $v_j(z)$ and $w_k(z)$ are the Poisson convolutions of $v_j(t)$ and $w_k(t)$. Then

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$$f_n(z) = (1 + i)V_n(z) + (1 - i)W_n(z) + AP(z, \alpha)$$
$$= \sum_{j,k=1}^{n} s_j t_k T_{jk}(z)$$

has a subsequence that converges to f. Consequently, if we show that $T_{jk} \in \mathscr{T}$ for all j and k, then it will follow that $f \in \mathscr{T}$.

To see that $T_{jk} \in \mathscr{T}$, recall that

$$\frac{A}{2\pi} = (1 - b - c) + i(c - b) \quad \text{where } 0 \leqq b, c \leqq \frac{1}{2}.$$

If $m = \max\{b, c\}$, then

$$T_{jk} = 2(b + c - m)\left[(1 + i)\frac{v_j}{2b} + (1 - i)\frac{w_k}{2c}\right]$$
$$+ 2(m - c)\left[(1 + i)\frac{v_j}{2b} + (1 - i)\pi P(z, \alpha)\right]$$
$$+ 2(m - b)\left[(1 + i)\pi P(z, \alpha) + (1 - i)\frac{w_k}{2c}\right]$$
$$+ (1 - 2m)[2\pi P(z, \alpha)]$$

is a convex decomposition of T_{jk} into functions of the form $T_{(\alpha,\beta,\gamma)}$ that appear in (4) and (5). If $b = 0$ or $c = 0$, the ambiguous terms are omitted.

In the proof of Theorem 3.3 the points α and β remained fixed. Therefore we proved also the following corollary. For $\lambda \leqq \kappa \leqq \lambda + 2\pi$, let $S_W(\kappa, \lambda)$ consist of those functions f in S_W such that the cluster set of f at $e^{i\kappa}$ contains the origin and the cluster set of f at $e^{i\lambda}$ contains infinity. That is, $S_W(\kappa, \lambda)$ contains those functions in S_W for which $e^{i\kappa}$ and $e^{i\lambda}$ correspond to 0 and ∞, respectively. As before, $HS_W(\kappa, \lambda)$ denotes the closed convex hull of $S_W(\kappa, \lambda)$. Next, let $\mathscr{T}(\kappa, \lambda)$ denote the set of all functions f of the form

$$f = \int_{K(\kappa,\lambda)} T_{(\lambda,\beta,\gamma)} d\mu(\lambda, \beta, \gamma)$$

where μ varies over all probability measures on the compact, convex set

$$K(\kappa, \lambda) = \{(\lambda, \beta, \gamma) : \lambda \leqq \beta \leqq \kappa \leqq \gamma \leqq 2\pi + \lambda\}.$$

COROLLARY 3.4. $HS_W(\kappa, \lambda) = \mathscr{T}(\kappa, \lambda)$.

We shall add the prefix E to denote the set of extreme points.

THEOREM 3.5.

$$EHS_W = \{T_{(\alpha,\beta,\gamma)} : 0 \leqq \alpha < 2\pi,\ \alpha \leqq \beta \leqq \gamma \leqq \alpha + 2\pi\}$$

and

$$EHS_W(\kappa, \lambda) = \{T_{(\lambda,\beta,\gamma)} : \lambda \leqq \beta \leqq \kappa \leqq \gamma \leqq \lambda + 2\pi\}.$$

Proof. That EHS_W and $EHS_W(\kappa, \lambda)$ are contained in the indicated sets is an immediate consequence of Theorem 3.3 and Corollary 3.4. For the opposite inclusions, it is sufficient to show that each $T_{(\alpha,\beta,\gamma)}$ is an extreme point of HS_W, for if it belongs also to the subset $HS_W(\kappa, \lambda)$, it will necessarily be an extreme point there, too.

Fix an arbitrary $T_{(\alpha_0,\beta_0,\gamma_0)}$, and denote it more simply by T. Assume that

$$T = \int_K T_{(\alpha,\beta,\gamma)} d\mu(\alpha, \beta, \gamma)$$

for some probability measure μ, where

$$K = \{ (\alpha, \beta, \gamma) : 0 \leqq \alpha < 2\pi,\ \alpha \leqq \beta \leqq \gamma \leqq \alpha + 2\pi\}.$$

We shall show that μ is a unit point mass at $(\alpha_0, \beta_0, \gamma_0)$. We may assume $0 \leqq \alpha_0 < 2\pi$.

First, we shall show that $\alpha = \alpha_0$, $\alpha_0 \leqq \beta \leqq \beta_0$, and $\gamma_0 \leqq \gamma \leqq \alpha_0 + 2\pi$ for μ-almost all points in K. Write

$$T = (1 + i)U_1 + (1 - i)U_2 \quad \text{and}$$

$$T_{(\alpha,\beta,\gamma)} = (1 + i)u_1 + (1 - i)u_2.$$

Then

$$(6) \qquad U_j = \int_{E_j} u_j d\mu + \int_{K\setminus E_j} \pi P(z, \alpha) d\mu$$

where E_1 consists of those points in K with $\alpha \neq \beta$, E_2 consists of those points in K with $\gamma \neq \alpha + 2\pi$, and we have suppressed the dependence of the functions u_j and the measure μ on (α, β, γ).

The integrands in (6) are nonnegative, and so Fatou's lemma implies

$$\hat{U}_1(e^{it}) \geqq \int_{E_1} \hat{u}_1(e^{it}) d\mu.$$

If $\alpha_0 < \beta_0$, then

$$\hat{U}_1(e^{it}) = 0 \quad \text{for } \beta_0 < t < \alpha_0 + 2\pi,$$

and it follows that $\hat{u}_1(e^{it}) = 0$ for a dense set of t's in the same interval for μ-almost every point in E_1. Thus $\alpha_0 \leqq \alpha < \beta \leqq \beta_0$ for μ-almost all points in E_1. In addition,

$$\int_{K\setminus E_1} \pi P(z, \alpha) d\mu$$

tends uniformly to zero on each subarc of $\{e^{it}:\beta_0 < t < \alpha_0 + 2\pi\}$. Therefore $\alpha_0 \leqq \alpha \leqq \beta_0$ for μ-almost every point in $K\backslash E_1$. If $\alpha_0 = \beta_0$, then

$$U_1 = \pi P(z, \alpha_0) = \int_K u_1 d\mu,$$

in which case μ-almost all u_1 satisfy

$$\hat{u}_1(e^{it}) = 0 \quad \text{for } t \neq \alpha_0.$$

The only u_1's with this property are the Poisson kernels. Thus

$$P(z, \alpha_0) = \int_K P(z, \alpha)d\mu,$$

which implies that $\alpha = \alpha_0$ for μ-almost all points of K. In all cases we have $\alpha_0 \leqq \alpha \leqq \beta \leqq \beta_0$ for μ-almost every point of K.

Similarly, by considering U_2, one concludes that $\gamma_0 \leqq \gamma \leqq \alpha + 2\pi \leqq \alpha_0 + 2\pi$ for μ-almost all points of K. Taken together, these and the previous inequalities imply also $\alpha = \alpha_0$.

Now, (6) can be rewritten as

$$U_j = \int_{E_j} u_j d\mu + \pi P(\,\cdot\,, \alpha_0)\mu(K\backslash E_j).$$

Consider the case where $\alpha_0 < \beta_0$. Then U_1 is bounded, and so both

$$\int_{E_1} u_1 d\mu \quad \text{and} \quad P(\,\cdot\,, \alpha_0)\mu(K\backslash E_1)$$

are bounded. Hence

$$\mu(K\backslash E_1) = 0 \quad \text{and} \quad U_1 = \int_{E_1} u_1 d\mu.$$

For μ-almost every point of E_1 we may write

$$u_1(z) = \frac{\pi}{\beta - \alpha_0} \int_{\alpha_0}^{\beta} P(z, t)dt.$$

In this form it is easy to see that the functions u_1 have a uniform bound when z is near $e^{i\beta_0}$. Therefore

$$\hat{U}_1(e^{i\beta_0}) = \int_{E_1} \hat{u}_1(e^{i\beta_0})d\mu = \hat{U}_1(e^{i\beta_0})\mu(K_1)$$

where

$$K_1 = \{\,(\alpha_0, \beta_0, \gamma) : \beta_0 \leqq \gamma \leqq \alpha_0 + 2\pi\}.$$

Since

$$\hat{U}_1(e^{i\beta_0}) = \frac{\pi}{2(\beta_0 - \alpha_0)} \neq 0,$$

we conclude that $\mu(K_1) = 1$. That is, $\beta = \beta_0$ for μ-almost all points.

In case $\alpha_0 = \beta_0$, we have

$$U_1(z) = \pi P(z, \alpha_0).$$

Then (6) and Fatou's lemma imply

$$0 = \pi\hat{P}(e^{it}, \alpha_0) \geqq \int_{E_1} \hat{u}_1(e^{it})d\mu \quad \text{for all } t \neq \alpha_0.$$

This, in turn, implies that

$$\hat{u}_1(e^{it}) = 0$$

for a dense set of t's for μ-almost all points of E_1. However, there are no functions u_1 with this property corresponding to points of E_1; that is, $\mu(E_1) = 0$. Consequently, $\beta = \alpha = \alpha_0 = \beta_0$ for μ-almost every point.

Finally, a similar treatment of the function U_2 leads to the conclusion that $\gamma = \gamma_0$ for μ-almost all points. As a result, μ is (equivalent to) a unit point mass at $(\alpha_0, \beta_0, \gamma_0)$, and the proof is complete.

As applications of the extreme point theory, we shall obtain coefficient bounds for functions in $\overline{S_W}$. The following lemmas will be useful for that purpose. Define

$$s(x) = \frac{\sin x}{x} \quad \text{for } x > 0 \quad \text{and} \quad s(0) = 1.$$

LEMMA 3.6. *The function*

$$G(x, y) = \frac{1}{2}s(x)^2 + \frac{1}{2}s(y)^2 - s(x)s(y)\sin(x + y)$$

satisfies $0 \leqq G(x, y) \leqq 1$ *for all* $x, y \geqq 0$, *and* $G(x, y) = 1$ *only when* $x = y = 0$.

Proof. Since $|\sin(x + y)| \leqq 1$, it follows that

$$G(x, y) \geqq \frac{1}{2}[\,|s(x)| - |s(y)|\,]^2 \geqq 0.$$

To derive the upper bound, we first selectively bound the absolute values of the s and sine functions by 1 to obtain

$$G(x, y) \leqq \frac{1}{2x^2} + \frac{1}{2} + \frac{1}{x} \quad \text{for } x \geqq 0.$$

Hence, $G(x, y) < 1$ for $x > 1 + \sqrt{2}$. By symmetry, the same is true whenever $y > 1 + \sqrt{2}$.

Next, assume $\pi/2 \leqq x, y \leqq 1 + \sqrt{2}$. Since s is decreasing for this range, we have

$$G(x, y) \leqq \frac{1}{2}s(\pi/2)^2 + \frac{1}{2}s(\pi/2)^2 + s(\pi/2)s(\pi/2) = 8/\pi^2 < 1.$$

Now, suppose that $\pi/2 \leqq x \leqq 1 + \sqrt{2}$ and $0 \leqq y \leqq \pi/2$. Then

$$-\sin(x + y) \leqq -\sin(x + \pi/2) = \sin(x - \pi/2) \leqq x - \pi/2$$

and

$$G(x, y) \leqq \frac{1}{2x^2} + \frac{1}{2} + \frac{1}{x}(x - \pi/2).$$

The latter expression is less than 1 when

$$|x - \pi/2| < \sqrt{(\pi/2)^2 - 1}.$$

Fortunately, this is the case when $\pi/2 \leqq x \leqq 1 + \sqrt{2}$. In summary, we have shown that $G(x, y) < 1$ whenever $x \geqq \pi/2$ and, by symmetry, whenever $y \geqq \pi/2$.

Finally, if $0 \leqq x, y \leqq \pi/2$, then

$$G(x, y) \leqq \frac{1}{2}s(x)^2 + \frac{1}{2}s(y)^2 \leqq 1,$$

and $G(x, y) = 1$ is possible only when $x = y = 0$.

LEMMA 3.7. *For $x, y \geqq 0$, the function*

$$H(x, y) = \frac{1}{2}s(x)^2 + \frac{1}{2}s(y)^2 + s(x)s(y)\sin(x + y)$$

is nonnegative and assumes its maximum at a unique point (x_0, x_0). In particular,

$$\sup_{x,y \geqq 0} H(x, y) = \max_{(\pi/8) \leqq x \leqq (\pi/4)} H(x, x) = H(x_0, x_0) \approx 1.7114$$

where $x_0 \approx 0.5875$.

Proof. As before,

$$H(x, y) \geqq \frac{1}{2}[\,|s(x)| - |s(y)|\,]^2 \geqq 0.$$

If $x \geqq \pi$, then

$$H(x, y) \leqq \frac{1}{2\pi^2} + \frac{1}{2} + \frac{1}{\pi},$$

which is less than the given $H(x_0, x_0)$, and so we may restrict attention to $0 \leqq x \leqq \pi$. In fact, since $s(x)$ is decreasing it follows that

$$H(x, y) \leqq \frac{1}{2}\sin^2(1) + \frac{1}{2} + \sin(1) = 1.6955 \ldots$$

for $1 \leqq x \leqq \pi$. This bound is also less than $H(x_0, x_0)$, and so by symmetry, we may further restrict attention to $0 \leqq x, y < 1$.

Now restrict (x, y) also to the line segment $x + y = c$, $0 \leqq x, y < 1$, and parametrize $x = t$ and $y = c - t$. Then $C = \sin c \geqq 0$ and

$$\frac{d^2H}{dt^2}(x, y) = \frac{1}{2}[s(x)^2]'' + \frac{1}{2}[s(y)^2]''$$

$$+ C[s''(x)s(y) - 2s'(x)s'(y) + s(x)s''(y)].$$

By considering power series, one observes that $s(x) > 0$, $s'(x) < 0$, $s''(x) < 0$, and $[s(x)^2]'' < 0$, at least for $0 < x < 1$. Therefore H is strictly concave on the line segment $x + y = c$, $0 \leqq x, y \leqq 1$. Since H is symmetric, the maximum of H on this segment occurs only when $x = y$. This is true for each line segment $x + y = c$, $0 \leqq x, y \leqq 1$. Therefore

$$\max_{x,y \geqq 0} H(x, y) = \max_{0 \leqq x \leqq 1} H(x, x) = \max_{0 \leqq x \leqq 1} s(x)^2[1 + \sin(2x)].$$

Since $\sin(2x)$ and $s(x)$ are decreasing for $\pi/4 < x \leqq 1$, the maximum cannot occur there. Similarly, if $0 \leqq x \leqq \pi/8$, we have

$$H(x, x) \leqq 1 + \frac{1}{\sqrt{2}},$$

which is less than $H(x_0, x_0)$. Therefore the maximum of $H(x, x)$ occurs in the interval $\pi/8 \leqq x \leqq \pi/4$.

The derivative of $H(x, x)$ is

$$2s(x)\{s'(x)[1 + \sin(2x)] + s(x)\cos(2x)\}.$$

This can be zero only if

$$w(x) = s'(x)[1 + \sin(2x)] + s(x)\cos(2x)$$

is zero. Since $w(\pi/8) > 0$, $w(\pi/4) < 0$, and the first two terms of

$$w'(x) = s''(x)[1 + \sin(2x)] - 2s(x)\sin(2x) + 3s'(x)\cos(2x)$$

are negative and the third is nonpositive, the equation $w(x) = 0$ has a unique solution x_0 in $[\pi/8, \pi/4]$. We conclude that

$$\max H(x, y) = H(x_0, x_0) = s(x_0)^2[1 + \sin(2x_0)].$$

Numerical computations give $x_0 \approx 0.5875$ and $\max H \approx 1.7114$.

THEOREM 3.8. *Let f belong to $\overline{S_W}$, and suppose that*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}.$$

https://doi.org/10.4153/CJM-1987-071-4 Published online by Cambridge University Press

Then

$$|a_n| \leqq \sqrt{H(x_0, x_0)} \approx 1.3082 \quad \textit{for all } n \geqq 1$$

where $H(x_0, x_0)$ *is defined in Lemma* 3.7. *Equality for any n is possible only for the functions* $T_{(\alpha,\beta,\gamma)}$ *where*

$$\beta = \alpha + \frac{2x_0}{n} \quad \textit{and} \quad \gamma = \alpha + 2\pi - \frac{2x_0}{n}.$$

In addition, $|b_n| \leqq 1$ *for all* $n \geqq 1$. *Equality in this case for any n is possible only for the functions* $2\pi P(\cdot, \alpha)$.

Proof. It is sufficient to consider the extreme points

$$T_{(\alpha,\beta,\gamma)}(z) = 1 + \sum_{n=1}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}.$$

If $\alpha < \beta < \gamma < \alpha + 2\pi$, then formula (4) can be rewritten as

$$\begin{aligned} &T_{(\alpha,\beta,\gamma)}(z) \\ &= -1 + \frac{(1+i)}{|I_1|} \arg \frac{e^{i\beta} - z}{e^{i\alpha} - z} + \frac{(1-i)}{|I_2|} \arg \frac{e^{i\alpha} - z}{e^{i\gamma} - z} \\ &= -1 + \left[\frac{(1-i)}{2|I_1|} \log \frac{e^{i\beta} - z}{e^{i\alpha} - z} - \frac{(1+i)}{2|I_2|} \log \frac{e^{i\alpha} - z}{e^{i\gamma} - z}\right] \\ &\quad - \overline{\left[\frac{(1+i)}{2|I_1|} \log \frac{e^{i\beta} - z}{e^{i\alpha} - z} - \frac{(1-i)}{2|I_2|} \log \frac{e^{i\alpha} - z}{e^{i\gamma} - z}\right]}. \end{aligned}$$

Therefore

$$A_n = \frac{(1-i)(e^{-in\alpha} - e^{-in\beta})}{2n|I_1|} - \frac{(1+i)(e^{-in\gamma} - e^{-in\alpha})}{2n|I_2|}$$

and

$$B_n = \frac{-(1+i)(e^{-in\alpha} - e^{-in\beta})}{2n|I_1|} + \frac{(1-i)(e^{-in\gamma} - e^{-in\alpha})}{2n|I_2|}.$$

Consequently,

$$(7) \qquad A_n = e^{-in\alpha}\left[\frac{(1+i)}{2} e^{-ix} s(x) + \frac{(1-i)}{2} e^{iy} s(y)\right]$$

and

$$(8) \qquad B_n = e^{-in\alpha}\left[\frac{(1-i)}{2} e^{-ix} s(x) + \frac{(1+i)}{2} e^{iy} s(y)\right]$$

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where

$$x = n|I_1|/2, \quad y = n|I_2|/2,$$

$$s(x) = \frac{\sin x}{x} \quad \text{for } x > 0, \quad \text{and } s(0) = 1.$$

In the other cases for (α, β, γ) the functions $T_{(\alpha,\beta,\gamma)}$ were defined by continuity. Therefore formulas (7) and (8) remain valid in the general case.

It follows from (7) and (8) that

$$|A_n|^2 = H(x, y) \quad \text{and} \quad |B_n|^2 = G(x, y)$$

where G and H are defined in Lemmas 3.6 and 3.7. These lemmas give the desired estimates and equality for the indicated functions. If f is any function in $\overline{S_W}$ for which $|a_n|$ is a maximum, then this function provides the maximum of $\text{Re}\{e^{i\theta}a_n\}$ for some θ. However, only one of the indicated extreme points solves this problem. Therefore f is this extreme point. In other words, the indicated extremal functions are the only ones for the $|a_n|$-problem. A similar statement holds for the $|b_n|$-problem.

For application to minimal surfaces we shall see in Section 5 that it will be useful to have estimates for $|a_1|$ from below. We saw already in the proof of Theorem 3.1 that

$$|a_1| = |f_z(0)| \geqq \frac{1}{1.72\sqrt{2}}$$

for all $f \in S_W$. This bound is not sharp; however, the following one is.

THEOREM 3.9. *If $f = h + \overline{g}$ belongs to $\overline{S_W}$, then*

$$|h'(0)| \geqq 2/\pi.$$

Equality occurs only when

$$f = T_{(\alpha,\alpha+\pi,\alpha+\pi)}, \quad 0 \leqq \alpha < 2\pi.$$

Proof. If $f = h + \overline{g}$ belongs to $\overline{S_W}$, then f belongs also to $S(\kappa, \lambda)$ for some κ and λ with $0 \leqq \lambda < 2\pi$ and $\lambda \leqq \kappa \leqq \lambda + 2\pi$. As a result, Corollary 3.4 and formula (7) imply that

$$(9) \quad |h'(0)| = \left| \int_{K(\kappa,\lambda)} e^{-i\lambda}\left[\frac{(1+i)}{2}e^{-ix}s(x) + \frac{(1-i)}{2}e^{iy}s(y)\right]d\mu(\beta, \gamma)\right|$$

$$= \frac{1}{2}\left|\int_{K(\kappa,\lambda)} [ie^{-ix}s(x) + e^{iy}s(y)]d\mu(\beta, \gamma)\right|$$

where

$$x = \frac{\beta - \lambda}{2}, y = \frac{\lambda + 2\pi - \gamma}{2}, \quad \text{and } \lambda \leqq \beta \leqq \kappa \leqq \gamma \leqq \lambda + 2\pi.$$

Let

$$\sigma = \frac{\kappa - \lambda}{2}.$$

Then $0 \leqq x \leqq \sigma$ and $0 \leqq y \leqq \pi - \sigma$, and it is sufficient to restrict $0 \leqq \sigma \leqq \pi/2$.

First, we shall show that $|h'(0)| > 2/\pi$ when $0 \leqq \sigma < \pi/8$. In this case

$$\begin{aligned} 2|h'(0)|^2 &= \left[\int [(\sin x)s(x) + (\cos y)s(y)]d\mu\right]^2 \\ &+ \left[\int [(\cos x)s(x) + (\sin y)s(y)]d\mu\right]^2 \\ &\geqq \left[\int (\cos x)s(x)d\mu\right]^2 = \left[\int \frac{\sin 2x}{2x} d\mu\right]^2 \\ &> \left[\int \frac{\sin \pi/4}{\pi/4} d\mu\right]^2 = 8/\pi^2. \end{aligned}$$

In other words, $|h'(0)| > 2/\pi$. So from now on, we may assume that $\pi/8 \leqq \sigma \leqq \pi/2$.

Let C_x and C_y denote the curves

$$C_x : ie^{-ix}s(x), \quad 0 \leqq x \leqq \sigma, \quad \text{and}$$
$$C_y : e^{iy}s(y), \quad 0 \leqq y \leqq \pi - \sigma.$$

These curves lie in the upper half-plane and are at least as far from the origin as the line segments L_x and L_y joining their endpoints. Therefore the convex average in (9) has modulus at least as large as that obtained from sums of points on L_x and L_y. The set of sums of points on L_x and L_y is a parallelogram Π with vertices

$$V_1 = ie^{-i\sigma}s(\sigma) + e^{i(\pi-\sigma)}s(\pi - \sigma), \quad V_2 = i + e^{i(\pi-\sigma)}s(\pi - \sigma),$$
$$V_3 = ie^{-i\sigma}s(\sigma) + 1, \quad V_4 = i + 1.$$

We shall show that V_1 is the nearest point of Π to the origin. One easily verifies that

$$|V_1|^2 \leqq |V_j|^2 \quad \text{for } j = 2, 3, 4,$$

and so the nearest point of Π to the origin is either on the segment joining V_1 to V_2 or on the segment joining V_1 to V_3. Therefore it is sufficient to show that the angle between the vectors $\overrightarrow{OV_1}$ and $\overrightarrow{V_1V_j}$ is at most $\pi/2$ for $j = 2, 3$. This is equivalent to showing that

$$\mathrm{Re}\{(V_j - V_1)\overline{V_1}\} \geqq 0 \quad \text{for } j = 2, 3.$$

First, we compute

$$\mathrm{Re}\{(V_2 - V_1)\overline{V_1}\} = s(\sigma)[\cos\sigma - s(\sigma)] + s(\pi - \sigma)\sin\sigma$$
$$= s^2(\sigma)\left[\sigma\cot\sigma - 1 + \frac{\sigma^2}{\pi - \sigma}\right].$$

We use the expansion

$$\sigma\cot\sigma - 1 = -2\sum_{k=1}^{\infty}\zeta(2k)(\sigma/\pi)^{2k}, \quad |\sigma| < \pi,$$

in terms of the function

$$\zeta(t) = \sum_{n=1}^{\infty} n^{-t}.$$

Since $\zeta(t) \leqq \zeta(2) = \pi^2/6$ for $t \geqq 2$, it follows that

$$\sigma\cot\sigma - 1 \geqq \frac{-\pi^2}{3}\sum_{k=1}^{\infty}(\sigma/\pi)^{2k} = \frac{-\pi^2\sigma^2}{3(\pi^2 - \sigma^2)}.$$

Therefore

$$\mathrm{Re}\{(V_2 - V_1)\overline{V_1}\} \geqq \frac{\sigma^2 s(\sigma)^2}{\pi^2 - \sigma^2}\left[\frac{-\pi^2}{3} + \pi + \sigma\right] > 0$$

for

$$\pi\left(\frac{\pi}{3} - 1\right) < \sigma < \pi.$$

This includes the interval $\pi/8 \leqq \sigma \leqq \pi/2$.

Next, consider

$$\mathrm{Re}\{(V_3 - V_1)\overline{V_1}\} = -s(\pi - \sigma)[\cos\sigma + s(\pi - \sigma)] + s(\sigma)\sin\sigma.$$

This is the same expression as for $\mathrm{Re}\{(V_2 - V_1)\overline{V_1}\}$ with σ replaced by $\pi - \sigma$. Since the expression for $\mathrm{Re}\{(V_2 - V_1)\overline{V_1}\}$ was proved to be positive for an interval containing $\pi/2 \leqq \sigma < \pi$, it follows that

$$\mathrm{Re}\{(V_3 - V_1)\overline{V_1}\} \geqq 0 \quad \text{for } 0 \leqq \sigma \leqq \pi/2.$$

Now we shall show that $|V_1|$ is a minimum when $\sigma = \pi/2$. Let

$$\varphi(\sigma) = |V_1|^2 = s(\sigma)^2 + s(\pi - \sigma)^2 = (\sin\sigma)^2\left[\frac{1}{\sigma^2} + \frac{1}{(\pi - \sigma)^2}\right]$$

for $\pi/8 \leqq \sigma \leqq \pi/2$. First, we use the fact that $s(\sigma)^2$ is decreasing to see that

$$\varphi(\sigma) > s(\sigma)^2 \geqq s(\pi/4)^2 = 8/\pi^2 = \varphi(\pi/2) \quad \text{for } \pi/8 \leqq \sigma \leqq \pi/4.$$

Next, since $s(\pi - \sigma)^2$ is increasing, there is a $\sigma_1 > \pi/4$ such that for $\pi/4 < \sigma < \sigma_1$ one has

$$\varphi(\sigma) > s(\sigma)^2 + s\left(\pi - \frac{\pi}{4}\right)^2 \geqq \varphi(\pi/2).$$

The number σ_1 satisfies the equation

$$s(\sigma_1)^2 + s\left(\pi - \frac{\pi}{4}\right)^2 = \varphi(\pi/2).$$

That is,

$$s(\sigma_1)^2 = \frac{64}{9\pi^2}$$

or

$$s(\sigma_1) = \frac{8}{3\pi}.$$

Since $s(\sigma)$ is a decreasing function, one easily approximates $\sigma_1 \approx .975$. Finally, using the inequality

$$(\cos \tau)^2 \geqq 1 - \tau^2,$$

we have

$$\varphi\left(\frac{\pi}{2} - \tau\right) = (\cos \tau)^2 \left[\frac{1}{\left(\frac{\pi}{2} - \tau\right)^2} + \frac{1}{\left(\frac{\pi}{2} + \tau\right)^2}\right]$$

$$\geqq (1 - \tau^2)\left[\frac{1}{\left(\frac{\pi}{2} - \tau\right)^2} + \frac{1}{\left(\frac{\pi}{2} + \tau\right)^2}\right].$$

It is easy to verify directly that the latter is larger than

$$\varphi(\pi/2) = 8/\pi^2 \quad \text{when } 0 < |\tau| < \frac{\pi}{2}\sqrt{\frac{12 - \pi^2}{4 + \pi^2}}.$$

That is,

$$\varphi(\sigma) > \varphi(\pi/2) \quad \text{for } .955 \approx \frac{\pi}{2}\left[1 - \sqrt{\frac{12 - \pi^2}{4 + \pi^2}}\right] < \sigma < \frac{\pi}{2}.$$

In any case, it follows that

$$\varphi(\sigma) > \varphi(\pi/2) \quad \text{for } \pi/8 \leqq \sigma < \pi/2.$$

In summary, we have shown that

$$2|h'(0)|^2 \geqq \varphi(\pi/2) = 8/\pi^2.$$

Equality can occur only when μ is a measure concentrated so that $x = y = \sigma = \pi/2$. In other words, equality in the estimates occurs only for functions in $\overline{S_W}$ that arise from a unit mass at a point (α, β, γ) with $\beta = \gamma = \alpha + \pi$.

The bounds for $|h'(0)|$ in Theorems 3.8 and 3.9 can be improved if we assume in addition that $g'(0) = 0$. In the applications of Section 5, this will mean geometrically that the tangent plane at a corresponding point of a certain minimal surface is horizontal. For that reason let S_W^0 be the subset of S_W consisting of functions $f = h + \overline{g}$ for which $g'(0) = 0$.

THEOREM 3.10. *If $f = h + \overline{g}$ belongs to $\overline{S_W^0}$, then*

$$\frac{4}{\pi + 2} \leqq |h'(0)| \leqq \frac{4}{\pi}.$$

The lower bound is sharp only for the functions

$$f = \frac{\pi}{\pi + 2} T_{(\alpha,\alpha+\pi,\alpha+\pi)} + \frac{4\pi}{\pi + 2} P(\,\cdot\,, \alpha), \quad 0 \leqq \alpha < 2\pi,$$

and the upper bound is sharp only for the functions

$$f = T_{(\alpha,\alpha+\pi/2,\alpha+\pi/2)}, \quad 0 \leqq \alpha < 2\pi.$$

Proof. We shall continue to use some of the notation from the proof of Theorem 3.9. In particular, we may represent

$$h'(0) = \int_{K(\kappa,\lambda)} e^{-i\lambda}\left[\frac{(1 + i)}{2} e^{-ix} s(x) + \frac{(1 - i)}{2} e^{iy} s(y)\right] d\mu(\beta, \gamma)$$

and, similarly,

$$g'(0) = \int_{K(\kappa,\lambda)} e^{-i\lambda}\left[\frac{(1 - i)}{2} e^{-ix} s(x) + \frac{(1 + i)}{2} e^{iy} s(y)\right] d\mu(\beta, \gamma)$$

where

$$x = \frac{\beta - \lambda}{2}, \quad y = \frac{\lambda + 2\pi - \gamma}{2}, \quad \lambda \leqq \kappa \leqq \lambda + 2\pi,$$

$$\sigma = \frac{\kappa - \lambda}{2}, \quad 0 \leqq x \leqq \sigma, \quad \text{and} \quad 0 \leqq y \leqq \pi - \sigma.$$

The additional restriction $g'(0) = 0$ is equivalent to

$$(10) \quad \int ie^{-ix}s(x)d\mu = \int e^{iy}s(y)d\mu,$$

and so

$$(11) \quad |h'(0)| = \sqrt{2}\left|\int e^{ix}s(x)d\mu\right| = \sqrt{2}\left|\int e^{iy}s(y)d\mu\right|.$$

If $0 \leqq \sigma \leqq 7\pi/16$, then

$$|h'(0)| = \sqrt{2}\left|\int e^{i(x-7\pi/32)}s(x)d\mu\right| \geqq \sqrt{2}\cos\left(\sigma - \frac{7\pi}{32}\right)s(\sigma)$$

$$\geqq \sqrt{2}\cos\left(\frac{7\pi}{32}\right)s\left(\frac{7\pi}{16}\right) = .780\ldots > \frac{4}{\pi + 2}.$$

A similar estimate holds for $9\pi/16 \leqq \sigma \leqq \pi$. Therefore, to derive the lower bound we may assume $7\pi/16 < \sigma < 9\pi/16$.

Since $s'(x) < 0$ and $s''(x) < 0$ for $0 < x < 9\pi/16$, the left side of (10) lies in the closed convex region R_x bounded by the curve

$$C_x : ie^{-ix}s(x), \quad 0 \leqq x \leqq \sigma,$$

and the line segment L_x joining its endpoints. Similarly, the right side of (10) lies in the convex region R_y bounded by the curve

$$C_y : e^{iy}s(y), \quad 0 \leqq y \leqq \pi - \sigma,$$

and the line segment L_y joining its endpoints. The constraint (10) restricts us to the intersection of these regions, and for arbitrary probability measures μ each point of $R_\sigma = R_x \cap R_y$ is attainable. The expressions in (11) involve the distance of points in R_σ from the origin. Thus we need to find the nearest point of R_σ to the origin for all σ.

It is evident that the nearest point of R_σ to the origin is on one of the segments L_x or L_y. We shall observe that it is at their intersection $I_\sigma = L_x \cap L_y$, at least for the minimizing value of σ. (The reader may find it useful to make a sketch.) If p is a point of $(L_x \cap R_\sigma)\backslash I_\sigma$, then p belongs to the interior of $R_{\sigma+\epsilon}$ for ϵ positive and sufficiently small; that is, p cannot be the nearest point of R_σ to the origin for all σ. Similarly, if p is a point of $(L_y \cap R_\sigma)\backslash I_\sigma$, then p belongs to the interior of $R_{\sigma-\epsilon}$ for ϵ positive and sufficiently small; that is, p does not minimize. Thus the solution to the minimum problem is

$$\min_{7\pi/16 \leqq \sigma \leqq 9\pi/16} \sqrt{2}|I_\sigma|.$$

A straightforward computation yields

$$I_\sigma = \frac{(\sin\sigma)\{s(\sigma)[1 + (\cos\sigma - \sin\sigma)s(\pi - \sigma)]}{1 - s(\sigma)s(\pi - \sigma) - (\cos\sigma)[s(\sigma) - s(\pi - \sigma)]}$$
$$+ \frac{is(\pi - \sigma)[1 - (\cos\sigma + \sin\sigma)s(\sigma)]\}}{1 - s(\sigma)s(\pi - \sigma) - (\cos\sigma)[s(\sigma) - s(\pi - \sigma)]}.$$

After substituting

$$s(\sigma) = (\sin \sigma)/\sigma \quad \text{and} \quad s(\pi - \sigma) = (\sin \sigma)/(\pi - \sigma),$$

we may write

$$|I_\sigma|^2 = N(\sigma)/D(\sigma)^2$$

where

$$N(\sigma) = (\sin \sigma)^4\{ (\pi - 2\sigma)(\sin 2\sigma) - 2(\pi - 1)(\sin \sigma)^2 + \sigma^2 + (\pi - \sigma)^2\}$$

and

$$D(\sigma) = \sigma(\pi - \sigma) - (\sin \sigma)^2 - \frac{1}{2}(\pi - 2\sigma)(\sin 2\sigma).$$

The function D is positive over the indicated interval, and since

$$D'(\sigma) = (\pi - 2\sigma)(1 - \cos 2\sigma),$$

it follows that D is largest when $\sigma = \pi/2$. In other words, $1/D^2$ is a minimum when $\sigma = \pi/2$. Next, we have

$$N'(\sigma) = -2(\pi - 2\sigma)(\sin \sigma)^4(1 - \cos 2\sigma) - 2(\sin 2\sigma)(\sin \sigma)^2[(3\pi - 2)(\sin \sigma)^2 - (\pi - 2\sigma)(\sin 2\sigma) - \sigma^2 - (\pi - \sigma)^2].$$

On the interval $7\pi/16 \leqq \sigma \leqq 9\pi/16$, the factor

$$(3\pi - 2)(\sin \sigma)^2 - (\pi - 2\sigma)(\sin 2\sigma) - \sigma^2 - (\pi - \sigma)^2$$

is bounded below by the positive number

$$(3\pi - 2)\left(\sin \frac{7\pi}{16}\right)^2 - \frac{\pi}{8}\left(\sin \frac{\pi}{8}\right) - \left(\frac{7\pi}{16}\right)^2 - \left(\frac{9\pi}{16}\right)^2$$
$$= 1.98 \ldots .$$

It follows that N has its minimum also at $\sigma = \pi/2$. As a result, we conclude that

$$\min_{7\pi/16 \leqq \sigma \leqq 9\pi/16} \sqrt{2}|I_\sigma| = \sqrt{2}|I_{\pi/2}| = \frac{4}{\pi + 2}$$

and that this number is the minimum of $|h'(0)|$.

Equality in the minimum problem occurs only when $\sigma = \pi/2$ and the expressions in (10) equal $I_{\pi/2}$. The measure μ must be concentrated at the points corresponding to the endpoints of C_x and C_y. That is, in the (x, y)-coordinates the measure μ has mass $2/(\pi + 2) - t$ at (0, 0), mass

$\pi/(\pi + 2) - t$ at $((\pi/2), (\pi/2))$, and mass t at each of the points $(0, (\pi/2))$ and $((\pi/2), 0)$, for some t, $0 \leqq t \leqq 2/(\pi + 2)$. The corresponding functions are

$$f = \left(\frac{2}{\pi + 2} - t\right)T_{(\alpha,\alpha,\alpha+2\pi)} + \left(\frac{\pi}{\pi + 2} - t\right)T_{(\alpha,\alpha+\pi,\alpha+\pi)}$$
$$+ tT_{(\alpha,\alpha,\alpha+\pi)} + tT_{(\alpha,\alpha+\pi,\alpha+2\pi)}.$$

Because of the identity

$$T_{(\alpha,\alpha,\alpha+2\pi)} + T_{(\alpha,\alpha+\pi,\alpha+\pi)} = T_{(\alpha,\alpha,\alpha+\pi)} + T_{(\alpha,\alpha+\pi,\alpha+2\pi)},$$

these are the functions given in the theorem.

The upper bound for $|h'(0)|$ is obtained more easily. Since the curves C_x and C_y move toward the origin, it is apparent that the point of R_σ farthest from the origin occurs at the intersection of the curves C_x and C_y when $\sigma \geqq \pi/4$ and $x = y = \pi/4$. That is, we have

$$|h'(0)| \leqq \sqrt{2}s(\pi/4) = 4/\pi.$$

Equality occurs only when the measure μ is concentrated at a point corresponding to $x = y = \pi/4$. These functions are given in the statement of the theorem.

4. Mappings onto a half-plane. Let $R = \{w:\text{Re } w > 0\}$, and denote by S_R the set of all univalent, harmonic, orientation-preserving mappings f of U onto R with normalization $f(0) = 1$. There are functions in S_R, such as $(1 + z)/(1 - z)$, that do not belong to h^1 and cannot be written in a form similar to that of Theorem 2.4 (c). Consequently, the treatment of S_R must differ from that of S_W. Instead of using radial limits, we shall focus attention on the behavior inside the unit disk as in [**8**] in order to obtain a representation theorem.

LEMMA 4.1. *If* $f \in S_R$, *then* $\text{Re } f = 2\pi P(\cdot, \lambda)$ *for some* λ, $0 \leqq \lambda < 2\pi$.

Proof. Since $f = h + \bar{g}$ maps onto a convex domain, it follows from [**2**, Theorem 5.7] that $h \pm g$ are univalent. Therefore $h \pm g$, h, g, and f all have finite radial limits almost everywhere. Just as in the proof of Theorem 2.4, we conclude that there is exactly one point $e^{i\lambda}$ which corresponds under f to ∞.

Since $\text{Re } f > 0$, we have $\text{Re } f \in h^1$ and

$$\text{Re } f = \int_0^{2\pi} P(\cdot, t)d\mu(t).$$

Except for the point $e^{i\lambda}$, the radial limits of $\text{Re } f$ all exist and are zero. Therefore μ is equivalent to a point mass at $t = \lambda$.

For $f \in S_R$ write

$$f = h + \bar{g} = 2\pi P(\cdot, \lambda) + iv$$

where $h(0) = 1$ and $g(0) = v(0) = 0$. Then, by differentiation, we have

$$h'(z) + g'(z) = 4\pi P_z(z, \lambda) = \frac{2e^{i\lambda}}{(e^{i\lambda} - z)^2}$$

and $h' - g' = 2iv_z$. So, if

$$p = \frac{h' - g'}{h' + g'},$$

then we find that

$$v(z) = \mathrm{Im}\left[\int_0^z [h'(\zeta) - g'(\zeta)]d\zeta\right] = \mathrm{Im}\left[\int_0^z \frac{2e^{i\lambda}p(\zeta)}{(e^{i\lambda} - \zeta)^2}d\zeta\right].$$

The orientation-preserving property of f implies that $|g'/h'| < 1$ and, hence, that $\mathrm{Re}\, p > 0$. Therefore, by the Herglotz formula, there is a probability measure μ so that

$$(12) \quad p(z) = b\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}d\mu(t) + ic$$

where $p(0) = b + ic$. By substitution, we have

$$\begin{aligned} v(z) &= \int_0^{2\pi} \mathrm{Im}\left\{\int_0^z \frac{2be^{i\lambda}(e^{it} + \zeta)}{(e^{i\lambda} - \zeta)^2(e^{it} - \zeta)}d\zeta\right\} d\mu(t) \\ &\quad + \mathrm{Re}\left\{\int_0^z \frac{2ce^{i\lambda}}{(e^{i\lambda} - \zeta)^2}d\zeta\right\} \\ &= \int_0^{2\pi} bK(z, t, \lambda)d\mu(t) + 2c\,\mathrm{Re}\left\{\frac{z}{e^{i\lambda} - z}\right\} \end{aligned}$$

where

$$K(z, t, \lambda) = \mathrm{Im}\left\{\int_0^z \frac{2e^{i\lambda}(e^{it} + \zeta)}{(e^{i\lambda} - \zeta)^2(e^{it} - \zeta)}d\zeta\right\}$$

can be integrated explicitly. The following representation is a consequence.

THEOREM 4.2. *If $f \in S_R$, then there are a probability measure μ and real numbers λ, $b > 0$, and c so that*

$$(13) \quad f(z) = 2\pi P(z, \lambda)$$

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$$+ i\left[\int_0^{2\pi} bK(z, t, \lambda)d\mu(t) + 2c \operatorname{Re}\left\{\frac{z}{e^{i\lambda} - z}\right\}\right].$$

Let $\mathscr{K}$ denote the set of all functions of the form (13) for real numbers λ, $b \geqq 0$, and c and probability measures μ. Then $\mathscr{K}$ is closed, but not compact since b and c are not bounded. However, the following is true.

THEOREM 4.3. *If $f \in \mathscr{K}$ with $b > 0$, then f is a univalent, harmonic, orientation-preserving mapping of U onto a convex domain.*

Proof. If $f \in \mathscr{K}$ and

$$\varphi(w) = e^{i\lambda}\frac{w - 1}{w + 1},$$

then f is harmonic and $f \circ \varphi$ maps vertical lines in R into themselves. For $u_0 > 0$, consider the vertical line

$$L_{u_0}\colon u_0 + iv, \quad -\infty < v < \infty.$$

Reversing some of the steps above, we obtain

$$\frac{\partial}{\partial v} \operatorname{Im}\{f \circ \varphi\} = \operatorname{Re}\{p \circ \varphi\}$$

where p has the form (12). Since Re p is positive, $f \circ \varphi$ maps each line L_{u_0} into itself in an increasing fashion. Hence $f \circ \varphi$ is one-to-one and orientation preserving in R. Thus the same properties hold for f in U.

To see that $f(U)$ is convex, we substitute

$$\Phi = ie^{-i\theta}(h - e^{2i\theta}g)$$

into expression (1) with $\mu = \lambda$ and $\nu = 0$. We obtain

$$\operatorname{Re}\{-ie^{i\lambda}(1 - e^{-i\lambda}z)^2\Phi'(z)\} = 2(\cos\theta)\operatorname{Re}\{p(z)\}.$$

This is positive for $-\pi/2 < \theta < \pi/2$, and it follows from [**13**, Theorem 1] that Φ is univalent and maps U onto a domain that is convex in the vertical direction. For $\theta = \pi/2$, the function

$$\Phi(z) = h(z) + g(z) = \frac{e^{i\lambda} + z}{e^{i\lambda} - z}$$

obviously has the same properties. Since Φ is convex in the vertical direction for all θ, it follows from [**2**, Theorem 5.7] that $f(U)$ is convex.

Remark 4.4. Given a function $a \in H(U)$ with $a(U) \subset U$, one obtains from (12) a measure μ representing

$$p = (1 - a)/(1 + a).$$

Since $b = \operatorname{Re}\{p(0)\} > 0$, the function (13) satisfies the conclusion of

Theorem 2.3 in case $D = R$. Part (c) of the theorem is clearly satisfied because Re f tends to zero at every point of $\partial U \setminus \{e^{i\lambda}\}$. The normalizations in (a) may be achieved by composing f with elementary mappings.

Using an argument similar to that in [**8**, Lemma 2.6 and Theorem 2.7], one obtains the following result. We omit the proof.

THEOREM 4.5. $\mathscr{K} = \overline{S_R}$.

In the remaining part of this section, we shall study the subclasses S_R^0 and $\mathscr{K}^0$ of S_R and $\mathscr{K}$, respectively, of functions $f = h + \bar{g}$ that satisfy the additional condition

$$f_{\bar{z}}(0) = g'(0) = 0$$

or, equivalently, $p(0) = 1$.

It is clear from the definitions that

$$S_R^0 \subset \mathscr{K}^0 \quad \text{and} \quad \overline{S_R^0} = \mathscr{K}^0.$$

Since $b = 1$ and $c = 0$ in (13), it is also clear that $\mathscr{K}^0$ is compact. Next, we characterize the extreme points of $\mathscr{K}^0$.

THEOREM 4.6. $E\mathscr{K}^0 = \{2\pi P(\cdot, \lambda) + iK(\cdot, t, \lambda) : 0 \leqq t, \lambda < 2\pi\}$.

Proof. If $f \in E\mathscr{K}^0$, then the probability measure μ in (13) must be a unit point mass. That is, the extreme points must have the given form. On the other hand, suppose that

$$2\pi P(\cdot, \lambda) + iK(\cdot, t, \lambda) = xf_1 + (1 - x)f_2$$

for some $f_1, f_2 \in \mathscr{K}^0$ and $0 < x < 1$. Then

$$\text{Re } f_1 = \text{Re } f_2 = 2\pi P(\cdot, \lambda)$$

simply by a comparison of singularities. Finally, the map

$$\mu \rightarrow \int_0^{2\pi} K(\cdot, t, \lambda)d\mu(t)$$

from the set $\mathscr{P}$ of probability measures is linear and one-to-one. Since the extreme points of $\mathscr{P}$ are unit masses, we conclude that Im f_1 = Im f_2 = $K(\cdot, t, \lambda)$, also.

Example 4.7. The function

$$f(z) = 2\pi P(z, \lambda) + iK(z, \lambda, \lambda)$$
$$= \text{Re}\left[\frac{e^{i\lambda} + z}{e^{i\lambda} - z}\right] + 2i \,\text{Im}\left[\frac{e^{i\lambda}z}{(e^{i\lambda} - z)^2}\right]$$

is an extreme point of $\mathscr{K}^0$ and belongs to S_R since it maps U onto R. Its boundary values are all zero except at the point $e^{i\lambda}$, which corresponds to the entire imaginary axis. Note that this function does not belong to h^1.

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Example 4.8. Suppose that $e^{it} \neq e^{i\lambda}$, and set

$$e^{i\xi} = e^{i(\lambda - t)}, \quad 0 < \xi < 2\pi.$$

Then

$$\begin{aligned} f(z) &= 2\pi P(z, \lambda) + iK(z, t, \lambda) \\ &= \operatorname{Re}\left[\frac{e^{i\lambda} + z}{e^{i\lambda} - z}\right] - \frac{i}{\left(\sin \frac{\xi}{2}\right)^2} \arg\left[\frac{e^{i\lambda} - z}{e^{i\lambda} - e^{i\xi}z}\right] \\ &\quad + 2i\left(\cot \frac{\xi}{2}\right) \operatorname{Re}\left[\frac{z}{e^{i\lambda} - z}\right] \end{aligned}$$

is an extreme point of $\mathcal{K}^0$. It maps U onto the intersection of R with an inclined infinite strip. The boundary values of f on one open arc between $e^{i\lambda}$ and e^{it} are all equal to

$$p_1 = i\frac{\xi - \sin \xi}{1 - \cos \xi}$$

and on the complementary open arc they equal

$$p_2 = i\frac{\xi - \pi - \sin \xi}{1 - \cos \xi}.$$

The cluster set of f at e^{it} is the closed segment of the imaginary axis joining p_1 and p_2. The cluster set of f at $e^{i\lambda}$ contains the rest of $\partial f(U)$. It consists of the half-lines

$$p_j + rie^{-i\xi/2}, \quad r \geqq 0, \quad j = 1, 2.$$

Finally, we give the following application of Theorem 4.6.

THEOREM 4.9. *If*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$$

belongs to $\mathcal{K}^0$, *then* $|a_1| = 2$, $|a_n| \leqq n + 1$, *and* $|b_n| \leqq n - 1$ *for* $n \geqq 2$. *Equality in either of the inequalities occurs only for the functions from Example* 4.7.

Proof. It is sufficient to estimate the coefficients of the extreme points in Theorem 4.6. For these functions we have

$$\sum_{n=1}^{\infty} na_n z^{n-1} = 2\pi P_z(z, \lambda) + iK_z(z, t, \lambda)$$

$$= \frac{e^{i\lambda}}{(e^{i\lambda} - z)^2} + \frac{e^{i\lambda}(e^{it} + z)}{(e^{i\lambda} - z)^2(e^{it} - z)}$$

and

$$\sum_{n=2}^{\infty} nb_n z^{n-1} = 2\pi P_z(z, \lambda) - iK_z(z, t, \lambda)$$

$$= \frac{e^{i\lambda}}{(e^{i\lambda} - z)^2} - \frac{e^{i\lambda}(e^{it} + z)}{(e^{i\lambda} - z)^2(e^{it} - z)}.$$

It follows that

$$na_n = 2e^{-int} \sum_{k=1}^{n} ke^{ik(t-\lambda)} \quad \text{and}$$

$$nb_n = -2e^{-int} \sum_{k=1}^{n-1} ke^{ik(t-\lambda)}.$$

From this one has

$$|a_1| = 2, \quad n|a_n| \leqq 2 \sum_{k=1}^{n} k = n(n + 1), \quad \text{and}$$

$$n|b_n| \leqq 2 \sum_{k=1}^{n-1} k = n(n - 1) \quad \text{for } n \geqq 2.$$

Equality in these inequalities occurs only if $e^{it} = e^{i\lambda}$, that is, the extreme point is of the form of Example 4.7. Finally, to see that equality occurs only for an extreme point, one argues as in the last paragraph of the proof of Theorem 3.8.

5. An application to minimal surfaces. Let S denote a nonparametric surface in $\mathbf{R}^3$ over a domain D in $\mathbf{C}$. Suppose that S is the graph of the function $\mathcal{F} = \mathcal{F}(u, v)$ where $u + iv \in D$. Then S is a nonparametric minimal surface if $\mathcal{F}$ belongs to C^2 and satisfies the differential equation

$$\left(1 + \left|\frac{\partial \mathcal{F}}{\partial v}\right|^2\right)\frac{\partial^2 \mathcal{F}}{\partial u^2} - 2\left(\frac{\partial \mathcal{F}}{\partial u} \cdot \frac{\partial \mathcal{F}}{\partial v}\right)\frac{\partial^2 \mathcal{F}}{\partial u \partial v} + \left(1 + \left|\frac{\partial \mathcal{F}}{\partial u}\right|^2\right)\frac{\partial^2 \mathcal{F}}{\partial v^2} = 0$$

in D [**12**, p. 17]. Proofs of the following proposition are indicated in [**11**, p. 120] and [**12**, p. 30].

PROPOSITION 5.1. *S is a nonparametric minimal surface over a simply-connected domain $D \neq \mathbf{C}$ if and only if there are analytic functions Φ_1, Φ_2, Φ_3 in U such that*

(a) $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 \equiv 0$ *in U,*

(b) $|\Phi_1^2(z)| + |\Phi_2^2(z)| + |\Phi_3^2(z)| \neq 0$ *for all $z \in U$,*

(c) *the mapping*

$$z \rightarrow \operatorname{Re} \int \Phi_1 dz + i \operatorname{Re} \int \Phi_2 dz$$

is harmonic and univalent from U onto D, and

(d) $S = \left\{\left(\operatorname{Re} \int \Phi_1 dz, \operatorname{Re} \int \Phi_2 dz, \operatorname{Re} \int \Phi_3 dz\right) : z \in U\right\}.$

Of course, in (c) and (d) appropriate constants of integration must be chosen for the antiderivatives $\int \Phi_j dz$.

If S is given as in (d), then the mapping

$$z \rightarrow \left(\operatorname{Re} \int \Phi_1 dz, \operatorname{Re} \int \Phi_2 dz, \operatorname{Re} \int \Phi_3 dz\right)$$

of U onto S is conformal, that is, angles are preserved, and z is an isothermal parameter. In other words, arc length on S can be obtained by $ds = \rho|dz|$ where $\rho > 0$. In fact, one has

$$\rho^2 = \frac{1}{2}(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2).$$

We are interested in the univalent harmonic mapping of U onto D that is given in (c). Denote

$$f = \operatorname{Re} \int \Phi_1 dz + i \operatorname{Re} \int \Phi_2 dz.$$

It is no loss of generality to assume that f is orientation-preserving. Hence, if we write $f = h + \bar{g}$, then $a = g'/h'$ satisfies $|a(z)| < 1$ for $z \in U$ and we also have

$$\Phi_1 = h' + g', \quad \Phi_2 = -i(h' - g'), \quad \text{and}$$
$$\Phi_3^2 = -4h'g' = -4a(h')^2.$$

The latter shows that a has a single-valued square root in U. For our purposes the following is a useful restatement of Proposition 5.1.

PROPOSITION 5.2. *S is a nonparametric minimal surface over a simply-connected domain $D \neq \mathbf{C}$ if and only if there is a univalent, harmonic, orientation-preserving mapping $f = h + \bar{g}$ of U onto D such that $a = g'/h'$ has a single-valued square root in U and*

$$S = \left\{\left(\operatorname{Re} f(z), \operatorname{Im} f(z), 2 \operatorname{Im} \int \sqrt{a} h' dz\right) : z \in U\right\}.$$

In terms of $f = h + \overline{g}$, the conformal factor ρ becomes simply

$$\rho = |h'| + |g'|.$$

Therefore the Gaussian curvature of S is

$$(14) \quad k = \frac{-\Delta \log \rho}{\rho^2} = \frac{-|h'g'' - h''g'|^2}{|h'g'|(|h'| + |g'|)^4} = \frac{-|a'|^2}{|a|(1 + |a|)^4|h'|^2}$$

where $a = g'/h'$. We summarize as follows.

LEMMA 5.3. *Let S be a nonparametric minimal surface over a simply-connected domain $D \neq \mathbf{C}$. Let $f = h + \overline{g}$ and a be as in Proposition 5.2. Then for $z \in U$ the Gaussian curvature of S at the point*

$$\left(\text{Re } f(z), \text{ Im } f(z), 2 \text{ Im} \int \sqrt{a}h'dz\right)$$

is

$$k(z) = \frac{-|a'(z)|^2}{|a(z)|(1 + |a(z)|)^4|h'(z)|^2}.$$

In addition, one has

$$(15) \quad |k(z)| \leqq \frac{4(1 - |a(z)|)^2}{(1 - |z|^2)^2(1 + |a(z)|)^4|h'(z)|^2} \leqq \frac{4}{(1 - |z|^2)^2(|g'(z)| + |h'(z)|)^2}$$

and

$$(16) \quad |k(0)| \leqq \frac{4}{(|g'(0)| + |h'(0)|)^2} \leqq \frac{4}{|g'(0)|^2 + |h'(0)|^2} \leqq \frac{4}{|h'(0)|^2}.$$

Proof. Since the function a has a single-valued analytic square root, which is bounded by one, the invariant form of Schwarz's lemma implies

$$|a'| \leqq 2\sqrt{|a|}(1 - |a|)/(1 - |z|^2).$$

Use this estimate in (14) to obtain the first inequality in (15). The second inequality in (15) follows from

$$1 - |a| \leqq 1 + |a|.$$

The inequalities in (16) are simple consequences.

Remark 5.4. If D is a convex domain, then the analytic part h of $f = h + \overline{g}$ maps U homeomorphically onto a domain Ω [**2**, Theorem 5.7]. Hence, the function $F = f \circ h^{-1}$ is a univalent harmonic mapping of Ω onto D, and in terms of F, the surface S has the attractive conformal parametrization

$$S = \left\{ \left(\text{Re } F(w), \text{ Im } F(w), \text{ 2 Im } \int \sqrt{A} dw \right) : w \in \Omega \right\}$$

where $A = \overline{F_{\bar{w}}}/F_w$ has a single-valued analytic square root and $|A| < 1$. Furthermore, the Gaussian curvature becomes

$$k = \frac{-|A'|^2}{|A| (1 + |A|)^4}.$$

For the rest of this section, we shall estimate the Gaussian curvature k for nonparametric minimal surfaces over certain domains D. E. Heinz **[6]** proved that if $D = \{w:|w| < R\}$, then the curvature of S at the point P above the origin satisfies

$$|k| \leqq 4\pi^2/(3R^2).$$

Improvements were considered by E. Hopf **[9]** and many others. This estimate can be improved to

$$|k| \leqq 16\pi^2/(27R^2)$$

by combining R. Hall's **[5]** determination of the "Heinz constant", which implies

$$|g'(0)|^2 + |h'(0)|^2 \geqq 27R^2/(4\pi^2),$$

and inequality (16). Unfortunately, this curvature estimate is still not sharp. However, if in addition, S has a horizontal tangent plane at P, then at this point

$$|k| \leqq \pi^2/(2R^2)$$

and this estimate is sharp **[12**, p. 108]. Furthermore, for a general domain D it is known **[12**, p. 108] that if S has a horizontal tangent plane at P, then

$$|k| \leqq 64/(9d^2)$$

where d is the distance along the surface from P to ∂S. This is also known to be sharp for a certain domain D.

Suppose, in the framework of Proposition 5.2, that the surface S has a horizontal tangent plane at the point P above $f(0)$. Then we have

$$\frac{\partial}{\partial x}\left[2 \text{ Im} \int \sqrt{a} h' dz\right] = \frac{\partial}{\partial y}\left[2 \text{ Im} \int \sqrt{a} h' dz\right] = 0$$

at $z = 0$. This implies that $\sqrt{a(0)}h'(0) = 0$ and, consequently, that $g'(0) = 0$. The following theorem is the result of combining the estimates for $|h'(0)|$ from below in Theorems 3.9, 3.10, 4.9 and **[8**, Theorem 2.3] with inequality (16) in Lemma 5.3.

THEOREM 5.5. *Let S be a nonparametric minimal surface in* $\mathbf{R}^3$ *that lies above a domain D in* $\mathbf{C}$. *Let k denote the Gaussian curvature of S at a point P that lies above* $w_0 \in D$.

(a) *If* $D = W = \{w : |\arg w| < \pi/4\}$ *and* $w_0 = 1$, *then* $|k| \leqq \pi^2$. *In addition, assume that S has a horizontal tangent plane at P.*

(b) *If* $D = W$ *and* $w_0 = 1$, *then* $|k| \leqq (\pi + 2)^2/4$.

(c) *If* $D = R = \{w : \operatorname{Re} w > 0\}$ *and* $w_0 = 1$, *then* $|k| \leqq 1$.

(d) *If* $D = \Omega = \{w : |\operatorname{Im} w| < \pi/4\}$ *and* $w_0 = 0$, *then* $|k| \leqq 4$.

The estimate (16) that we have used in proving Theorem 5.5 is sharp only when

$$a(z) = e^{i\alpha}z^2 \quad \text{for some } \alpha.$$

Therefore the curvature estimates in this theorem are sharp only if the lower bounds for $|h'(0)|$ arise from harmonic mappings $f = h + \overline{g}$ with

$$g'(z)/h'(z) = e^{i\alpha}z^2.$$

This is not the case in (a) or (b). For (c) equality occurs for the surface

$$S = \left\{\left(\operatorname{Re}\left[\frac{1+z}{1-z}\right], \operatorname{Im}\left[\frac{z}{(1-z)^2}\right] + \frac{1}{2}\arg\left[\frac{1+z}{1-z}\right], \operatorname{Re}\left[\frac{z}{(1-z)^2}\right] - \frac{1}{2}\log\left|\frac{1+z}{1-z}\right|\right) : z \in U\right\}$$

and for (d) equality occurs for the surface

$$S = \left\{\left(\operatorname{Re}\left[\frac{z}{1-z^2}\right], \frac{1}{2}\arg\left[\frac{1+z}{1-z}\right], \operatorname{Im}\left[\frac{z^2}{1-z^2}\right]\right) : z \in U\right\}.$$

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