Compositio Math. 142 (2006) 1263–1285
doi:10.1112/S0010437X06002302

Gromov–Witten theory and Donaldson–Thomas theory, I

D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande

Abstract

We conjecture an equivalence between the Gromov–Witten theory of 3-folds and the holomorphic Chern–Simons theory of Donaldson and Thomas. For Calabi–Yau 3-folds, the equivalence is defined by the change of variables $e^{iu} = -q$, where u is the genus parameter of Gromov–Witten theory and q is the Euler characteristic parameter of Donaldson–Thomas theory. The conjecture is proven for local Calabi–Yau toric surfaces.

1. Introduction

1.1 Overview

Let X be a nonsingular, projective, Calabi–Yau 3-fold. Gromov–Witten theory concerns counts of maps of curves to X. The counts are defined in terms of a canonical 0-dimensional virtual fundamental class on the moduli space of maps. The discrete invariants of a map are the genus g of the domain and the degree $\beta \in H_2(X, \mathbb{Z})$ of the image. For every g and β, the Gromov–Witten (GW) invariant is the virtual number of genus g, degree β maps. We sum the contributions of all genera with weight u^{2g-2}, where u is a parameter.

The Gromov–Witten invariants have long been expected to be expressible in terms of appropriate curve counts in the target X. A curve in X corresponds to an ideal sheaf on X. The discrete invariants of the ideal sheaf are the holomorphic Euler characteristic χ and the fundamental class $\beta \in H_2(X, \mathbb{Z})$ of the associated curve. Donaldson and Thomas have constructed a canonical 0-dimensional virtual fundamental class on the moduli space of ideal sheaves on X. For every χ and β, the Donaldson–Thomas (DT) invariant is the virtual number of the corresponding ideal sheaves. We sum the contributions over χ with weight q^{χ}, where q is a parameter.

We present here a precise mathematical conjecture relating the Gromov–Witten and Donaldson–Thomas theories of X. Our conjecture is motivated by the description of Gromov–Witten theory via crystals in [ORV03]. A connection between Gromov–Witten theory and integration over the moduli space of ideal sheaves is strongly suggested there. A related physical conjecture is formulated in [INOV03].

Conjecture. The change of variables $e^{iu} = -q$ equates the Gromov–Witten and Donaldson–Thomas theories of X.

The moduli of maps and sheaves have been related previously by the Gopakumar–Vafa conjecture equating Gromov–Witten invariants to BPS state counts determined by the *cohomology* of the moduli of sheaves [GV98a, GV98b]. The Gopakumar–Vafa conjecture has been verified in several cases and has been a significant source of motivation. However, there have been difficulties on the

Received 5 January 2005, accepted in final form 18 April 2006.
2000 Mathematics Subject Classification 14N35, 14H81.
Keywords: Gromov–Witten, Donaldson maps, sheaves.

mathematical side in selecting an appropriate cohomology theory for the singular moduli of sheaves which arise (see [HST99]).

Donaldson–Thomas theory concerns *integration* over the moduli of sheaves. The subject was defined, along with a construction of the virtual class, by Donaldson and Thomas in [DT98, Tho00] with motivation from several sources (see [AOS97, BKS97, Wit92]). As the Donaldson–Thomas invariant is similar to the Euler characteristic of the moduli of sheaves, a philosophical connection between Gromov–Witten invariants and the cohomology of the moduli of sheaves is implicit in our work. However, the Donaldson–Thomas invariant is *not* the Euler characteristic.

As evidence for our conjecture, we present a proof in the toric local Calabi–Yau case via the virtual localization formula for Donaldson–Thomas theory. The proof depends upon evaluations of the topological vertex on the Gromov–Witten side.

1.2 General 3-folds

We believe the Gromov–Witten/Donaldson–Thomas correspondence holds for all 3-folds. Donaldson–Thomas theory has a natural supply of observables constructed from the Chern classes of universal sheaves. These Chern classes should correspond to insertions in Gromov–Witten theory (see [LMN03]). The degree 0 case, where no insertions are required, is discussed in § 2 below. For primary fields, a complete GW/DT correspondence for all 3-folds is conjectured in [MNOP06]. For descendent fields, the correspondence is precisely formulated for the descendents of a point in [MNOP06].

We conjecture that the *equivariant vertex*, discussed in § 4.9 below, has the same relation to general cubic Hodge integrals as the topological vertex does to Calabi–Yau Hodge integrals [DF03]. A closely related issue is the precise formulation of the GW/DT correspondence for *all* descendent fields. We will pursue the topic in a future paper.

1.3 Gromov–Witten theory

Gromov–Witten theory is defined via integration over the moduli space of stable maps. Let X be a nonsingular, projective, Calabi–Yau 3-fold. Let $\overline{M}_g(X,\beta)$ denote the moduli space of stable maps from connected genus g curves to X representing the class $\beta \in H_2(X,\mathbb{Z})$, and let

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1$$

denote the corresponding Gromov–Witten invariant. Foundational aspects of the theory are treated, for example, in [Beh97, BF97, LT98].

Let $\mathsf{F}'_{\mathrm{GW}}(X;u,v)$ denote the reduced Gromov–Witten potential of X,

$$\mathsf{F}'_{\mathrm{GW}}(X;u,v) = \sum_{\beta\neq 0}\sum_{g\geqslant 0} N_{g,\beta}\ u^{2g-2}v^{\beta},$$

omitting the constant maps. The reduced partition function,

$$\mathsf{Z}'_{\mathrm{GW}}(X;u,v) = \exp \mathsf{F}'_{\mathrm{GW}}(X;u,v),$$

generates disconnected Gromov–Witten invariants of X excluding constant contributions.

Let $\mathsf{Z}'_{\mathrm{GW}}(X;u)_\beta$ denote the reduced partition function of degree β invariants,

$$\mathsf{Z}'_{\mathrm{GW}}(X;u,v) = 1 + \sum_{\beta\neq 0} \mathsf{Z}'_{\mathrm{GW}}(X;u)_\beta v^{\beta}.$$

1.4 Donaldson–Thomas theory

Donaldson–Thomas theory is defined via integration over the moduli space of ideal sheaves. Let X be a nonsingular, projective, Calabi–Yau 3-fold. An *ideal sheaf* is a torsion-free sheaf of rank 1 with trivial determinant. Each ideal sheaf $\mathcal{I}$ injects into its double dual,

$$0 \to \mathcal{I} \to \mathcal{I}^{\vee\vee}.$$

As $\mathcal{I}^{\vee\vee}$ is reflexive of rank 1 with trivial determinant,

$$\mathcal{I}^{\vee\vee} \cong \mathcal{O}_X$$

(see [OSS80]). Each ideal sheaf $\mathcal{I}$ determines a subscheme $Y \subset X$,

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

We will consider only ideal sheaves of subschemes Y with components of dimension at most 1. The dimension 1 components of Y (weighted by their intrinsic multiplicities) determine an element

$$[Y] \in H_2(X, \mathbb{Z}).$$

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$\chi(\mathcal{O}_Y) = n$$

and

$$[Y] = \beta \in H_2(X, \mathbb{Z}).$$

Here, χ denotes the holomorphic Euler characteristic. The moduli space $I_n(X, \beta)$ is isomorphic to the Hilbert scheme of curves of X (see [MP06]).

The Donaldson–Thomas invariant is defined via integration against the dimension 0 virtual class,

$$\tilde{N}_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1.$$

Foundational aspects of the theory are treated in [MP06, Tho00].

Let $\mathsf{Z}_{\mathrm{DT}}(X; q, v)$ be the partition function of the Donaldson–Thomas theory of X,

$$\mathsf{Z}_{\mathrm{DT}}(X; q, v) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{n \in \mathbb{Z}} \tilde{N}_{n,\beta}\, q^n v^\beta.$$

An elementary verification shows that, for fixed β, the invariant $\tilde{N}_{n,\beta}$ vanishes for sufficiently negative n since the corresponding moduli spaces of ideal sheaves are empty.

The degree 0 moduli space $I_n(X, 0)$ is isomorphic to the Hilbert scheme of n points on X. The degree 0 partition function,

$$\mathsf{Z}_{\mathrm{DT}}(X; q)_0 = \sum_{n \geqslant 0} \tilde{N}_{n,0}\, q^n,$$

plays a special role in the theory. The McMahon function

$$M(q) = \prod_{n \geqslant 1} \frac{1}{(1 - q^n)^n}$$

is the generating series for 3-dimensional partitions (see [Sta99]).

CONJECTURE 1. The degree 0 partition function is determined by

$$\mathsf{Z}_{\mathrm{DT}}(X; q)_0 = M(-q)^{\chi(X)},$$

where $\chi(X)$ is the topological Euler characteristic.

The reduced partition function $\mathsf{Z}'_{\mathrm{DT}}(X;q,v)$ is defined by dividing by the degree 0 function,

$$\mathsf{Z}'_{\mathrm{DT}}(X;q,v) = \mathsf{Z}_{\mathrm{DT}}(X;q,v)/\mathsf{Z}_{\mathrm{DT}}(X;q)_0.$$

Let $\mathsf{Z}'_{\mathrm{DT}}(X;q)_\beta$ denote the reduced partition function of degree $\beta \neq 0$ invariants,

$$\mathsf{Z}'_{\mathrm{DT}}(X;q,v) = 1 + \sum_{\beta\neq 0} \mathsf{Z}'_{\mathrm{DT}}(X;q)_\beta\, v^\beta.$$

Conjecture 2. The reduced series $\mathsf{Z}'_{\mathrm{DT}}(X;q)_\beta$ is a rational function of q symmetric under the transformation $q \mapsto 1/q$.

We now state our main conjecture relating the Gromov–Witten theory and the Donaldson–Thomas theory of a Calabi–Yau 3-fold X.

Conjecture 3. The change of variables $e^{iu} = -q$ equates the reduced partition functions:

$$\mathsf{Z}'_{\mathrm{GW}}(X;u,v) = \mathsf{Z}'_{\mathrm{DT}}(X;-e^{iu},v).$$

The change of variables in Conjecture 3 is well defined by Conjecture 2. Gromov–Witten theory and Donaldson–Thomas theory may be viewed as expansions of a single partition function at different points. Conjecture 3 can be checked order by order in u and q only if an effective bound on the degree of the rational function in Conjecture 2 is known.

1.5 Integrality

The Gopakumar–Vafa conjecture for Calabi–Yau 3-folds predicts the following form for the reduced Gromov–Witten partition function via the change of variables $e^{iu} = -q$:

$$\mathsf{Z}'_{\mathrm{GW}}(X;u)_\beta = q^r \frac{f(q)}{\prod_{i=1}^{k}(1-(-q)^{s_i})^2}, \quad f \in \mathbb{Z}[q], \quad r \in \mathbb{Z}, \quad s_i > 0.$$

In particular, by the Gopakumar–Vafa conjecture, $\mathsf{Z}'_{\mathrm{GW}}(X;u)_\beta$ defines a series in q with *integer* coefficients.

Conjecture 3 identifies the q series with the reduced partition function of the Donaldson–Thomas theory of X. Integrality of the Donaldson–Thomas invariants holds by construction (as no orbifolds occur). We may refine Conjecture 2 above to fit the form of the Gopakumar–Vafa conjecture.

1.6 Gauge/string dualities

In spirit, our conjecture is similar to a gauge/string duality with Donaldson–Thomas theory on the gauge side and Gromov–Witten theory on the string side. While at present we are not aware of a purely gauge-theoretic interpretation of Donaldson–Thomas theory, there are various indications that such an interpretation should exist. Most importantly, the equivariant vertex measure, which appears in the equivariant localization formula for Donaldson–Thomas theory (see § 4.9), is identical to the equivariant vertex in noncommutative Yang–Mills theory. We plan to investigate the issue further.

The interplay between gauge fields and strings is one of the central themes in modern theoretical and mathematical physics [Pol87]. In particular, the conjectural Chern–Simons/string duality of Gopakumar and Vafa [GV99] was a source of many insights into the Gromov–Witten theory of Calabi–Yau 3-folds. As a culmination of these developments, the *topological vertex* was introduced in [AKMV03]. The topological vertex is a certain explicit function of three partitions λ, μ, ν, and the genus expansion parameter u, which is an elementary building block for constructing the Gromov–Witten invariants of arbitrary local toric Calabi–Yau 3-folds. The gauge/string duality seems to hold in a broader context; see [LMN03] and [Pan02] for evidence in the Fano case.

In [ORV03], the topological vertex was interpreted as counting 3-dimensional partitions π with asymptotics λ, μ, ν along the coordinate axes. The variable $q = e^{iu}$ couples to the volume of the partition π in the enumeration. The global data obtained by gluing such 3-dimensional partitions according to the gluing rules of the topological vertex were observed in [INOV03, ORV03] to correspond naturally to torus invariant ideal sheaves in the target 3-fold X. The main mathematical result of our paper is the identification of the topological vertex expansion with the equivariant localization formula for the Donaldson–Thomas theory of the local Calabi–Yau geometry.

The GW/DT correspondence is conjectured to hold for *all* Calabi–Yau 3-folds. While several motivations for the correspondence came from local Calabi–Yau geometry, new methods of attack will be required to study the full GW/DT correspondence.

A relation between Gromov–Witten theory and gauge theory on the same space X has been observed previously in four (real) dimensions in the context of Seiberg–Witten invariants [Tau00]. There, a deformation of the Seiberg–Witten equations by a 2-form yields solutions concentrated near the zero locus of the 2-form, an embedded curve.

We expect, in our case, that the sheaf-theoretic description of curves will be identified with a deformed version of solutions to some gauge-theoretic problem. An outcome should be a natural method of deriving the Donaldson–Thomas measure. The gauge theory in question is a deformation of the twisted maximally supersymmetric Yang–Mills theory compactified on our 3-fold X. The theory, discussed in [BKS97], has BPS solutions and generalized instantons. The expansion of the super-Yang–Mills action about these solutions gives rise to a quadratic form with bosonic and fermionic determinants which should furnish the required measure [MNS97, MNS98].

In the case $X = \mathbf{C}^3$, the deformation in question is the passage to the noncommutative $\mathbb{R}^6$ (see [NS98, Wit00]). Ordinary gauge theories have typically noncompact moduli spaces of BPS solutions. It is customary in mathematics to compactify these spaces by replacing holomorphic bundles by coherent sheaves. The physical consequences of such a replacement are usually quite interesting and lead to many insights [HM96, KON94, LMNS95]. Sometimes the 'compactified' space is nonempty while the original space is empty. Our problem corresponds to $U(1)$ gauge fields which do not support nontrivial instantons, while the compactified moduli space of instantons is nonempty and coincides with the Hilbert scheme of curves of given topology on X.

2. Degree 0

2.1 Gromov–Witten theory

Let X be a nonsingular, projective 3-fold (not necessarily Calabi–Yau). The degree 0 potential $\mathsf{F}_{\mathrm{GW}}(X;u)_0$ may be separated as

$$\mathsf{F}_{\mathrm{GW}}(X;u)_0 = \mathsf{F}^0_{X,0} + \mathsf{F}^1_{X,0} + \sum_{g\geqslant 2} \mathsf{F}^g_{X,0}.$$

The genus 0 and 1 contributions in degree 0 are not constants; the variables of the classical cohomology appear explicitly. Formulas can be found, for example, in [Pan02].

We will be concerned here with the higher genus terms. For $g \geqslant 2$, a virtual class calculation yields

$$\mathsf{F}^g_{X,0} = (-1)^g \frac{u^{2g-2}}{2} \int_X (c_3(X) - c_1(X)c_2(X)) \cdot \int_{\overline{M}_g} \lambda^3_{g-1},$$

where c_i and λ_i denote the Chern classes of the tangent bundle T_X and the Hodge bundle $\mathbb{E}_g$

respectively. Define the degree 0 partition function of Gromov–Witten theory by

$$\mathsf{Z}_{\mathrm{GW}}(X;u)_0 = \exp\biggl(\sum_{g\geqslant 2} \mathsf{F}^g_{X,0}\biggr).$$

The Hodge integrals which arise have been computed in [FP00],

$$\int_{\overline{M}_g} \lambda^3_{g-1} = \frac{|B_{2g}|}{2g}\frac{|B_{2g-2}|}{2g-2}\frac{1}{(2g-2)!}, \tag{1}$$

where B_{2g} and B_{2g-2} are Bernoulli numbers.

Using the Euler–Maclaurin formula, the asymptotic relation

$$\mathsf{Z}_{\mathrm{GW}}(X;u)_0 \sim M(e^{iu})^{\frac{1}{2}\int_X c_3(X)-c_1(X)c_2(X)} \tag{2}$$

may be derived from (1). The precise meaning of (2) is the following: the logarithms of both sides have identical $o(1)$ tails in their $u \to 0$ asymptotic expansion.

2.2 Donaldson–Thomas theory

We now turn to the degree 0 partition function for the Donaldson–Thomas theory of X. The first issue is the construction of the virtual class in Donaldson–Thomas theory.

In [Tho00], the Donaldson–Thomas theory of X is defined only in the Calabi–Yau and Fano cases. Since the arguments of [Tho00] use only the existence of an anticanonical section on X, the result can be stated in the following form.

Lemma 1. *Let X be a nonsingular, projective 3-fold with*

$$H^0(X, \wedge^3 T_X) \neq 0.$$

Then $I_n(X,\beta)$ carries a canonical perfect obstruction theory.

Under the hypotheses of Lemma 1, the Donaldson–Thomas theory of X is constructed for *higher* rank sheaves as well as the rank 1 case of ideal sheaves. The connection, if any, between Gromov–Witten theory and the higher rank Donaldson–Thomas theories is not clear to us.

A sufficient condition for the construction of the perfect obstruction theory and the virtual class $[I_n(X,\beta)]^{\mathrm{vir}}$ in [Tho00] is the vanishing of traceless $\mathrm{Ext}^3_0(\mathcal{I},\mathcal{I})$ for all $[\mathcal{I}] \in I_n(X,\beta)$. See [Tho00] for the definitions and properties of tracelessness used here.

For simplicity, let us assume the vanishing of the higher cohomology of the structure sheaf,

$$H^i(X, \mathcal{O}_X) = 0, \tag{3}$$

for $i \geqslant 1$. Then, $\mathrm{Ext}_0(\mathcal{I},\mathcal{I})$ equals $\mathrm{Ext}(\mathcal{I},\mathcal{I})$.

Lemma 2. *Let X be a nonsingular, projective, 3-fold satisfying (3). Then,*

$$\mathrm{Ext}^3(\mathcal{I},\mathcal{I}) = 0,$$

for all $[\mathcal{I}] \in I_n(X,\beta)$.

Proof. By Serre duality for Ext,

$$\mathrm{Ext}^3(\mathcal{I},\mathcal{I}) = \mathrm{Ext}^0(\mathcal{I},\mathcal{I}\otimes K_X)^\vee,$$

where K_X denotes the canonical bundle. We must therefore prove that

$$\mathrm{Hom}(\mathcal{I},\mathcal{I}\otimes K_X) = 0.$$

Let $U \subset X$ be the complement of the support of Y. Since $\mathcal{I}$ restricts to $\mathcal{O}_U$ on U,

$$\mathrm{Hom}(\mathcal{I}|_U, \mathcal{I}|_U \otimes K_U) = \Gamma(U, K_U) = H^0(X, K_X).$$

The last equality is obtained from the extension of sections since Y has at most 1-dimensional support. Since $\mathcal{I}$ is torsion-free, the restriction

$$\mathrm{Hom}(\mathcal{I}, \mathcal{I} \otimes K_X) \to \mathrm{Hom}(\mathcal{I}|_U, \mathcal{I}|_U \otimes K_U)$$

is injective. Since $h^0(X, K_X) = h^3(X, \mathcal{O}_X)$, the lemma is proven. □

The proof of the vanishing of $\mathrm{Ext}^3_0(\mathcal{I}, \mathcal{I})$ in the presence of higher cohomology of the structure sheaf is similar [MP06]. Hence, Donaldson–Thomas theory is well defined in rank 1 for *all* 3-folds X, not just the Calabi–Yau and Fano cases.

The virtual dimension of $I_n(X, 0)$ is 0 for general 3-folds X. A simple calculation from the definitions yields the following result.

Lemma 3. *We have* $\tilde{N}_{1,0} = -\int_X c_3(X) - c_1(X)c_2(X)$.

Proof. The moduli space $I_1(X, 0)$ is the nonsingular 3-fold X. The tangent bundle is $\mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I})$, and the obstruction bundle is $\mathrm{Ext}^2_0(\mathcal{I}, \mathcal{I})$. Using Serre duality and the local-to-global spectral sequence for Ext, we find that the obstruction bundle is isomorphic to $(T_X \otimes K_X)^\vee$. Then,

$$\tilde{N}_{1,0} = -\int_X c_3(T_X \otimes K_X) = -\int_X c_3(X) - c_1(X)c_2(X),$$

completing the proof. □

The degree 0 Gromov–Witten and Donaldson–Thomas theories are already related by Lemma 3. However, we make a stronger connection generalizing Conjecture 1.

Conjecture 1′. The degree 0 Donaldson–Thomas partition function for a 3-fold X is determined by

$$\mathsf{Z}_{\mathrm{DT}}(X; q)_0 = M(-q)^{\int_X c_3(T_X \otimes K_X)}.$$

We present a proof of Conjecture 1′ in the case that X is a nonsingular toric 3-fold in [MNOP06].

The series $M(q)$ arises naturally in the computation of the Euler characteristic of the Hilbert scheme of points of a 3-fold [Che96]. It would be interesting to find a direct connection between the degree 0 Donaldson–Thomas invariants and the Euler characteristics of $I_n(X, 0)$ in the Calabi–Yau case.

3. Local Calabi–Yau geometry

3.1 Gromov–Witten theory

Let S be a nonsingular, projective, toric, Fano surface with canonical bundle K_S. The Gromov–Witten theory of the local Calabi–Yau geometry of S is defined via an excess integral. Denote the universal curve and universal map over the moduli space of stable maps to S by

$$\pi : U \to \overline{M}_g(S, \beta),$$
$$\mu : U \to S.$$

Then,

$$N_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{\mathrm{vir}}} e(R^1\pi_*\mu^* K_S),$$

for $0 \neq \beta \in H_2(S, \mathbb{Z})$. The reduced partition function $\mathsf{Z}'_{\mathrm{GW}}(X; u, v)$ is defined in terms of the local invariants $N_{g,\beta}$ as before.

https://doi.org/10.1112/S0010437X06002302 Published online by Cambridge University Press

3.2 Donaldson–Thomas theory

Let X be the projective bundle $\mathbf{P}(K_S \oplus \mathcal{O}_S)$ over the surface S. The Donaldson–Thomas theory of X is well defined in every rank by the following observation.

Lemma 4. *X has an anticanonical section.*

Proof. Consider the fibration $\pi : X \to S$. We have

$$\wedge^3 T_X = T_\pi \otimes \pi^*(\wedge^2 T_S),$$

where T_π is the π-vertical tangent line.

Let V denote the vector bundle $K_S \oplus \mathcal{O}_S$ on S. The π-relative Euler sequence is

$$0 \to \mathcal{O}_X \to \pi^*(V) \otimes \mathcal{O}_{\mathbf{P}(V)}(1) \to T_\pi \to 0.$$

Hence,

$$T_\pi = \wedge^2 \pi^*(V) \otimes \mathcal{O}_{\mathbf{P}(V)}(2).$$

We conclude that

$$\wedge^3 T_X = \wedge^2 \pi^*(V) \otimes \mathcal{O}_{\mathbf{P}(V)}(2) \otimes \pi^*(\wedge^2 T_S) = \mathcal{O}_{\mathbf{P}(V)}(2).$$

However, since

$$H^0(X, \mathcal{O}_{\mathbf{P}(V)}(2)) = H^0(S, \mathrm{Sym}^2 V^*) \neq 0,$$

the lemma is proven. □

For classes $\beta \in H_2(S, \mathbb{Z})$, we define the reduced partition function for the Donaldson–Thomas theory of the Calabi–Yau geometry of S by

$$\mathsf{Z}'_{\mathrm{DT}}(S; q)_\beta = \mathsf{Z}'_{\mathrm{DT}}(X; q)_\beta.$$

While X is not Calabi–Yau, the Donaldson–Thomas theory of X is still well defined by Lemma 1 or Lemma 2.

We will prove that Conjectures 2 and 3 are true for the local Calabi–Yau geometry of toric Fano surfaces by virtual localization.

3.3 Local curves

The constructions above also define the local Calabi–Yau theory of the curve $\mathbf{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The proof of Conjectures 2 and 3 for local surfaces given in § 4 below is valid for the local $\mathbf{P}^1$ case.

The Gromov–Witten theory of a local Calabi–Yau curve of arbitrary genus has been defined in [BP03]. We believe that the GW/DT correspondence holds for these geometries as well [BP06].

4. Localization in Donaldson–Thomas theory

4.1 Toric geometry

Let X be a nonsingular, projective, toric 3-fold. Let $\mathbf{T}$ be the 3-dimensional complex torus acting on X. Let $\Delta(X)$ denote the Newton polyhedron of X determined by a polarization. The polyhedron $\Delta(X)$ is the image of X under the moment map.

The vertices of $\Delta(X)$ correspond to fixed points $X^{\mathbf{T}} = \{X_\alpha\}$ of the $\mathbf{T}$-action. For each X_α, there is a canonical, $\mathbf{T}$-invariant, affine open chart,

$$U_\alpha \cong \mathbf{A}^3,$$

centered at X_α. We may choose coordinates t_i on $\mathbf{T}$ and coordinates x_i on U_α for which the $\mathbf{T}$-action on U_α is determined by

$$(t_1, t_2, t_3) \cdot x_i = t_i x_i. \tag{4}$$

In these coordinates, the tangent representation X_α has character

$$t_1^{-1} + t_2^{-1} + t_3^{-1}.$$

We will use the covering $\{U_\alpha\}$ of X to compute Čech cohomology.

The $\mathbf{T}$-invariant lines of X correspond to the edges of $\Delta(X)$. More precisely, if

$$C_{\alpha\beta} \subset X$$

is a $\mathbf{T}$-invariant line incident to the fixed points X_α and X_β, then $C_{\alpha\beta}$ corresponds to an edge of $\Delta(X)$ joining the vertices X_α and X_β.

The geometry of $\Delta(X)$ near the edge is determined by the normal bundle $\mathcal{N}_{C_{\alpha\beta}/X}$. If

$$\mathcal{N}_{C_{\alpha\beta}/X} = \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}),$$

then the transition functions between the charts U_α and U_β can be taken in the form

$$(x_1, x_2, x_3) \mapsto (x_1^{-1}, x_2\, x_1^{-m_{\alpha\beta}}, x_3\, x_1^{-m'_{\alpha\beta}}). \tag{5}$$

The curve $C_{\alpha\beta}$ is defined in these coordinates by $x_2 = x_3 = 0$.

4.2 Moduli of ideal sheaves

The $\mathbf{T}$-action on X canonically lifts to the moduli space of ideal sheaves $I_n(X, \beta)$. The perfect obstruction theory constructed by Thomas is canonically $\mathbf{T}$-equivariant [MP06, Tho00]. The virtual localization formula reduces integration against $[I_n(X, \beta)]^{\mathrm{vir}}$ to a sum of fixed point contributions [GP99].

The first step is to determine the $\mathbf{T}$-fixed points of $I_n(X, \beta)$. If

$$[\mathcal{I}] \in I_n(X, \beta)$$

is $\mathbf{T}$-fixed, then the associated subscheme $Y \subset X$ must be preserved by the torus action. Hence, Y must be supported on the $\mathbf{T}$-fixed points X_α and the $\mathbf{T}$-invariant lines connecting them.

Since $\mathcal{I}$ is $\mathbf{T}$-fixed on each open set, $\mathcal{I}$ must be defined on U_α by a monomial ideal,

$$I_\alpha = \mathcal{I}\big|_{U_\alpha} \subset \mathbf{C}[x_1, x_2, x_3],$$

and may also be viewed as a 3-dimensional partition π_α,

$$\pi_\alpha = \left\{ (k_1, k_2, k_3),\ \prod_1^3 x_i^{k_i} \notin I_\alpha \right\} \subset \mathbb{Z}^3_{\geqslant 0}. \tag{6}$$

The associated subscheme of I_α is 1-dimensional. The corresponding partitions π_α may be infinite in the direction of the coordinate axes. If the 3-dimensional partition π is viewed as a box diagram, the vertices (6) are determined by the *interior* corners of the boxes, the corners closest to the origin.

The asymptotics of π_α in the coordinate directions are described by three ordinary 2-dimensional partitions. In particular, in the direction of the $\mathbf{T}$-invariant curve $C_{\alpha\beta}$, we have the partition $\lambda_{\alpha\beta}$ with the following diagram:

$$\begin{aligned} \lambda_{\alpha\beta} &= \{(k_2, k_3),\ \forall k_1\ x_1^{k_1} x_2^{k_2} x_3^{k_3} \notin I_\alpha\} \\ &= \{(k_2, k_3), x_2^{k_2} x_3^{k_3} \notin I_{\alpha\beta}\}, \end{aligned}$$

where

$$I_{\alpha\beta} = \mathcal{I}\big|_{U_\alpha \cap U_\beta} \subset \mathbf{C}[x_1^{\pm 1}, x_2, x_3].$$

The vertices of $\lambda_{\alpha\beta}$ defined above are the interior corners of the squares of the associated Young diagram.

In summary, a $\mathbf{T}$-fixed ideal sheaf $\mathcal{I}$ can be described in terms of the following data:

(i) a 2-dimensional partition $\lambda_{\alpha\beta}$ assigned to each edge of $\Delta(X)$;

(ii) a 3-dimensional partition π_α assigned to each vertex of $\Delta(X)$ such that the asymptotics of π_α in the three coordinate directions is given by the partitions $\lambda_{\alpha\beta}$ assigned to the corresponding edges.

4.3 Melting crystal interpretation

The partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}$ corresponding to a $\mathbf{T}$-fixed ideal sheaf $\mathcal{I}$ can be given a melting crystal interpretation [ORV03]. Consider the weights of the $\mathbf{T}$-action on

$$H^0(X, \mathcal{O}_Y(d)).$$

For large d, the corresponding points of $\mathbb{Z}^3$ can be described as follows.

Scale the Newton polyhedron $\Delta(X)$ by a factor of d. Near each corner of $d\Delta(X)$, the intersection $\mathbb{Z}^3 \cap d\Delta(X)$ looks like a standard $\mathbb{Z}^3_{\geqslant 0}$, so we can place the corresponding partition π_α there. Since d is large and since, by construction, π_α and π_β agree along the edge joining them, a global combinatorial object emerges.

One can imagine that the points of $\mathbb{Z}^3 \cap d\Delta(X)$ are atoms in a crystal and, as the crystal is melting or dissolving, some of the atoms near the corners and along the edges are missing. These missing atoms are described by the partitions π_α and $\lambda_{\alpha\beta}$. They are precisely the weights of the $\mathbf{T}$-action on $H^0(X, \mathcal{O}_Y(d))$.

4.4 Degree and Euler characteristic

Let $[\mathcal{I}] \in I_n(X, \beta)$ be a $\mathbf{T}$-fixed ideal sheaf on X described by the partition data $\{\pi_\alpha, \lambda_{\alpha\alpha'}\}$. We see

$$\beta = \sum_{\alpha,\alpha'} |\lambda_{\alpha\alpha'}| [C_{\alpha\alpha'}],$$

where $|\lambda|$ denotes the size of a partition λ, the number of squares in the diagram.

For 3-dimensional partitions π, one can similarly define their size $|\pi|$ by the number of cubes in their diagram. Since the partitions π_α may be infinite along the coordinate axes, the number $|\pi_\alpha|$ so defined will often be infinite. We define the renormalized volume $|\pi_\alpha|$ as follows. Let $\lambda_{\alpha\beta_i}$, $i = 1, 2, 3$, be the asymptotics of π_α. We set

$$|\pi_\alpha| = \#\{\pi_\alpha \cap [0, \ldots, N]^3\} - (N+1) \sum_1^3 |\lambda_{\alpha\beta_i}|, \quad N \gg 0.$$

The renormalized volume is independent of the cut-off N as long as N is sufficiently large. The number $|\pi_\alpha|$ so defined may be negative.

Given $m, m' \in \mathbb{Z}$ and a partition λ, we define

$$f_{m,m'}(\lambda) = \sum_{(i,j) \in \lambda} (-mi - m'j + 1),$$

where the sum is over the interior corners of the Young diagram of λ. Each edge of $\Delta(X)$ is assigned a pair of integers $(m_{\alpha\beta}, m'_{\alpha\beta})$ from the normal bundle of the associated $\mathbf{T}$-invariant line and a partition $\lambda_{\alpha\beta}$ from the $\mathbf{T}$-fixed ideal sheaf $\mathcal{I}$. By definition, we set

$$f(\alpha, \beta) = f_{m_{\alpha\beta}, m'_{\alpha\beta}}(\lambda_{\alpha\beta}). \tag{7}$$

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Lemma 5. $\chi(\mathcal{O}_Y) = \sum_{\alpha} |\pi_\alpha| + \sum_{\alpha,\beta} f(\alpha, \beta)$.

Proof. The result is an elementary calculation in toric geometry. For example, a computation of $\chi(\mathcal{O}_Y)$ using the Čech cover defined by $\{U_\alpha\}$ immediately yields the result. $\square$

4.5 The T-fixed obstruction theory

The moduli space $I_n(X, \beta)$ carries a $\mathbf{T}$-equivariant perfect obstruction theory,

$$E_0 \to E_1$$

(see [MP06, Tho00]). Assume that the virtual dimension of $I_n(X, \beta)$ is 0. The virtual localization formula [GP99] may be stated as follows:

$$\int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 = \sum_{[\mathcal{I}] \in I_n(X,\beta)^{\mathbf{T}}} \int_{[S(\mathcal{I})]^{\mathrm{vir}}} \frac{e(E_1^m)}{e(E_0^m)}.$$

Here, $S(\mathcal{I})$ denotes the $\mathbf{T}$-fixed subscheme of $I_n(X, \beta)$ supported at the point $[\mathcal{I}]$, and E_0^m, E_1^m denote the nonzero $\mathbf{T}$-weight spaces. The virtual class, $[S(\mathcal{I})]^{\mathrm{vir}}$, is determined by the $\mathbf{T}$-fixed obstruction theory.

We first prove that $S(\mathcal{I})$ is the reduced point $[\mathcal{I}]$. It suffices to prove that the Zariski tangent space to $I_n(X, \beta)$ at $[\mathcal{I}]$ contains no trivial subrepresentations. Since X is toric, all the higher cohomologies of $\mathcal{O}_X$ vanish,

$$H^i(X, \mathcal{O}_X) = 0,$$

for $i \geqslant 0$. Hence, the traceless condition is satisfied, and the Zariski tangent space of $I_n(X, \beta)$ at $[\mathcal{I}]$ is $\mathrm{Ext}^1(\mathcal{I}, \mathcal{I})$.

Lemma 6. *Let $[\mathcal{I}] \in I_n(X, \beta)$ be a $\mathbf{T}$-fixed point. The $\mathbf{T}$-representation*

$$\mathrm{Ext}^1(\mathcal{I}, \mathcal{I})$$

contains no trivial subrepresentations.

Proof. From the $\mathbf{T}$-equivariant ideal sheaf sequence,

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0, \tag{8}$$

we obtain a sequence of $\mathbf{T}$-representations,

$$\to \mathrm{Ext}^0(\mathcal{I}, \mathcal{O}_Y) \to \mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) \to \mathrm{Ext}^1(\mathcal{I}, \mathcal{O}_X) \to .$$

The left-hand term, $\mathrm{Ext}^0(\mathcal{I}, \mathcal{O}_Y)$, does not contain trivial representations by Lemma 7 below.

We will prove that the right-hand term, $\mathrm{Ext}^1(\mathcal{I}, \mathcal{O}_X)$, also does not contain trivial representations. By Serre duality, it suffices to study the representation

$$\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{I} \otimes K_X) = H^2(X, \mathcal{I} \otimes K_X).$$

The long exact sequence in cohomology obtained from (8) by tensoring with K_X and the vanishings

$$H^1(X, K_X) = H^2(X, K_X) = 0$$

together yield a $\mathbf{T}$-equivariant isomorphism

$$H^1(X, \mathcal{O}_Y \otimes K_X) \xrightarrow{\sim} H^2(X, \mathcal{I} \otimes K_X).$$

The first Čech cohomology of $\mathcal{O}_Y \otimes K_X$ is computed via the representations

$$H^0(U_{\alpha\beta}, \mathcal{O}_Y \otimes K_X),$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Here we use the Čech cover defined in § 4.1. An elementary argument shows that these representations contain no trivial subrepresentations. $\square$

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Lemma 7. $\mathrm{Ext}^k(\mathcal{I},\mathcal{O}_Y)$ *contains no trivial subrepresentations.*

Proof. By the local-to-global spectral sequence, it suffices to prove that

$$H^i(\mathcal{E}xt^j(\mathcal{I},\mathcal{O}_Y))$$

contains no trivial subrepresentations for all i and j. By a Čech cohomology calculation, it then suffices to prove that

$$H^0(U_\alpha,\mathcal{E}xt^j(\mathcal{I}_\alpha,\mathcal{O}_{Y_\alpha})) \quad \text{and} \quad H^0(U_{\alpha\beta},\mathcal{E}xt^j(\mathcal{I}_{\alpha\beta},\mathcal{O}_{Y_{\alpha\beta}}))$$

contain no trivial subrepresentations. Triple intersections need not be considered since $\mathcal{O}_{Y_{\alpha\beta\gamma}}$ vanishes.

We will study $\mathcal{E}xt^j(\mathcal{I}_\alpha,\mathcal{O}_{Y_\alpha})$ on U_α via the $\mathbf{T}$-equivariant Taylor resolution of the monomial ideal $\mathcal{I}_\alpha$. The argument for $\mathcal{E}xt^j(\mathcal{I}_{\alpha\beta},\mathcal{O}_{Y_{\alpha\beta}})$ on $U_{\alpha\beta}$ is identical.

Let $\mathcal{I}_\alpha$ be generated by the monomials $m_1,\dots,m_s$. For each subset

$$T\subset\{1,\dots,s\},$$

let

$$m_T = x^{r(T)} = \text{least common multiple of } \{m_i \mid i\in T\}.$$

For $1\leqslant t\leqslant s$, let F_t be the free $\Gamma(U_\alpha)$-module with basis e_T indexed by subsets $T\subset\{1,\dots,s\}$ of size t.

A differential $d: F_t\to F_{t-1}$ is defined as follows. Given a subset T of size t, let $T=\{i_1,\dots,i_t\}$ where $i_1<\dots<i_t$. Let

$$d(e_T)=\sum_{T'=T\setminus\{i_k\}}(-1)^k x^{r_T-r_{T'}}e_{T'}.$$

The Taylor resolution,

$$0\to F_s\to\cdots\to F_2\to F_1\to\mathcal{I}_\alpha\to 0,$$

is exact [Tay66]. Moreover, the resolution is equivariant with $\mathbf{T}$-weight $r(T)$ on the generator $e(T)$.

The weights of the generators of F_t are weights of monomials in $\mathcal{I}_\alpha$. However, the weights of the $\mathbf{T}$-representation $\mathcal{O}_{Y_\alpha}$ are precisely not equal to weights of monomials in $\mathcal{I}_\alpha$. Hence, $\mathcal{H}om(F_t,\mathcal{O}_{Y_\alpha})$ contains no trivial subrepresentations. We then conclude that $\mathcal{E}xt^j(\mathcal{I}_\alpha,\mathcal{O}_Y)$ contains no trivial subrepresentations by computing via the Taylor resolution of $\mathcal{I}_\alpha$. □

The obstruction space at $[\mathcal{I}]\in I_n(X,\beta)$ of the perfect obstruction theory is $\mathrm{Ext}^2(\mathcal{I},\mathcal{I})$. The following lemma implies that the $\mathbf{T}$-fixed obstruction theory at $[\mathcal{I}]$ is trivial.

Lemma 8. $\mathrm{Ext}^2(\mathcal{I},\mathcal{I})$ *contains no trivial subrepresentations.*

Proof. From the $\mathbf{T}$-equivariant ideal sheaf sequence, we obtain

$$\to\mathrm{Ext}^1(\mathcal{I},\mathcal{O}_Y)\to\mathrm{Ext}^2(\mathcal{I},\mathcal{I})\to\mathrm{Ext}^2(\mathcal{I},\mathcal{O}_X)\to .$$

The left-hand term, $\mathrm{Ext}^1(\mathcal{I},\mathcal{O}_Y)$, does not contain trivial representations by Lemma 7 above.

We will prove that the right-hand term, $\mathrm{Ext}^2(\mathcal{I},\mathcal{O}_X)$, also does not contain trivial representations. By Serre duality, it suffices to study the representation

$$\mathrm{Ext}^1(\mathcal{O}_X,\mathcal{I}\otimes K_X)=H^1(X,\mathcal{I}\otimes K_X).$$

The long exact sequence in cohomology obtained by tensoring the ideal sheaf sequence with K_X and the vanishings

$$H^0(X,K_X)=H^1(X,K_X)=0$$

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together yield a $\mathbf{T}$-equivariant isomorphism

$$H^0(X, \mathcal{O}_Y \otimes K_X) \xrightarrow{\sim} H^1(X, \mathcal{I} \otimes K_X).$$

The space of global sections of $\mathcal{O}_Y \otimes K_X$ is computed via the representations

$$H^0(U_\alpha, \mathcal{O}_Y \otimes K_X).$$

As before, an elementary argument shows that these representations contain no trivial subrepresentations. □

The virtual localization formula may then be written as

$$\int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 = \sum_{[\mathcal{I}] \in I_n(X,\beta)^{\mathbf{T}}} \frac{e(\operatorname{Ext}^2(\mathcal{I},\mathcal{I}))}{e(\operatorname{Ext}^1(\mathcal{I},\mathcal{I}))}.$$

A calculation of the virtual representation

$$\operatorname{Ext}^1(\mathcal{I},\mathcal{I}) - \operatorname{Ext}^2(\mathcal{I},\mathcal{I})$$

is required for the evaluation of the virtual localization formula.

4.6 Virtual tangent space

The virtual tangent space at $\mathcal{I}$ is given by

$$\mathcal{T}_{[\mathcal{I}]} = \operatorname{Ext}^1(\mathcal{I},\mathcal{I}) - \operatorname{Ext}^2(\mathcal{I},\mathcal{I}) = \chi(\mathcal{O},\mathcal{O}) - \chi(\mathcal{I},\mathcal{I}),$$

where

$$\chi(\mathcal{F},\mathcal{G}) = \sum_{i=0}^{3} (-1)^i \operatorname{Ext}^i(\mathcal{F},\mathcal{G}).$$

We can compute each Euler characteristic using the local-to-global spectral sequence

$$\begin{aligned}\chi(\mathcal{I},\mathcal{I}) &= \sum_{i,j=0}^{3} (-1)^{i+j} H^i(\mathcal{E}xt^j(\mathcal{I},\mathcal{I}))\\ &= \sum_{i,j=0}^{3} (-1)^{i+j} \mathfrak{C}^i(\mathcal{E}xt^j(\mathcal{I},\mathcal{I})),\end{aligned}$$

where, in the second line, we have replaced the cohomology terms with the Čech complex with respect to the open affine cover $\{U_\alpha\}$. Though these modules are infinite-dimensional, they have finite-dimensional weight spaces and, therefore, their $\mathbf{T}$-character is well defined as a formal power series.

Since Y is supported on the curves $C_{\alpha\beta}$, we have $\mathcal{I} = \mathcal{O}_X$ on the intersection of three or more U_α. Therefore, only the $\mathfrak{C}^0$ and $\mathfrak{C}^1$ terms contribute to the calculation. We find

$$\mathcal{T}_{[\mathcal{I}]} = \bigoplus_\alpha \left(\Gamma(U_\alpha) - \sum_i (-1)^i \Gamma(U_\alpha, \mathcal{E}xt^i(\mathcal{I},\mathcal{I})) \right) - \bigoplus_{\alpha,\beta} \left(\Gamma(U_{\alpha\beta}) - \sum_i (-1)^i \Gamma(U_{\alpha\beta}, \mathcal{E}xt^i(\mathcal{I},\mathcal{I})) \right). \quad (9)$$

The calculation is reduced to a sum over all the vertices and edges of the Newton polyhedron. In each case, we are given an ideal

$$I = I_\alpha, I_{\alpha\beta} \subset \Gamma(U),$$

and we need to compute

$$\left(\Gamma(U) - \sum_i (-1)^i \operatorname{Ext}^i(I,I) \right)$$

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over the ring $\Gamma(U)$, which is isomorphic to $\mathbf{C}[x,y,z]$ in the vertex case and is isomorphic to $\mathbf{C}[x,y,z,z^{-1}]$ in the edge case. We treat each case separately.

4.7 Vertex calculation

Let R be the coordinate ring,

$$R = \mathbf{C}[x_1,x_2,x_3] \cong \Gamma(U_\alpha).$$

As before, we can assume that the $\mathbf{T}$-action on R is the standard action (4). Consider a $\mathbf{T}$-equivariant graded free resolution of I_α,

$$0 \to F_s \to \cdots \to F_2 \to F_1 \to I_\alpha \to 0, \tag{10}$$

such as, for example, the Taylor resolution [Tay66]. Each term in (10) has the form

$$F_i = \bigoplus_j R(d_{ij}), \quad d_{ij} \in \mathbb{Z}^3.$$

The Poincaré polynomial

$$P_\alpha(t_1,t_2,t_3) = \sum_{i,j} (-1)^i \, t^{d_{ij}}$$

does not depend on the choice of the resolution (10). In fact, from the resolution (10) we see that the Poincaré polynomial P_α is related to the $\mathbf{T}$-character of R/I_α as follows:

$$\begin{aligned} Q_\alpha(t_1,t_2,t_3) &:= \operatorname{tr}_{R/I_\alpha}(t_1,t_2,t_3) \\ &= \sum_{(k_1,k_2,k_3)\in\pi_\alpha} t_1^{k_1} t_2^{k_2} t_3^{k_3} \\ &= \frac{1 + P_\alpha(t_1,t_2,t_3)}{(1-t_1)(1-t_2)(1-t_3)}, \end{aligned} \tag{11}$$

where 'tr' in the first line denotes the trace of the $\mathbf{T}$-action on R/I_α.

The virtual representation $\chi(I_\alpha, I_\alpha)$ is given by the alternating sum

$$\begin{aligned} \chi(I_\alpha, I_\alpha) &= \sum_{i,j,k,l} (-1)^{i+k} \operatorname{Hom}_R(R(d_{ij}), R(d_{kl})) \\ &= \sum_{i,j,k,l} (-1)^{i+k} R(d_{kl} - d_{ij}), \end{aligned}$$

and, therefore,

$$\operatorname{tr}_{\chi(I_\alpha,I_\alpha)}(t_1,t_2,t_3) = \frac{P_\alpha(t_1,t_2,t_3)\, P_\alpha(t_1^{-1},t_2^{-1},t_3^{-1})}{(1-t_1)(1-t_2)(1-t_3)}.$$

We find that the character of the $\mathbf{T}$-action on the α summand of (9) is given by

$$\frac{1 - P_\alpha(t_1,t_2,t_3)\, P_\alpha(t_1^{-1},t_2^{-1},t_3^{-1})}{(1-t_1)(1-t_2)(1-t_3)}.$$

Using (11), we may express the answer in terms of the generating function Q_α of the partition π_α,

$$\operatorname{tr}_{R-\chi(I_\alpha,I_\alpha)}(t_1,t_2,t_3) = Q_\alpha - \frac{\overline{Q}_\alpha}{t_1t_2t_3} + Q_\alpha \overline{Q}_\alpha \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}, \tag{12}$$

where

$$\overline{Q}_\alpha(t_1,t_2,t_3) = Q_\alpha(t_1^{-1},t_2^{-1},t_3^{-1}).$$

The rational function (12) should be expanded in ascending powers of the t_i.

4.8 Edge calculation

We now consider the summand of (9) corresponding to a pair (α, β). Our calculations will involve modules over the ring

$$R = \Gamma(U_{\alpha\beta}) = \mathbf{C}[x_2, x_3] \otimes_{\mathbf{C}} \mathbf{C}[x_1, x_1^{-1}].$$

The $\mathbf{C}[x_1, x_1^{-1}]$ factor will result only in the overall factor

$$\delta(t_1) = \sum_{k\in\mathbb{Z}} t_1^k,$$

the formal δ-function at $t_1 = 1$, in the $\mathbf{T}$-character. Let

$$Q_{\alpha\beta}(t_2, t_3) = \sum_{(k_2,k_3)\in\lambda_{\alpha\beta}} t_2^{k_2} t_3^{k_3}$$

be the generating function for the edge partition $\lambda_{\alpha\beta}$. Arguing as in the vertex case, we find

$$-\mathrm{tr}_{R-\chi(I_{\alpha\beta}, I_{\alpha\beta})}(t_1, t_2, t_3) = \delta(t_1)\biggl(-Q_{\alpha\beta} - \frac{\overline{Q}_{\alpha\beta}}{t_2 t_3} + Q_{\alpha\beta}\overline{Q}_{\alpha\beta}\frac{(1-t_2)(1-t_3)}{t_2 t_3}\biggr). \tag{13}$$

Note that, because of the relations

$$\delta(1/t) = \delta(t) = t\delta(t),$$

the character (13) is invariant under the change of variables (5).

4.9 The equivariant vertex

The formulas (12) and (13) express the Laurent polynomial $\mathrm{tr}_{\mathcal{I}_{[\mathcal{I}]}}(t_1, t_2, t_3)$ as a linear combination of infinite formal power series. Our goal now is to redistribute the terms in these series so that both the vertex and edge contributions are finite.

The edge character (13) can be written as

$$\frac{F_{\alpha\beta}(t_2, t_3)}{1 - t_1} + t_1^{-1}\frac{F_{\alpha\beta}(t_2, t_3)}{1 - t_1^{-1}}, \tag{14}$$

where

$$F_{\alpha\beta}(t_2, t_3) = -Q_{\alpha\beta} - \frac{\overline{Q}_{\alpha\beta}}{t_2 t_3} + Q_{\alpha\beta}\overline{Q}_{\alpha\beta}\frac{(1-t_2)(1-t_3)}{t_2 t_3},$$

and the first (respectively second) term in (14) is expanded in ascending (respectively descending) powers of t_1.

Let us denote the character (12) by F_α and define

$$\mathsf{V}_\alpha = F_\alpha + \sum_{i=1}^{3} \frac{F_{\alpha\beta_i}(t_{i'}, t_{i''})}{1 - t_i},$$

where $C_{\alpha\beta_1}, C_{\alpha\beta_2}, C_{\alpha\beta_3}$ are the three $\mathbf{T}$-invariant rational curves passing through the point $X_\alpha \in X^{\mathbf{T}}$, and $\{t_i, t_{i'}, t_{i''}\} = \{t_1, t_2, t_3\}$.

Similarly, we define

$$\mathsf{E}_{\alpha\beta} = t_1^{-1}\frac{F_{\alpha\beta}(t_2, t_3)}{1 - t_1^{-1}} - \frac{F_{\alpha\beta}(t_2\, t_1^{-m_{\alpha\beta}}, t_3\, t_1^{-m'_{\alpha\beta}})}{1 - t_1^{-1}}.$$

The term $\mathsf{E}_{\alpha\beta}$ is canonically associated to the edge. Formulas (12) and (13) yield the following result.

Theorem 1. *The* $\mathbf{T}$*-character of* $\mathcal{T}_{[\mathcal{I}]}$ *is given by*

$$\mathrm{tr}_{\mathcal{T}_{[\mathcal{I}]}}(t_1,t_2,t_3)=\sum_{\alpha}\mathsf{V}_\alpha+\sum_{\alpha\beta}\mathsf{E}_{\alpha\beta}. \tag{15}$$

Lemma 9. *Both* V_α *and* $\mathsf{E}_{\alpha\beta}$ *are Laurent polynomials.*

Proof. The numerator of $\mathsf{E}_{\alpha\beta}$ vanishes at $t_1=1$, whence it is divisible by the denominator. The claim for V_α follows from

$$Q_\alpha=\frac{Q_{\alpha\beta}}{1-t_1}+\cdots,$$

where the dots stand for terms regular at $t_1=1$. □

From V_α, the equivariant localization formula defines a natural 3-parameter family of measures w on 3-dimensional partitions π_α. Namely, the measure of π_α equals

$$\mathsf{w}(\pi_\alpha)=\prod_{k\in\mathbb{Z}^3}(s,k)^{-\mathsf{v}_k},$$

where $s=(s_1,s_2,s_3)$ are parameters, $(\,\cdot\,,\,\cdot\,)$ denotes the standard inner product, and v_k is the coefficient of t^k in V_α. We call the measure w the *equivariant vertex measure*.

The equivariant vertex measure simplifies dramatically in the local Calabi–Yau case to a signed volume; the simplification plays a basic role in the calculation of § 4.10. The full equivariant vertex measure is discussed further in [MNOP06].

4.10 Local Calabi–Yau and the topological vertex

We now specialize to the local Calabi–Yau geometry discussed in § 3.

Let S be a nonsingular, toric, Fano surface with canonical bundle K_S. We view the total space of K_S as an open toric Calabi–Yau 3-fold. Let X be the toric compactification defined in § 3. By definition,

$$\mathsf{Z}'_{\mathrm{DT}}(S;q)_\beta=\mathsf{Z}_{\mathrm{DT}}(X;q)_\beta/\mathsf{Z}_{\mathrm{DT}}(X;q)_0, \tag{16}$$

for $\beta\in H_2(S,\mathbb{Z})$.

We may compute the right-hand side of (16) by localization. Let

$$D=X\setminus K_S$$

denote the divisor at infinity. Let $[\mathcal{I}]\in I_n(X,\beta)$ be a $\mathbf{T}$-fixed ideal sheaf. We have seen that the weights of the virtual tangent representation of $[\mathcal{I}]$ are determined by the vertices and edges of the support of Y. Since β is a class on S, the support of Y lies in K_S except for possibly a finite union of 0-dimensional subschemes supported on D. Therefore, as a consequence of the virtual localization formula for the Donaldson–Thomas theory of X, we find

$$\mathsf{Z}'_{\mathrm{DT}}(S;q)_\beta=\frac{\sum_n q^n\sum_{[\mathcal{I}]\in I_n(K_S,\beta)}e(\mathrm{Ext}^2(\mathcal{I},\mathcal{I}))/e(\mathrm{Ext}^1(\mathcal{I},\mathcal{I}))}{\sum_n q^n\sum_{[\mathcal{I}]\in I_n(K_S,0)}e(\mathrm{Ext}^2(\mathcal{I},\mathcal{I}))/e(\mathrm{Ext}^1(\mathcal{I},\mathcal{I}))}. \tag{17}$$

Here, only the ideal sheaves $\mathcal{I}$ for which Y has compact support in K_S are considered. In particular, the local Donaldson–Thomas theory should be viewed as independent of the compactification X.

The open set K_S has a canonical Calabi–Yau 3-form Ω. There is a 2-dimensional subtorus

$$\mathbf{T}_0\subset\mathbf{T}$$

which preserves Ω. We will evaluate the formula (17) on the subtorus $\mathbf{T}_0$.

Let $U_\alpha\subset K_S$ be a chart with coordinates (4). The subgroup $\mathbf{T}_0$ is defined by

$$t_1t_2t_3=1.$$

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By Serre duality for a compact Calabi–Yau 3-fold, we obtain a canonical isomorphism

$$\mathrm{Ext}^1_0(\mathcal{I},\mathcal{I}) = \mathrm{Ext}^2_0(\mathcal{I},\mathcal{I})^*.$$

We will find the $\mathbf{T}_0$-representations to be dual in the local Calabi–Yau geometry as well. Formula (17) will be evaluated by canceling the dual weights and counting signs.

The following functional equation for the character (15) expresses Serre duality. On the subtorus $t_1t_2t_3 = 1$, the character is odd under the involution $f \mapsto \overline{f}$ defined by

$$(t_1, t_2, t_3) \mapsto (t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

Below we will see, in fact, that each term in (15) is an anti-invariant of this transformation.

A crucial technical point is that no term of (15) specializes to 0 weight under the restriction to $\mathbf{T}_0$. Since the specializations are all nonzero, the localization formula for $\mathbf{T}$ may be computed after restriction to $\mathbf{T}_0$. We leave the straightforward verification to the reader.

We will split the edge contributions of (15) into two pieces,

$$\mathsf{E}_{\alpha\beta} = \mathsf{E}^+_{\alpha\beta} + \mathsf{E}^-_{\alpha\beta},$$

satisfying

$$\overline{\mathsf{E}}^+_{\alpha\beta}|_{t_1t_2t_3=1} = -\mathsf{E}^-_{\alpha\beta}|_{t_1t_2t_3=1}, \tag{18}$$

where

$$\overline{\mathsf{E}}_{\alpha\beta}(t_1,t_2,t_3) = \mathsf{E}(t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

The total count of (-1)s contributing to $\mathsf{E}_{\alpha\beta}$ is then determined by the parity of the evaluation of $\mathsf{E}^+_{\alpha\beta}$ at the point $(t_1,t_2,t_3) = (1,1,1)$ so long as the constant term of

$$\mathsf{E}^+_{\alpha\beta}|_{t_1t_2t_3=1}$$

is even. Concretely, we set

$$F^+_{\alpha\beta} = -Q_{\alpha\beta} - Q_{\alpha\beta}\overline{Q}_{\alpha\beta}\frac{1-t_2}{t_2}$$

and define $\mathsf{E}^+_{\alpha\beta}$ in terms of $F^+_{\alpha\beta}$ using the same formulas as before. A straightforward check verifies (18). The constant term will be discussed in § 4.11.

Observe that

$$\mathsf{E}^+_{\alpha\beta}|_{t_1=1} = \left(m_{\alpha\beta}\, t_2\frac{\partial}{\partial t_2} + m'_{\alpha\beta}\, t_3\frac{\partial}{\partial t_3} - 1\right)F^+_{\alpha\beta}.$$

Hence, we conclude that

$$\mathsf{E}_{\alpha\beta}(1,1,1) \equiv f(\alpha,\beta) + m_{\alpha\beta}|\lambda_{\alpha\beta}| \mod 2, \tag{19}$$

where the function $f(\alpha,\beta)$ was defined in (7). The second term in (19) comes from applying $\partial/\partial t_2$ to the $(1-t_2)$ factor in the $Q\overline{Q}$ term.

Naively, a similar splitting of the vertex term

$$\mathsf{V}_\alpha = \mathsf{V}^+_\alpha + \mathsf{V}^-_\alpha,$$

satisfying

$$\overline{\mathsf{V}}^+_\alpha|_{t_1t_2t_3=1} = -\mathsf{V}^-_\alpha|_{t_1t_2t_3=1},$$

is obtained by defining F^+_α to be equal to

$$Q_\alpha - Q_\alpha\overline{Q}_\alpha\frac{(1-t_1)(1-t_2)}{t_1t_2}.$$

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However, the definition is not satisfactory since rational functions and not polynomials are obtained. The factor

$$\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} \tag{20}$$

in the $Q\overline{Q}$ term in (12) can be split in three different ways and no single choice can serve all terms in the $Q\overline{Q}$ product. The correct choice of the splitting is the following. Define the polynomial Q'_α by the equality

$$Q_\alpha = Q'_\alpha + \sum_{i=1}^{3} \frac{Q_{\alpha\beta_i}}{1-t_i}.$$

Now for each set of bar-conjugate terms in the expansion of the $Q\overline{Q}$ product, we pick its own splitting of (20), so that, for example, the term

$$-\frac{Q_{\alpha\beta_1}\overline{Q}_{\alpha\beta_1}}{(1-t_1)(1-t_1^{-1})}\frac{(1-t_1)(1-t_2)}{t_1t_2}$$

cancels the corresponding contribution of $F^+_{\alpha\beta_1}$, and for $i \neq j$ the terms

$$-\bigg(\frac{Q_{\alpha\beta_i}\overline{Q}_{\alpha\beta_j}}{(1-t_i)(1-t_j^{-1})} + \frac{\overline{Q}_{\alpha\beta_i}Q_{\alpha\beta_j}}{(1-t_i^{-1})(1-t_j)}\bigg)\frac{(1-t_i)(1-t_j)}{t_it_j}$$

are regular and even at $(1,1,1)$. The constant term of

$$\mathsf{V}^+_\alpha|_{t_1t_2t_3=1}$$

will be shown to be even in § 4.11.

Using the splitting defined above, we easily compute

$$\mathsf{V}^+_\alpha(1,1,1) \equiv Q'_\alpha(1,1,1) \mod 2. \tag{21}$$

From the discussion in § 4.4, we find

$$Q'_\alpha(1,1,1) = |\pi_\alpha|. \tag{22}$$

Equations (19) and (22) together with Lemma 5 yield the following result.

Theorem 2. *Let $\mathcal{I}$ be a $\mathbf{T}$-fixed ideal sheaf in $I_n(K_S,\beta)$. Then*

$$\frac{e(\mathrm{Ext}^2(\mathcal{I},\mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I},\mathcal{I}))} = (-1)^{\chi(\mathcal{O}_Y)+\sum_{\alpha\beta} m_{\alpha\beta}|\lambda_{\alpha\beta}|},$$

where the sum in the exponent is over all edges and

$$O(m_{\alpha\beta}) \oplus O(m'_{\alpha\beta})$$

is the normal bundle to the edge curve $C_{\alpha\beta}$.

As a corollary of Theorem 2, we prove the Gromov–Witten/Donaldson–Thomas correspondence for toric local Calabi–Yau surfaces.

Theorem 3. *For toric local Calabi–Yau surfaces S,*

$$\mathsf{Z}'_{\mathrm{GW}}(S;u,v) = \mathsf{Z}'_{\mathrm{DT}}(S;-e^{iu},v)$$

holds.

Proof. The proof is obtained from Theorem 2 by direct comparison with the topological vertex calculation of $\mathsf{Z}'_{\mathrm{GW}}(S;u,v)$.

The topological vertex [AKMV03] is a conjectural evaluation of the Gromov–Witten theory of all toric Calabi–Yau 3-folds. In the case of local toric Calabi–Yau surfaces, the topological vertex conjecture has been proven in [LLZ03].

To match our Donaldson–Thomas calculation, the melting crystal interpretation of the topological vertex is required [ORV03]. In the melting crystal interpretation, the Gromov–Witten contribution of a $\mathbf{T}$-fixed ideal sheaf $\mathcal{I}$ in $I_n(K_S, \beta)$ is

$$\mathbf{Contribution}_{\mathcal{I}}(\mathsf{Z}'_{\mathrm{GW}}(S; u, v)) = e^{iu\chi(\mathcal{O}_Y)}(-1)^{\sum_{\alpha\beta} m_{\alpha\beta}|\lambda_{\alpha\beta}|}v^{\beta}.$$

The Donaldson–Thomas contribution of $\mathcal{I}$ is

$$\mathbf{Contribution}_{\mathcal{I}}(\mathsf{Z}'_{\mathrm{DT}}(S; q, v)) = (-q)^{\chi(\mathcal{O}_Y)}(-1)^{\sum_{\alpha\beta} m_{\alpha\beta}|\lambda_{\alpha\beta}|}v^{\beta}$$

by Theorem 2. The number $1 + m_{\alpha\beta}$ has the same parity as the *framing* of corresponding edge in the topological vertex formalism. □

4.11 Constant terms

The calculation of § 4.10 requires the constant terms after restriction to $t_1t_2t_3 = 1$ of the vertex and edge splittings V^+_α and $\mathsf{E}^+_{\alpha\beta}$ to be even.

Consider first the constant term of the vertex splitting. The finite case is immediate.

LEMMA 10. *Let γ be a finite 3-dimensional partition. Then, the constant term of*

$$Q_\gamma - Q_\gamma \overline{Q}_\gamma \frac{(1-t_1)(1-t_2)}{t_1t_2} \tag{23}$$

after the restriction $t_1t_2t_3 = 1$ is even.

Proof. Assume that the result holds for partitions with fewer boxes than γ. Let $b \in \gamma$ be an extreme box on the highest level in the t_3 direction. We show that the change in the constant term of (23) after removing b is even.

Let (b_1, b_2, b_3) be the coordinates of the box b indexed by the corner closest to the origin. A box $b' \in \gamma$ can interact with b in the constant term of the second summand of (23) in two ways:

(i) $\mathbf{Constant}\left(b\,\overline{b'}\frac{(1-t_1)(1-t_2)}{t_1t_2}\right)$;

(ii) $\mathbf{Constant}\left(b'\,\overline{b}\frac{(1-t_1)(1-t_2)}{t_1t_2}\right)$.

Here, **Constant** denotes the constant term after restriction to $t_1t_2t_3 = 1$. If $b' = b$, only contribution (i) is included.

The type (i) contribution for the box (b'_1, b'_2, z) exactly equals the type (ii) contribution for the box $(b'_1, b'_2, z-1)$. The cancelation mod 2 is therefore perfect except for:

(a) the $b' = (b'_1, b'_2, b'_3)$ contributions to (ii) for which $(b'_1, b'_2, b'_3 + 1)$ does not lie in γ;

(b) the $b' = (b'_1, b'_2, 0)$ contributions to (i).

There are no contributions of type (a). If $b'_3 = b_3$, there are no contributions since b is extremal and $b' \neq b$. If $b'_3 < b_3$, there are no contributions since either $b'_1 > b_1$ or $b'_2 > b_2$.

We study now the $b'_3 = 0$ contributions to (i). If b is not on the main diagonal (x, x, x), then the $b'_3 = 0$ contributions to (i) are either 0, 2 or 4. If b is on the main diagonal, then the $b'_3 = 0$ contribution is 1.

The box b contributes 1 to the constant term of Q_γ if and only if b is on the main diagonal. Hence, the change of the constant term of (23) after the removal of b is even. □

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Let A, B be 3-dimensional partitions which are cylinders in *distinct* directions t_i, t_j with cross sections given by the 2-dimensional partitions $\lambda(A), \lambda(B)$,

$$Q_A = \frac{Q_{\lambda(A)}}{1-t_i}, \quad Q_B = \frac{Q_{\lambda(B)}}{1-t_j}.$$

Let C_A be a suitably large cut-off of A, and let C_B be a suitably large cut-off of B. For any Laurent polynomial $F(t_1, t_2, t_3)$,

$$\mathbf{Constant}\biggl(F\overline{Q}_A + \overline{F}Q_A\frac{(1-t_i)(1-t_k)}{t_it_k}\biggr) = \mathbf{Constant}\biggl(F\overline{Q}_{C_A} + \overline{F}Q_{C_A}\frac{(1-t_i)(1-t_k)}{t_it_k}\biggr), \quad (24)$$

since the extreme parts of A cannot contribute to the constant after restriction to $t_1t_2t_3 = 1$. Similarly,

$$\mathbf{Constant}\biggl(Q_A\overline{Q}_B + \overline{Q}_AQ_B\frac{(1-t_i)(1-t_j)}{t_it_j}\biggr) = \mathbf{Constant}\biggl(Q_{C_A}\overline{Q}_{C_B} + \overline{Q}_{C_A}Q_{C_B}\frac{(1-t_i)(1-t_j)}{t_it_j}\biggr), \quad (25)$$

since the extreme parts of A and B cannot combine to form constants after $t_1t_2t_3 = 1$. We conclude that only cut-offs are needed to calculate the constants.

The following observation is crucial for the constant calculation of the vertex splitting.

Lemma 11. *We have*

$$\mathbf{Constant}\biggl(F\overline{Q}_{C_A} + \overline{F}Q_{C_A}\frac{(1-t_i)(1-t_k)}{t_it_k}\biggr) \mod 2$$
$$= \mathbf{Constant}\biggl(F\overline{Q}_{C_A} + \overline{F}Q_{C_A}\frac{(1-t_1)(1-t_2)}{t_1t_2}\biggr) \mod 2.$$

Proof. Without loss of generality, we assume $i = 1$ and $k = 3$. Then

$$\frac{(1-t_1)(1-t_3)}{t_1t_3} - \frac{(1-t_1)(1-t_2)}{t_1t_2} = (t_2t_3 - 1)\biggl(\frac{1}{t_3} - \frac{1}{t_2}\biggr)$$
$$= \biggl(t_2 + \frac{1}{t_2}\biggr) - \biggl(t_3 + \frac{1}{t_3}\biggr),$$

where we have used $t_1t_2t_3 = 1$.

The difference of the constants in the lemma is

$$\mathbf{Constant}\biggl(F\overline{Q}_{C_A} + \overline{F}Q_{C_A}\biggl(t_2 + \frac{1}{t_2} - t_3 - \frac{1}{t_3}\biggr)\biggr)$$
$$= \mathbf{Constant}\biggl(F\overline{Q}_{C_A}t_2 + \overline{F}Q_{C_A}\frac{1}{t_2}\biggr) + \mathbf{Constant}\biggl(F\overline{Q}_{C_A}\frac{1}{t_2} + \overline{F}Q_{C_A}t_2\biggr)$$
$$- \mathbf{Constant}\biggl(F\overline{Q}_{C_A}t_3 + \overline{F}Q_{C_A}\frac{1}{t_3}\biggr) - \mathbf{Constant}\biggl(F\overline{Q}_{C_A}\frac{1}{t_3} + \overline{F}Q_{C_A}t_3\biggr).$$

Since each line on the right-hand side is of the form $G + \overline{G}$, the right-hand side is even. □

The same argument yields

$$\mathbf{Constant}\biggl(Q_A\overline{Q}_B + \overline{Q}_AQ_B\frac{(1-t_i)(1-t_j)}{t_it_j}\biggr) \mod 2$$
$$= \mathbf{Constant}\biggl(Q_{C_A}\overline{Q}_{C_B} + \overline{Q}_{C_A}Q_{C_B}\frac{(1-t_1)(1-t_2)}{t_1t_2}\biggr) \mod 2.$$

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We can now show that the vertex splitting has even constant term after the restriction $t_1t_2t_3 = 1$. Following the notation of § 4.10, the vertex splitting V^+_α is

$$\begin{aligned} &Q'_\alpha - Q'_\alpha \overline{Q}'_\alpha \frac{(1-t_1)(1-t_2)}{t_1t_2} \\ &\quad - \sum_{i=1}^{3} \left(Q'_\alpha \frac{\overline{Q}_{\alpha\beta_i}}{1-t_i^{-1}} + \overline{Q}'_\alpha \frac{Q_{\alpha\beta_i}}{1-t_i} \right) \frac{(1-t_i)(1-t_{\hat{\imath}})}{t_it_{\hat{\imath}}} \\ &\quad - \sum_{i<j} \left(\frac{Q_{\alpha\beta_i}\overline{Q}_{\alpha\beta_j}}{(1-t_i)(1-t_j^{-1})} + \frac{Q_{\alpha\beta_j}\overline{Q}_{\alpha\beta_i}}{(1-t_i^{-1})(1-t_j)} \right) \frac{(1-t_i)(1-t_j)}{t_1t_j}. \end{aligned}$$

To calculate the constant term after restriction, replace all occurrences of

$$\frac{Q_{\alpha\beta_i}}{1-t_i}$$

by cut-offs $Q_{C_{\alpha\beta_i}}$ satisfying (24) with $F = Q'_\alpha$ and (25). Then, replace all occurrences of

$$\frac{(1-t_i)(1-t_{\hat{\imath}})}{t_it_{\hat{\imath}}}, \quad \frac{(1-t_i)(1-t_j)}{t_it_j}$$

by $(1-t_1)(1-t_2)/t_1t_2$. The moves do not change the value mod 2 of the constant term after restriction.

Let γ be the cut-off partition so

$$Q_\gamma = Q'_\alpha + \sum_{i=1}^{3} Q_{C_{\alpha\beta_i}}.$$

By Lemma 10, we have

$$\begin{aligned} \mathbf{Constant}\bigg(&Q_\gamma - Q_\gamma \overline{Q}_\gamma \frac{(1-t_1)(1-t_2)}{t_1t_2} \\ &- \sum_{i=1}^{3} Q_{C_{\alpha\beta_i}} - Q_{C_{\alpha\beta_i}} \overline{Q}_{C_{\alpha\beta_i}} \frac{(1-t_1)(1-t_2)}{t_1t_2} \bigg) = 0 \mod 2. \end{aligned} \tag{26}$$

After expanding, we find that (26) equals

$$\begin{aligned} \mathbf{Constant}\bigg(&Q'_\alpha - Q'_\alpha \overline{Q}'_\alpha \frac{(1-t_1)(1-t_2)}{t_1t_2} \\ &- \sum_{i=1}^{3} (Q'_\alpha \overline{Q}_{C_{\alpha\beta_i}} + \overline{Q}'_\alpha Q_{C_{\alpha\beta_i}}) \frac{(1-t_1)(1-t_2)}{t_1t_2} \\ &- \sum_{i<j} (Q_{C_{\alpha\beta_i}} \overline{Q}_{C_{\alpha\beta_j}} + Q_{C_{\alpha\beta_i}} \overline{Q}_{C_{\alpha\beta_j}}) \frac{(1-t_1)(1-t_2)}{t_1t_2} \bigg). \end{aligned}$$

Since the latter is the constant term after restriction of V^+_α, we have proven that the constant term of the vertex splitting is even.

The constant term of the edge splitting $\mathsf{E}^+_{\alpha\beta}$ after restriction to $t_1t_2t_3 = 1$ is much more easily studied. A direct analysis from the definitions shows that the edge splitting constant is even. We leave the details to the reader.

Acknowledgements

We thank J. Bryan, A. Iqbal, M. Kontsevich, Y. Soibelman, R. Thomas, and C. Vafa for related discussions. DM was partially supported by a Princeton Centennial graduate fellowship.

AO was partially supported by DMS-0096246 and fellowships from the Sloan and Packard Foundations. RP was partially supported by DMS-0071473 and fellowships from the Sloan and Packard Foundations.

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D. Maulik dmaulik@math.princeton.edu
Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

N. Nekrasov nikita@ihes.fr
Institut de Hautes Études Scientifiques, Bures-sur-Yvette, F-91440, France

A. Okounkov okounkov@math.princeton.edu
Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

R. Pandharipande rahulp@math.princeton.edu
Department of Mathematics, Princeton University, Princeton, NJ 08544, USA