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TRANSVERSALS IN PERMUTATION GROUPS AND FACTORISATIONS OF COMPLETE GRAPHS

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Let G be a transitive permutation group acting on a finite set of order n. We discuss certain types of transversals for a point stabiliser A in G: *free transversals* and *global transversals*. We give sufficient conditions for the existence of such transversals, and show the connection between these transversals and combinatorial problems of decomposing the complete directed graph K_n^* into edge disjoint cycles. In particular, we classify all the inner-transitive Oberwolfach factorisations of the complete directed graph. We mention also a connection to Frobenius theorem.

1. INTRODUCTION

Let G be a transitive permutation group acting on a set X and let A be a point stabiliser. We define a *free transversal* for A in G to be a right transversal T containing 1 such that all the elements of $T - \{1\}$ are fixed-point-free. We define a *global transversal* in G to be a set T which is a right transversal for *all* the point stabilisers of G. It is easily proved that a global transversal containing 1 must be a free transversal. Such a transversal will be called a *free global transversal.*

Free and global transversals do not always exist, and one of our aims in this paper is to find conditions for their existence. Global transversals in a transitive permutation group of degree n are closely related to edge disjoint factorisations of K_n^*, the complete directed graph on n vertices. The corresponding definition is the following:

DEFINITION 1: Let K_n^* be the complete directed graph on n vertices (that is, every pair of vertices of K_n^* is connected by two arcs (directed edges) of opposite directions). A *factor* of K_n^* is a spanning subgraph of K_n^*. A *factorisation* $\{F_1, F_2, \ldots, F_k\}$ of K_n^* is a partition of the arc set of K_n^* into arc disjoint factors $F_1, F_2, \ldots, F_k$. An F*-factorisation* of K_n^* is a factorisation of K_n^* all whose factors are isomorphic to the factor F.

We shall be interested in factorisations of K_n^* whose factors are vertex disjoint union of cycles (we shall include the possibility of 2-cycles (double edges)). Of particular interest will be the *Oberwolfach factorisation*, which is an F-factorisation where F is a vertex disjoint union of cycles. For a brief review on graph facorisations see **[1]** and **[2]**.

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In this paper we shall use a natural correspondence between certain subgraphs of K_n^* and permutations in the symmetric group S_n. More precisely, let F be a spanning subgraph of K_n^* such that F is a vertex disjoint union of cycles (1-cycles, that is, isolated vertices, and 2-cycles are permitted). Then in a natural way there corresponds to F a permutation f on the vertices of K_n^*: the cycle decomposition of f is induced by the cycles in F (that is, we have $i^f = j$ if and only if (i,j) is an arc of F). When we fix a labeling of the vertices of K_n^* by the numbers $1, 2, \dots, n$, then we can consider f as a permutation on $\{1, 2, \dots, n\}$, that is, $f \in S_n$. The following lemma shows the connection between free global transversals and factorisations of K_n^* (its simple proof is omitted).

Lemma 1.1. *Let $F_1, F_2, \dots, F_{n-1}$ be factors of K_n^*, each consisting of a vertex disjoint union of cycles of length at least 2. Let $f_1, f_2, \dots, f_{n-1} \in S_n$ be the corresponding permutations. Then*

(i) *$\{F_1, \dots, F_{n-1}\}$ is a factorisation of K_n^* if and only if the set $\{1\} \cup \{f_1, \dots, f_{n-1}\}$ is a free global transversal in S_n;*

(ii) *$\{F_1, \dots, F_{n-1}\}$ is an Oberwolfach factorisation of K_n^* if and only if the set $\{1\} \cup \{f_1, \dots, f_{n-1}\}$ is a free global transversal in S_n and all the f_is belong to the same conjugacy class of S_n.*

Suppose that, for a given Oberwolfach factorisation α of K_n^*, there exists a permutation group G on the vertices of K_n^* which acts transitively on the factor set α. In this case we say that α is a G-transitive factorisation. Let F_1 and F_2 be two factors in α with corresponding permutations $f_1, f_2 \in S_n$, and let $g \in G$. Then one can verify that g sends F_1 to F_2 if and only if the equality $f_1^g = f_2$ holds in S_n.

A special type of G-regular Oberwolfach factorisations of K_n^*, in which $|G| = n - 1$ and G is regular on α, was investigated in [**12**]. It was shown in [**12**] that this type of factorisations is connected to the concept of sequenceable groups. We define now another special type of transitive Oberwolfach factorisations. In this definition, the induced permutations $f_1, f_2, \dots, f_{n-1}$, besides being members of the same conjugacy class of S_n, are also members of the same conjugacy class of the automorphism group of the factorisation.

Definition 2: Let $\alpha = \{F_1, F_2, \dots, F_{n-1}\}$ be an Oberwolfach factorisation of K_n^*, and let $f_1, f_2, \dots, f_{n-1} \in S_n$ be the corresponding permutations. We shall say that α is *inner-transitive* with a corresponding group G, $G \leqslant S_n$, if G acts transitively on α and $f_1, f_2, \dots, f_{n-1} \in G$.

Notice that $\alpha = \{F_1, \dots, F_{n-1}\}$ is inner-transitive with a corresponding group G, if and only if $\{1\} \cup \{f_1, \dots, f_{n-1}\}$ is a free global transversal in G and $\{f_1, \dots, f_{n-1}\}$ is a conjugacy class of G. Thus the problem of finding inner-transitive Oberwolfach factorisations of K_n^* is equivalent to the problem of finding transitive groups G of degree n, with a free global transversal T, such that $T - \{1\}$ is a conjugacy class of G. Using the classification of the finite simple groups, we prove the following.

THEOREM C. *Let G be a transitive permutation group of degree n and let T be a right transversal of a point stabiliser A, such that $1 \in T$. Then $T - \{1\}$ is a conjugacy class of G if and only if $n = p^m$, p a prime, G is 2-transitive, and T is a regular elementary Abelian normal subgroup of G.*

This theorem enables us to classify all the inner-transitive Oberwolfach factorisations.

COROLLARY C1. *An Oberwolfach factorisation $\alpha = \{F_1, \ldots, F_{n-1}\}$ of K_n^* is inner-transitive if and only if n is a power of a prime p and the set of permutations $\{1\} \cup \{f_1, \ldots, f_{n-1}\}$ is an elementary Abelian p-group.*

PROOF: The *only if* part is a direct consequence of Theorem C. For the *if* part, suppose n is a prime power and $\{1\} \cup \{f_1, \ldots, f_{n-1}\}$ is an elementary Abelian group. Then clearly there exists a 2-transitive permutation group G of degree n in which $\{1\} \cup \{f_1, \ldots, f_{n-1}\}$ is a regular normal subgroup (for instance, a Frobenius group of order $n(n-1)$). Thus this part also follows by Theorem C. □

Another interesting case in transitive permutation groups arises when a transversal T of a point stabiliser is a conjugacy class of G (in this case T is a global transversal which is not free). This case was treated recently in **[14]** and **[15]**. We shall discuss it briefly at the end of Section 4.

Let G be a transitive permutation group on a set X. When a free transversal exists in G, we shall say that G is *freely transitive.* A connection between this concept and the well known Frobenius theorem is discussed in Section 2. In Section 2 we discuss also the basic properties of freely transitive groups. We show that solvable groups and simple groups may not be freely transitive. In Section 3 we prove that any supersolvable transitive permutation group is freely transitive. Furthermore, in nilpotent transitive groups there always exists a free global transversal. Thus it follows that in any transitive permutation group of prime power degree there exists a free global transversal. In Section 4 we return to inner-transitive Oberwolfach factorisations and prove Theorem C above.

2. FREELY TRANSITIVE GROUPS AND THEIR BASIC PROPERTIES

Let G be a transitive permutation group on a set X. An ordered pair (x, y) of different points from X is called a *free* pair, if there exists a fixed-point-free element $g \in G$ such that $x^g = y$. Denote the point stabiliser G_x by A and let $y \in X$, $y \neq x$. For an element $u \in G$ such that $x^u = y$, it holds that the coset Au is the set of all $g \in G$ such that $x^g = y$. Thus the pair (x, y) is free if and only if Au contains a fixed-point-free element.

It follows that the existence of a free transversal for A in G is equivalent to the condition that for each $y \in X$, $y \neq x$, the pair (x, y) is free. Now since G is transitive, for every distinct points $z, w \in X$ there exist $h \in G$ and $y \in X$, $y \neq x$ such that

$(x^h, y^h) = (z, w)$. Then $x^g = y$ if and only if $z^{h^{-1}gh} = w$; hence (x, y) is a free pair if and only if (z, w) is a free pair. From this it follows that the following conditions are equivalent for G:

(i) Each ordered pair of distinct points from X is free;

(ii) A point stabiliser has a free transversal in G.

A transitive permutation group satisfying these conditions will be called in this paper a *freely transitive group*. A trivial group is considered as freely transitive. Let G be a transitive permutation group with a point stabiliser A. Then G is freely transitive if and only if besides A, no right coset of A is contained in $\bigcup_{h \in G} A^h$.

Throughout the rest of the paper, *all groups are finite*.

The following definition is related to the concept of free transitivity. Let G be an abstract group with a subgroup A. We shall say that A is *closed* in G if A is a maximal subgroup in the set $\bigcup_{g \in G} A^g$. That is, if for any subgroup B such that $B > A$ it holds that $B \not\subseteq \bigcup_{g \in G} A^g$. Notice that maximal subgroups and normal subgroups are always closed. Now let G be a transitive permutation group with a point stabiliser A, and assume that A is not closed in G. Let $B > A$ satisfy $B \subseteq \bigcup_{g \in G} A^g$. Then for $b \in B - A$ we have $Ab \subseteq \bigcup_{g \in G} A^g$, yielding that G is not freely transitive.

For example, let P be an elementary Abelian p-group, p a prime, such that $|P| > p$, and let $H = \mathrm{Aut}(P)$. Let $G = [P]H$ be the natural semidirect product and let $A < P$, $|A| = p$. Then $\bigcup_{g \in G} A^g = P$, whence A is not closed in G. If we consider G as a transitive permutation group on the right cosets of A, then we obtain a transitive group which is not freely transitive.

The last example belongs to a wide family of examples, as follows: Let K be a group with a subgroup $A < K$ such that $K = \bigcup_{g \in \mathrm{Aut}(K)} A^g$. Such groups K were researched by Brandl [5], and were called there **-groups*. Let $G = [K]\,\mathrm{Aut}(K)$, the natural semidirect product, then clearly A is not closed in G. Furthermore, if the centre of K is trivial then we can consider K as a subgroup of $\mathrm{Aut}(K)$, and then A is not closed in $\mathrm{Aut}(K)$.

Occasionally, we can obtain examples for non-freely transitive groups by using a proper subgroup of $\mathrm{Aut}(K)$. For instance, let P be elementary Abelian as before, and let $h \in \mathrm{Aut}(K)$ be an element of order $|P| - 1$ acting transitively on $P - \{1\}$ (a Singer cycle). Let $G = [P]\langle h \rangle$ and let $A < P$, $|A| = p$. Then A is not closed in G. Considering G as a transitive group on the right cosets of A, we obtain a transitive solvable group which is not freely transitive.

We notice that there exist simple transitive groups which are non-freely transitive. For example, let p be a prime and let P be an elementary Abelian group of order p^m, $m \geqslant 2$. We can embed P as a regular subgroup in the symmetric group $S = S_{p^m}$.

Note that the normaliser $N_S(P)$ is isomorphic to the semidirect product $[P]\operatorname{Aut}(P)$ (the holomorph of P; see [8, Exercise 2.5.6 on p. 45]. Let A be a subgroup of P with order p, then we saw above that A is not closed in $N_S(P)$. We can embed S in the alternating group $G = A_{p^m+2}$, by identifying S with the group of all even permutations in $S_{p^m} \times S_2$. Hence it follows that A is a non-closed subgroup of G. Now G is simple, and as a transitive permutation group on the right cosets of A, it is not freely transitive.

Frobenius permutation groups are clearly freely transitive, since they contain a regular subgroup, by Frobenius theorem. An interesting observation concerning the concept of free transitivity is the following

Assertion 1. *Let G be a Frobenius group, and assume that we have an elementary (that is, character free) proof that G is freely transitive. Then we can prove in an elementary way the validity of Frobenius theorem for G, that is, that the set of fixed-point-free elements with the identity is a subgroup of G.*

Proof: Let n be the degree of G and let A be a point stabiliser. Then $[G : A] = n$. We know that $A \cap A^g = 1$ for every $g \in G - A$. An elementary computation shows that the set of fixed-point-free elements, that is, the set $S = G - \bigcup_{g\in G} A^g$, has order $n - 1$. Since G is freely transitive, each coset Ag different from A contains an element from S, yielding that $S \cup \{1\}$ is a right transversal for A. By the same reasoning, $S \cup \{1\}$ is a right transversal for A^g, for each $g \in G$. It follows that every $u, v \in S \cup \{1\}$ satisfy $uv^{-1} \in S \cup \{1\}$. Hence $S \cup \{1\}$ is a subgroup of G. ∎

We do not know whether there exist primitive non-freely transitive groups. If such groups fail to exist, the proof for that is, not expected to be easy. The reason is the following assertion.

Assertion 2. *Assume there exists an elementary (that is, character free) proof that all primitive groups are freely transitive. Then there exists an elementary proof for Frobenius theorem.*

Proof: Let G be a Frobenius permutation group and let $A = G_x$, the stabiliser of the point x. By Assertion 1 we may assume that G is not primitive. Then we may choose a subgroup $L \leqslant G$ such that $A < L$ and A is maximal in L. Now L, as a permutation group on the orbit x^L, is clearly a Frobenius permutation group. Furthermore L is primitive and so freely transitive. Thus we can prove in an elementary way (by Assertion 1) that the set of fixed-point-free elements of L with the identity is a subgroup of L. This subgroup is normal in L and so normalised by A. Frobenius theorem follows now for G in an elementary way, by [9, Lemma 2.2 (iv)]. ∎

We introduce now some basic results on freely transitive groups. These results will be used later.

Basic fact 2.1. Let G be a transitive permutation group with a transitive subgroup H. Then

(i) A free transversal in H is also a free transversal in G. Thus, if H is freely transitive then also G is freely transitive.

(ii) A free global transversal in H is also a free global transversal in G.

BASIC FACT 2.2. Let G be a transitive permutation group with a regular subgroup R. Then R is a free global transversal, and in particular G is freely transitive.

Since a solvable primitive group contains a regular normal subgroup (see [**16**, Theorem 11.5]), it follows that such a group has a free global transversal.

Some parts of the next item were explained above.

BASIC FACTS 2.3. Let G be a transitive permutation group acting on a set X, let $x \in X$ and let $A = G_x$, the stabiliser of x.

1. Let $y \in X$, $y \neq x$, and let $g \in G$ satisfy $x^g = y$. Then the following are equivalent:
   (i) (x, y) is a free pair;
   (ii) Ag contains a fixed-point-free element;
   (iii) Ag is not contained in $\bigcup_{h \in G} A^h$.

2. Let $y \in X$, $y \neq x$, and let $k \in G$. Then (x, y) is free if and only if (x^k, y^k) is free. In particular, (x, y) is free if and only if, for each $a \in A$, (x, y^a) is free.

3. (Follows from 2.) G is freely transitive if and only if for each $y \in X - \{x\}$ the pair (x, y) is free.

4. (Follows from 3.) G is freely transitive if and only if, besides A, no right coset of A is contained in $\bigcup_{h \in G} A^h$.

5. (Follows from 1, 2 and 4.) G is freely transitive if and only if, besides A, no double coset AkA of A is contained in $\bigcup_{h \in G} A^h$.

COROLLARY 2.4. *Any 2-transitive permutation group is freely transitive.*

PROOF: Let G be 2-transitive and let A be a point stabiliser. Then there exists exactly one double coset of A which is different from A, and since $G \neq \bigcup_{h \in G} A^h$, the result follows from Basic fact 2.3.5. ∎

Consider a transitive group of prime degree. It is primitive, and by [**16**, Theorem 11.7], it is either solvable or 2-transitive. Thus, by the remark after Basic fact 2.2, and by Corollary 2.4, such a group must be freely transitive (we shall prove more in Section 3, Corollary B2).

In the following proposition we describe a simple condition ensuring that a transitive group is freely transitive. However, this is clearly not a necessary condition.

PROPOSITION 2.5. *Let G be a transitive permutation group which contains a transposition (that is, a 2-cycle). Then G is freely transitive.*

PROOF: Let X be the set on which G acts and let $x \in X$. Suppose on the contrary that G is not freely transitive. Then (Basic fact 2.3.3) there exists $y \in X - \{x\}$ such

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that if $g \in G$ and $x^g = y$ then g is not fixed-point-free. Choose an element $h \in G$ with minimal number of fixed points such that $x^h = y$. Let z be a fixed point of h. Then, since G is transitive, G contains a transposition of the form (z, w), where $w \in X$. Denote this transposition by t. Now, if $w \neq x$ then $x^{th} = y$, and th fixes less points than h, since z and w are not fixed by th. This implies a contradiction. If $w \neq y$ then $x^{ht} = y$ and ht fixes less points than h (by a similar argument), which again implies a contradiction. Thus the proof is completed. ∎

3. Supersolvable groups, nilpotent groups and groups of prime power degree

We have the following result on supersolvable transitive permutation groups. As the examples in Section 2 show, this result can not be extended to solvable transitive groups.

Theorem A. *All supersolvable transitive permutation groups are freely transitive.*

Proof: Let $G \neq 1$ be a supersolvable transitive permutation group acting on a set X. We apply induction on the degree of G. Since G is supersolvable, there exists $N \trianglelefteq G$, a minimal normal subgroup of G, such that N is cyclic of prime order p. If N is transitive on X then it is regular and the assertion is true by Basic fact 2.2. Assume then that N is intransitive, and let $X_1, X_2, \ldots, X_k$ (here $|X| > k > 1$) be all the N-orbits on X. Then the X_is form a system of imprimitivity blocks. Fix $x \in X_1$ and let $y \in X$, $y \neq x$. By Basic fact 2.3.3, it suffices to show the existence of a fixed-point-free element $g \in G$ such that $x^g = y$.

Case (i). $y \in X_j$, $j \neq 1$. Consider the transitive action of G on the block system $X^* = \{X_1, X_2, \ldots, X_k\}$. Let M be the kernel of this action (clearly $M \geqslant N$). Then $G^* = G/M$ is a supersolvable transitive permutation group on X^*. By the induction hypothesis G^* is freely transitive on X^*. Thus there exists a fixed-point-free element $h^* \in G^*$ satisfying $X_1^{h^*} = X_j$. Let $h \in G$ be in the inverse image of h^*, then $X_1^h = X_j$ and $X_i^h \neq X_i$ for each i. Since $x^h \in X_j$, there exists $u \in N$ such that $x^{hu} = y$. We have $X_i^{hu} = X_i^h \neq X_i$ for each i, yielding that hu is a fixed-point-free element of G as required.

Case (ii). $y \in X_1$. Then there exists $v \in N$ satisfying $x^v = y$. Furthermore, since $|N| = p$, the action of N on each of the X_is is faithful and regular. Hence v is fixed-point-free on X_i for each i, yielding that v is a fixed-point-free element of G as required. ∎

The following corollary follows from Theorem A and Basic fact 2.1 (i).

Corollary A1: *Every transitive group with a supersolvable transitive subgroup is freely transitive.*

When we consider nilpotent transitive permutation groups, we have a stronger result, as follows.

THEOREM B. *All nilpotent transitive permutation groups have a free global transversal.*

PROOF: Let $G \neq 1$ be a nilpotent transitive permutation group with a point stabiliser A and apply induction on $[G : A]$, the degree of G. Let N be a minimal normal subgroup of G, then N is contained in the centre of G and $A \cap N = 1$. Denote $N = \{n_1 = 1, n_2, \dots, n_p\}$ (this is, of course, a right transversal for A in AN). Consider now the transitive action of G on the cosets of AN in G. Let K be the kernel of this action, then G/K is a nilpotent transitive permutation group of degree $[G : AN]$ and by induction it has a free global transversal. It follows that there exists a set T of elements of G such that $1 \in T$ and T is a right transversal in G for all the conjugates $(AN)^h$, $h \in G$. Denote $T = \{g_1 = 1, g_2, \dots, g_s\}$. Let $U = \{n_i g_j \mid 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant s\}$. This is a right transversal for A in G which contains 1, and it remains to show that it is global.

Suppose that $n_i g_j (n_k g_l)^{-1} \in A^h$ for some $h \in G$. Then (recall that N is central) $g_j g_l^{-1} n_i n_k^{-1} \in A^h$, yielding $g_j g_l^{-1} \in (AN)^h$ and so $j = l$. Thus $n_i n_k^{-1} \in A^h$, yielding $n_i n_k^{-1} \in A$ and $i = k$. It follows that U is a right transversal for A^h in G, for each $h \in G$. Thus it is a global transversal as required. ∎

The following is immediate by Basic fact 2.1 (ii).

COROLLARY B1. *Let G be a transitive permutation group with a nilpotent transitive subgroup. Then there exists a free global transversal in G.*

COROLLARY B2. *Let G be a transitive permutation group of degree p^m, p a prime. Then there exists a free global transversal in G.*

PROOF: Immediate by Corollary B1, since a Sylow p-subgroup of G is transitive. ∎

Call an abstract group G *a global group* if for every subgroup $A \leqslant G$ there exists a right transversal T in G, such that T is a right transversal for all the conjugates A^g in G. Thus G is a global group if and only if in every transitive permutation representation of G there exists a global transversal. Theorem B above shows that if G is nilpotent then it is global. The converse is not true, since if G is a group satisfying that $|G|$ is a square-free number, then clearly G is global (a subgroup of G is a Hall π-subgroup for a set of primes π, and then a Hall π'-subgroup is the required right transversal). However we have the following result.

PROPOSITION 3.1. *Any global group is solvable.*

PROOF: Suppose G is a global group. It is easily proved that any subgroup and any quotient of G are also global. Thus, by applying induction on $|G|$, it suffices to show that G is not a simple non-Abelian group. If $|G|$ is odd then by Feit-Thompson theorem G is solvable. Assume then that $|G|$ is even and let $u \in G$ be an involution. We may assume that u is not in the centre of G. Since G is global, there exists a set T which is a right transversal for all the conjugates $\langle u^g \rangle$ in G. Notice that for any $g \in G$, $u^g T = G - T$, whence for any $g, h \in G$ we have $u^g T = u^h T$ and $u^g u^h T = T$. Following [4], we denote

$\mathrm{Ker}(T) = \{k \in G \mid kT = T\}$. This is a proper subgroup of G (whose order divides $|T|$), and it contains the set $S = \{u^g u^h \mid g, h \in G\}$. Obviously the set S contains a non-trivial conjugacy class of G, which implies that $\mathrm{Ker}(T)$ contains a non-trivial normal subgroup of G, as required. ∎

4. Inner-transitive Oberwolfach factorisations

We return now to Theorem C and give its proof.

Proof of Theorem C: For the *if* part of the theorem, suppose that G is 2-transitive and T is a regular normal subgroup as described. Let x be the point stabilised by A, then the action of A on the other points is transitive, and it is equivalent to the action of A by conjugation on $T - \{1\}$ (this is a well known property of regular normal subgroups; see, for example, [**16**, Theorem 11.2]). It follows that $T - \{1\}$ is indeed a conjugacy class of G.

For the *only if* part of the theorem, suppose that $D = T - \{1\}$ is a conjugacy class of G. We shall prove first that G is 2-transitive. Let $t \in D$. Then $[G : C_G(t)] = n - 1$ and $[G : A] = n$, whence A and $C_G(t)$ are subgroups with coprime indices in G. Consequently $G = AC_G(t)$. It follows that A acts transitively on D (by conjugation), and so $AtA = AD = G - A$, implying that G is 2-transitive as claimed.

By [**6**, p. 110], there are exactly two possibilities for G, either

(i) G is an affine 2-transitive group, or

(ii) G is an almost simple 2-transitive group.

Suppose (i) holds. Then, by [**6**, p. 194], $n = p^m$, p a prime, and G has a regular elementary Abelian normal subgroup N, $|N| = p^m$. It remains to show that $T = N$. For $t \in D$ the index $[G : C_G(t)]$ is coprime to $|N|$ and so $N \leqslant C_G(t)$ must hold. But since N is Abelian and regular, N is its own centraliser in G (see [**16**, Proposition 4.4]), implying $t \in N$. Thus $T \subseteq N$, and since $|T| = |N|$ we obtain $T = N$.

Suppose (ii) holds and let N denote the socle of G, which is a simple non-Abelian group in this case. Then N, like G, is transitive of degree n. For $t \in D$ we have $\left[N : N \cap C_G(t)\right] = \left[NC_G(t) : C_G(t)\right]$ and so $\left[N : N \cap C_G(t)\right]$ divides $n - 1$. Notice however that $\left[N : N \cap C_G(t)\right] > 1$, since t, as a non-identity element of the almost simple group G, does not centralise N. Now, by the classification of the finite simple groups, it is known (see [**6**, p. 196]) that N itself is 2-transitive, except one case in which $n = 28$ and N is isomorphic to $PSL(2,8)$. However in this case N does not have a proper subgroup with index dividing 27, by [**11**].

We consider now all the other cases (see [**6**, Table 7.4], or [**8**, Section 7.7]). In these cases N is a simple non-Abelian 2-transitive group with degree n. We shall show that there does not exist $t \in G$ such that $\left[N : N \cap C_G(t)\right] > 1$ and $\left[N : N \cap C_G(t)\right]$ divides $n - 1$.

1. $N = A_n$, the alternating group on n letters, $n \geqslant 5$. Then N does not have a proper subgroup of index less than n, since N can not be embedded in S_{n-1}.

In Cases 2 and 3 below N does not have a proper subgroup with index dividing $n-1$, by [**13**, Table 5.2.A on p. 175].

2. $N = PSL(m,q)$ for a prime power q, $n = (q^m-1)/(q-1)$, and $(m,q) \neq (2,5),(2,7),(2,9),(4,2),(2,11)$. These five exceptional cases are eliminated by verifying that there does not exist $K < N$ such that $|N:K|$ divides $n-1$ and $K = N \cap C_G(t)$ for some $t \in G$. For instance, $PSL(2,11)$ has two 2-transitive actions: on $n = 11$ and on $n = 12$ points. The case $n = 11$ is excluded since $PSL(2,11)$ has no proper subgroup of index less then 11 ([**7**]). It remains to check the case $n = 12$. We have $N = PSL(2,11) \leqslant G \leqslant PGL(2,11)$, and $PSL(2,11)$ has a subgroup H of index 11, which is isomorphic to A_5 (it is the stabiliser of a point in the action on $n = 11$ points). Since no non-trivial element in $PGL(2,11)$ centralises H, this case is excluded.

3. $N = Sp(2m,2)$, where $m \geqslant 3$, and $n = 2^{m-1}(2^m-1)$.

Another case in which $Sp(2m,2)$ is involved is as follows.

4. $N = Sp(2m,2)$, where $m \geqslant 3$, has another 2-transitive action, on $n = 2^{m-1}(2^m+1)$ points. Then by [**13**, Theorem 5.2.4 on p. 176], we conclude that any subgroup H whose order divides $n-1$ is contained in a member of the family of subgroups $C(G)$. This family of subgroups is described in [**13**, Table 3.5. Chapter on p. 72]. Checking the rows of this table which are relevant to our case (the underlying field has two elements) implies that the only proper subgroups H of N with index less than n are isomorphic to subgroups of $O_n^+(2)$ or $O_n^-(2)$. But in this case we have that $[G:H]$ is even, while $n-1$ is odd, and in particular $[G:H]$ does not divide $n-1$, as required.

In the following three cases $n-1$ is a prime power, and N does not have a proper subgroup with index dividing $n-1$, by [**11**].

5. $N = PSU(3,q)$, where q is a prime power, $q \geqslant 3$, and $n = q^3+1$.

6. $N = Sz(q)$, where $q = 2^{2m+1} > 2$ and $n = q^2+1$.

7. $N = R_1(q)$, where $q = 3^{2m+1} > 3$ and $n = q^3+1$.

Further cases:

8. $N = A_7$, $n = 15$. In this case $G = N$ (see [**6**, Table 7.4, on p. 197]. Clearly G does not have an element $t \neq 1$ such that $[G : C_G(t)]$ divides 14.

9. The remaining possibilies are when N is one of the sporadic groups M_{11} (two 2-transitive actions, on $n = 11$ and $n = 12$ points), M_{12}, M_{22}, M_{23}, M_{24}, Co_3, and HS. By [**7**] we have that the only case where N has a proper subgroup of index dividing $n-1$ is $N = M_{11}$ where $n = 12$. In this case N has the subgroup M_{10} (which is the stabiliser of a point in the action of M_{11} on $n = 11$ points) whose index in N is 11. Since all the automorphisms of $N = M_{11}$ are inner, $G = N$ in this case. Thus we only have to check that M_{10} is not a centraliser of an element of N. This is true since the centre of M_{10} is trivial.

We checked all the cases, and so the proof is completed. □

We end this section with a note on a related subject. Theorem C deals with the case when a transversal T of a subgroup A of G satisfies that $T - \{1\}$ is a conjugacy class of G. Another case of interest is when a transversal T of a subgroup A of G is a conjugacy class of G. This case was treated recently in [**14**] and [**15**]. One can easily verify that in this case each conjugate of A contains exactly one element of T. Furthermore, each $t \in T$ lies in the centre of a unique conjugate of A. Considering the transitive action of G on the right cosets of A, we obtain a transitive permutation group isomorphic to a quotient of G. The image of T in this quotient is a global transversal in which each element has exactly one fixed point. Particular cases of the above appear in the celebrated Z^*-Theorem of Glauberman [**10**], and in an extension of it proved by Artemovich [**3**] using the classification of the finite simple groups. Stein [**15**] proved, using the classification of the finite simple groups, that in the situation above the group generated by the conjugacy class T is solvable. He gave an application of this result to quasigroups ([**15**, Theorem 1.4]). Similarly to the free global transvesrals discussed in the current paper, a transversal which is a conjugacy class is related to (extended) Oberwolfach factorisations (in which each factor contains, apart from the non-trivial cycles, an isolated vertex). A special case of such factorisations is the class of inner-transitive Hering configurations studied in [**14**].

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