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ON POLYLOGARITHMS*)

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§ 0. Introduction

Some functions related to the complex dilogarithmic function

$$\mathrm{Li}_2(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^2}\,, \qquad |z| < 1 \tag{0.1}$$

(in the notation of Lewin [9]) are known to occur in connection with algebraic K-theory and characteristic classes (see e.g. Bloch [1], Gelfand-MacPherson [7], Dupont [5], and the references given there). Recently MacPherson and Hain (see [10]) has announced results of a similar kind for some higher polylogarithmic functions. Also Ramakrishnan [11] and [12] has recently studied the classical polylogarithms, which for $|z| < 1$ are given by

$$\mathrm{Li}_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n}\,. \tag{0.2}$$

In this note we shall pursue an idea in Bloch [1] and [2] where the dilogarithm takes values in the tensor-product (over $\mathbb{Z}$) $\mathbb{C} \otimes \mathbb{C}^*$, with $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ being respectively the additive and multiplicative group of complex numbers. Thus let

$$L_n^*\colon \mathbb{C}\backslash\{0, 1\} \longrightarrow \mathbb{C}^* \otimes \cdots \otimes \mathbb{C}^* \qquad (n \text{ copies})$$

be given by

$$L_n^*(z) = (1 - z) \otimes z \otimes \cdots \otimes z \qquad (n - 1\ z\text{'s}) \tag{0.3}$$

and consider the exponential map $e\colon \mathbb{C} \otimes \cdots \otimes \mathbb{C} \to \mathbb{C}^* \otimes \cdots \otimes \mathbb{C}^*$

$$e(a_1 \otimes \cdots \otimes a_n) = \exp(2\pi i a_1) \otimes \cdots \otimes \exp(2\pi i a_n)\,, \quad a_1, \cdots, a_n \in \mathbb{C}\,. \tag{0.4}$$

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Then we shall define a natural lift L_n in the diagram

$$\begin{tikzcd} \mathbb{C}\backslash\{0,1\} \arrow[r, "L_n"] \arrow[dr, "L_n^*"'] & \mathbb{C}\otimes\cdots\otimes\mathbb{C} \arrow[d, "e"] \\ & \mathbb{C}^*\otimes\cdots\otimes\mathbb{C}^* \end{tikzcd} \tag{0.5}$$

where L_n is an expression involving the polylogarithms $\mathrm{Li}_1, \cdots, \mathrm{Li}_n$. The explicit formula is given in Corollary 4.7. However, a lift in (0.5) is not unique. Thus e.g. for $n = 2$ one can add terms of the form

$$\frac{\log z}{2\pi i}\otimes 1 - 1\otimes\frac{\log z}{2\pi i}$$

without changing the image in $\mathbb{C}^*\otimes\mathbb{C}^*$. One might therefore ask what motivates our particular choice. First of all the formula for L_n is a natural generalization of the dilogarithm as it occurs in [1] or [5] and, in spite of the length of the formula, it seems difficult to produce anything shorter which is well-defined. More importantly there is a certain recursiveness in the formula for L_n: If one considers the reduction of L_n in the group

$$\mathbb{C}\otimes\cdots\otimes\mathbb{C}/\mathbb{Q}\otimes\cdots\otimes\mathbb{C}/\mathbb{Q}\otimes\cdots\otimes\mathbb{C}$$

(where $\mathbb{C}$ is reduced to $\mathbb{C}/\mathbb{Q}$ in two places) then the image is a tensor product of 2 similar formulas of lower order (formula (5.14) below). This can be exploited when one tries to prove relations for these polylogarithms. It clearly follows from the diagram (0.5) that a relation involving L_n's must necessarily be satisfied by the algebraic symbols L_n^*. We show that one can also work backwards — at least one can reduce the proof of such a relation to some involving only polylogarithms of lower order. As an example we mention 2 relations among trilogarithms which has occurred in our work with C.-H. Sah [6] on the homology of the discrete group $\mathrm{Gl}(n, \mathbb{C})$. (See also [14, Section 4].) There the algebraically defined maps L_n^* naturally occurs in expressions of certain differentials in a spectral sequence. It is expected that the analogous expressions involving the lifts L_n should also give homological invariants for $\mathrm{Gl}(n, \mathbb{C})$ similar to the situation for the dilogarithm in the case $n = 2$.

Finally let us observe that we obtain well-defined *real-valued* polylogarithmic functions as in Ramakrishnan [12] if we compose L_n with one of the n maps $\mathbb{C}\otimes\cdots\otimes\mathbb{C}\to\mathbb{R}$ defined by

$$a_1 \otimes \cdots \otimes a_n \longrightarrow (\mathrm{Re}\, a_1) \cdots (\mathrm{Re}\, a_{i-1})(\mathrm{Im}\, a_i)(\mathrm{Re}\, a_{i+1}) \cdots (\mathrm{Re}\, a_n)$$

$(i = 1, \cdots, n)$ where Re and Im denotes the real and imaginary parts.

§ 1. Iterated integrals

Polylogarithms are defined by iterated integrals in the sense of K.-T. Chen (see [3] and the references given there; see also Hain [8]). Thus let M be a Riemann surface (e.g. $M = \mathbb{C} - \{z_1, \cdots, z_m\}$) and $\omega_1, \cdots, \omega_n$ some holomorphic 1-forms (not necessarily different) on M. Let $\pi\colon \tilde{M} \to M$ be the universal covering and $z_0 \in \tilde{M}$ a base point. The *iterated integral* is defined inductively as a solution on $\tilde{M}$ to the differential equation

$$d\int \omega_1 \cdots \omega_n = \Big(\int \omega_1 \cdots \omega_{n-1}\Big)\omega_n\,. \tag{1.1}$$

More precisely given initial values $c_1, \cdots, c_n \in \mathbb{C}$ the vector $\underline{a} = (1, a_1, \cdots, a_n)$ with

$$a_j = \int \omega_1 \cdots \omega_j \tag{1.2}$$

is a solution to the system of differential equations

$$\begin{aligned} &da_j = a_{j-1}\omega_j, \quad j = 1, \cdots, n, \quad (a_0 = 1) \\ &a_j(z_0) = c_j\,. \end{aligned} \tag{1.3}$$

Explicitly, if γ is a path in $\tilde{M}$ from z_0 to $z \in \tilde{M}$ then

$$a_j(z) = \int^z \omega_1 \cdots \omega_j = \sum_{i=0}^{j-1} c_i \int_\gamma \omega_{i+1} \cdots \omega_j + c_j \qquad (c_0 = 1) \tag{1.4}$$

where $\int_\gamma \omega_{i+1} \cdots \omega_j$, $i < j$, are iterated path-integrals as in Chen [3, Ch. 1]. Thus we are here using the notation $\int \omega_1 \cdots \omega_n$ $\Big($or $\int^z \omega_1 \cdots \omega_n$ when we indicate the variable $z\Big)$ slightly more generally than used by Chen. Therefore in the following when specifying $\int \omega_1 \cdots \omega_n$ it is necessary also to consider all the integrals $\int \omega_{i+1} \cdots \omega_j$, $0 \leqq i < j \leqq n$ at the same time:

Let $N(n, \mathbb{C}) \subseteqq \mathrm{Gl}(n+1, \mathbb{C})$ be the subgroup of unipotent matrices $A = (a_{ij})$ with $a_{ii} = 1$, $a_{ij} = 0$ if $j < i$. Then given $C = (c_{ij}) \in N(n, \mathbb{C})$ we have a unique holomorphic solution $A = A(z)$, $z \in \tilde{M}$, to the matrix equation

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$$\begin{aligned} dA &= A\Omega \\ A(z_0) &= C \end{aligned} \tag{1.5}$$

where Ω is the $(n+1)\times(n+1)$ matrix

$$\Omega = \begin{bmatrix} 0 & \omega_1 & 0 \cdots\cdots 0 \\ 0 & 0 & \omega_2 \cdots\cdots 0 \\ \vdots & & \vdots \\ \vdots & & \omega_n \\ 0 \cdots & \cdots & \cdots\cdots 0 \end{bmatrix}. \tag{1.6}$$

Then $a_{ij} = \int \omega_{i+1} \cdots \omega_j$, $i<j$ with initial condition $a_{ij}(z_0) = c_{ij}$. In particular for γ a path in $\tilde{M}$ starting at z_0 we denote the solution to (1.5) along γ with initial value $A(z_0) = I$ by A_γ. If the endpoint of γ is z then clearly the solution to (1.5) is given by

$$A(z) = CA_\gamma. \tag{1.7}$$

It follows that for two composable paths α and β we have for the composite path $\alpha*\beta$ (first α then β):

$$A_{\alpha*\beta} = A_\alpha A_\beta. \tag{1.8}$$

This is equivalent to the equation in Chen [3, 1.6.1.]

$$\int_{\alpha*\beta} \omega_{i+1} \cdots \omega_j = \sum_{s=i}^{j} \left(\int_\alpha \omega_{i+1} \cdots \omega_s\right)\left(\int_\beta \omega_{s+1} \cdots \omega_j\right) \quad 0 \leqq i < j \leqq n. \tag{1.9}$$

In particular $\gamma \mapsto A_\gamma$ defines a homomorphism of the fundamental group $\pi_1(M, \pi z_0)$ to $N(n, \mathbb{C})$. Let $\Gamma_{z_0} \subseteqq N(n, \mathbb{C})$ be the image. Then for α a closed curve in M at $\pi(z_0)$ the lift of α starting at z_0 defines the monodromy transformation $A_\alpha \in \Gamma_{z_0}$. It follows from (1.7) that if g denotes the covering transformation of $\tilde{M}$ corresponding to α then the solution to (1.5) satisfies

$$A(g_\alpha(z)) = CA_\alpha C^{-1}A(z). \tag{1.10}$$

Hence A gives a well-defined holomorphic mapping

$$A\colon M \longrightarrow C\Gamma_{z_0}C^{-1}\backslash N(n, \mathbb{C}). \tag{1.11}$$

It is useful to see how the monodromy transformation change with change of basepoint. Thus for g a covering transformation of $\tilde{M}$ and $z \in \tilde{M}$ let $A_g(z) \in N(n, \mathbb{C})$ be the monodromy transformation for a curve

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starting at z and representing g in $\pi_1(M, \pi z)$. That is, if $z = z_0$ and g is represented by α then $\Lambda_g(z_0) = A_\alpha$ in the above notation. We now have

PROPOSITION 1.12. i) *Λ_g is a holomorphic solution to the equation*

$$d\Lambda = [\Lambda, \Omega] \tag{1.13}$$

ii) *If A is a solution to* (1.5) *and Λ is a solution to* (1.13) *then $A\Lambda A^{-1}$ is constant, i.e.,*

$$A(z)\Lambda(z)A^{-1}(z) \equiv CA_\alpha C^{-1} \equiv A(g(z))A(z)^{-1}, \qquad \forall z \in \tilde{M}. \tag{1.14}$$

iii) *Suppose $M = \overline{M} - \{z_1, \cdots, z_m\}$ for $\overline{M}$ another Riemann surface and suppose that the singularity of Ω in z_j is at most a simple pole. Then for $g_j \in \pi_1(M)$ corresponding to going once around z_j in the positive direction we have that Λ_{g_j} is holomorphic in z_j and*

$$\Lambda_{g_j}(z_j) = \exp(2\pi i \operatorname{Res}_{z_j}\Omega). \tag{1.15}$$

Proof. ii) follows simply because the left hand side of (1.10) is independent of z_0.

i) follows by setting $C = I$ in (1.14) so that

$$\Lambda(z) = A(z)^{-1}A_\alpha A(z)$$

which is easily seen to satisfy (1.13).

iii) is proved in Deligne [4, théorème II, 1.17].

Remark. Let the constant matrix $T = A(g(z))A(z)^{-1} \in N(n, \mathbb{C})$ have entries t_{ij}. Then t_{ij} only depends on $\omega_{i+1}, \cdots, \omega_j$, together with the constants $c_{i',j'}$, $i \leqq i' \leqq j' \leqq j$. Similar to the notation for iterated integrals we put

$$t_{ij} = t_g(\omega_{i+1} \cdots \omega_j).$$

$\Big($In the case $c_{ij} = 0$ for $i < j$ and $g = g_\alpha$ for α a closed curve at $\pi(z)$ we clearly have $t_g(\omega_{i+1} \cdots \omega_j) = \int_\alpha \omega_{i+1} \cdots \omega_j.\Big)$ Then (1.10) is written

$$\int^{gz} \omega_{i+1} \cdots \omega_j = \sum_{i=i}^{j} t_g(\omega_{i+1} \cdots \omega_s) \int^{z} \omega_{s+1} \cdots \omega_j, \qquad 0 \leqq i < j \leqq n \tag{1.16}$$

similar to (1.9).

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§ 2. Polylogarithms

We now specialize the situation in the previous section to the following case: Let $M = \mathbb{C} - \{0, 1\}$ and let z_0 lie over some point in the domain $U = \{z \,|\, |z| < 1, |1 - z| < 1\}$. Let υ_0 and υ_1 be the 1-forms

(2.1) $$\upsilon_0 = \frac{1}{2\pi i}\frac{dz}{z}, \qquad \upsilon_1 = \frac{1}{2\pi i}\frac{dz}{z-1}.$$

Then as ω_i in the previous section we shall take either υ_0 or υ_1, and in order to specify a particular solution it is enough to exhibit the solution in U.

Case 0. Suppose $\omega_i = \upsilon_0$ for all $i = 1, \cdots, n$. Then we define

(2.2) $$l_n^0(z) = \int^z \upsilon_0^n = \frac{1}{n!}\left(\int^z \upsilon_0\right)^n = \frac{1}{n!\,(2\pi i)^n}(\log z)^n, \quad z \in U$$

where $\log z$ is the usual branch of the logarithm. Notice that all integrals in the matrix $A \in N(n, \mathbb{C})$ given by $a_{ij} = \int \omega_{i+1} \cdots \omega_j$, $0 \leqq i < j \leqq n$, are of the same form.

Case 1. Let $\omega_1 = \upsilon_1$ and $\omega_i = \upsilon_0$ for $i = 2, \cdots, n$. This is the case studied by Ramakrishnan [11]. We define

(2.3) $$l_n^1(z) = \int^z \upsilon_1 \upsilon^{n-1} = -\frac{1}{(2\pi i)^n}\mathrm{Li}_n(z) = -\frac{1}{(2\pi i)^n}\sum_{m=1}^{\infty}\frac{z^m}{m^n}, \quad z \in U.$$

(In particular $\mathrm{Li}_1(z) = -\log(1 - z)$.) Again all integrals in the matrix A with $a_{ij} = \int \omega_{i+1} \cdots \omega_j$, $0 \leqq i < j \leqq n$ are either of the form (2.2) (for $i > 0$) or of the form (2.3).

In general for some fixed k, $1 \leqq k \leqq n$, we shall consider

Case k. Let $\omega_k = \upsilon_1$ and $\omega_i = \upsilon_0$ for $i \neq k$, $i = 1, \cdots, n$. We then define

(2.4) $$l_n^k(z) = \int^z \upsilon_0^{k-1}\upsilon_1\upsilon_0^{n-k} = \sum_{s=1}^{k}(-1)^{s+1}\binom{n-k+s-1}{n-k} l_{n-k+s}^1(z)\, l_{k-s}^0(z).$$

It is straight forward to verify by induction that these functions satisfy (1.1). Again all integrals in the matrix A with $a_{ij} = \int \omega_{i+1} \cdots \omega_j$, $0 \leqq i < j \leqq n$, are of the form (2.2) or (2.4) In fact

$$(2.5) \qquad a_{ij} = \begin{cases} l^0_{j-i}, & \text{for } j < k \text{ or } i \geqq k, \\ l^{k-i}_{j-i}, & \text{for } i < k \leqq j. \end{cases}$$

For later use let us notice the asymptotic behaviour of l^k_n for $z \to 0$ and $z \to 1$:

$$(2.6) \qquad l^k_n(z) = O(|z|(-\log|z|)^{k-1}) \qquad \text{as } z \to 0 \quad (\text{if } 1 \leqq k \leqq n)$$

$$(2.7) \qquad l^k_n(z) = \text{const.} + \begin{cases} O(-|z-1|^{n-1} \log|z-1|) & \text{as } z \to 1 \quad (\text{if } k = 1) \\ O(|z-1|) & \text{as } z \to 1 \quad (\text{if } 1 < k \leqq n). \end{cases}$$

In particular

$$(2.8) \qquad l^k_n(z) \to 0 \qquad \text{for } z \to 0 \quad (\text{if } 1 \leqq k \leqq n)$$

$$(2.9) \qquad l^k_n(z) \to \text{const.} \qquad \text{for } z \to 1 \quad (\text{if } 1 < k \text{ or } 1 < n).$$

We will now determine the monodromy in the above cases: Thus let g_0 and g_1 be the covering transformations of the universal covering of $\mathbb{C} - \{0, 1\}$ corresponding to going once around 0 respectively 1 in the positive direction. The following generalizes Ramakrishnan [11, Theorem]:

THEOREM 2.10. *Given $k = 0, 1, \cdots, n$ let $A \in N(n, \mathbb{C})$ be the matrix $A = (a_{ij})$ given by (2.5). Then the matrices T_{g_0}, $T_{g_1} \in N(n, \mathbb{C})$*

$$T_{g_0} = A(g_0(z))A(z)^{-1} \quad \textit{and} \quad T_{g_1} = A(g_1(z))A(z)^{-1}$$

are given by $T_{g_0} = (t^0_{ij})$, $T_{g_1} = (t^1_{ij})$, where

$$(2.11) \qquad t^0_{ij} = \begin{cases} 1/(j-i)! & \textit{if } i \leqq j < k \textit{ or } k \leqq i \leqq j \\ 0 & \textit{otherwise} \end{cases}$$

$$(2.12) \qquad t^1_{ij} = \begin{cases} 1 & \textit{if } i = j \textit{ or } i = k-1,\ j = k \\ 0 & \textit{otherwise.} \end{cases}$$

In particular A defines a well-defined holomorphic map $A\colon \mathbb{C} - \{0, 1\} \to \Gamma \backslash N(n, \mathbb{C})$ where Γ is a discrete subgroup of matrices with rational entries.

Proof. In the notation of Section 1 consider

$$\Lambda_{g_0} = (\lambda^0_{ij}), \qquad \Lambda_{g_1} = (\lambda^1_{ij}).$$

Then Λ_{g_0} and Λ_{g_1} are determined by Proposition 1.12. That is, they satisfy (1.13) and the initial conditions

$$(2.13) \qquad \Lambda_{g_0}(0) = \exp(2\pi i \operatorname{Res}_0 \Omega), \qquad \Lambda_{g_1}(1) = \exp(2\pi i \operatorname{Res}_1 \Omega).$$

Then

$$T_{g_0} = A\Lambda_{g_0}A^{-1}, \qquad T_{g_1} = A\Lambda_{g_1}A^{-1}. \tag{2.14}$$

Case $k = 0$. In this case A is nonsingular at 1 so that

$$T_{g_1} = \Lambda_{g_1} = I.$$

Also Λ_{g_0} is constant and

$$\lambda^0_{ij} = \begin{cases} 1/(j-i)! & \text{if } i \leqq j \\ 0 & \text{otherwise.} \end{cases}$$

Letting $z \to 1$, $A(z) \to I$ so also

$$t^0_{ij} = \begin{cases} 1/(j-i)! & \text{if } i \leqq j \\ 0 & \text{otherwise.} \end{cases}$$

Case $k > 0$. The statements in (2.11) for $i \leqq j < k$ or $k \leqq i \leqq j$ follows from the case $k = 0$ by the remark following Proposition 1.12. So we only need to determine t^0_{ij} and t^1_{ij} for $i < k \leqq j$. By (2.14)

$$t^\nu_{ij} = \sum_{i \leqq r \leqq s \leqq j} a_{ir}\lambda^\nu_{rs}a^*_{sj}, \qquad \nu = 0, 1 \tag{2.15}$$

where $A = (a_{ij})$ is given by (2.5) and $A^{-1} = (a^*_{sj})$ is the inverse matrix. Notice that in (2.15) the left hand side is constant in z whereas the terms on the right might vary. Also notice that the terms for $r = s$ add up to zero since $\lambda^\nu_{rr} = 1$ and $i \neq j$. Hence

$$t^\nu_{ij} = \sum_{i \leqq r < s \leqq j} a_{ir}\lambda^\nu_{rs}a^*_{sj}, \qquad \nu = 0, 1. \tag{2.16}$$

For $\nu = 1$ notice that λ^1_{rs} is analytic near 1 and that by (2.13)

$$\lambda^1_{rs}(1) = \begin{cases} 1 & \text{if } r = s \text{ or } r = k-1,\ j = k \\ 0 & \text{otherwise.} \end{cases}$$

By (2.5) and (2.7) a_{ir} is bounded near 1 except for $r = k$, $i = k - 1$ where

$$a_{k-1,k}(z) = l^1_1(z) = \frac{1}{2\pi i}\log(1-z).$$

Also a^*_{sj} grows at most as a constant times $-\log|1-z|$ as $z \to 1$, so it follows that

$$t^1_{ij} = \lim_{z \to 1} a_{ik-1}a^*_{kj}.$$

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Here if $i < k-1$, $a_{ik-1} = l^0_{k-i-1}$ has 1 as zero and similarly if $j > k$, a^*_{kj} $(-1)^{j-k}l^0_{j-k}$ also has 1 as a zero so $t^1_{ij} = 0$ unless $i = k-1$, $j = k$, in which case $t^1_{k-1k} = a_{k-1\,k-1}a^*_{kk} = 1$, which proves (2.12).

For $\nu = 0$ in (2.16) we have λ^0_{rs} analytic near 0 and

$$\lambda^0_{rs}(0) = \begin{cases} 1/(s-r)! & \text{if } r \leqq s < k \text{ or } k \leqq r \leqq s \\ 0 & \text{otherwise.} \end{cases}$$

Also it follows from (2.5) and (2.6) that a_{ir} and a^*_{sj} grows at most like a constant times a power of $-\log|z|$ as $z \to 0$. Hence

$$t^0_{ij} = \lim_{z\to 0}\Big[\sum_{i\leqq r<s<k} a_{ir}a^*_{sj}/(s-r)! + \sum_{k\leqq r<s\leqq j} a_{ir}a^*_{sj}/(s-r)!\Big]. \tag{2.17}$$

Using (2.5) and (2.6) it follows that in the first term of (2.17) a_{ir} grows as a constant times $(-\log|z|)^{r-i}$ whereas a^*_{sj} grows as a constant times $|z|(-\log|z|)^{j-s-1}$, that is, the limit is zero for $z \to 0$. Similarly also the second term has limit zero for $z \to 0$ so $t^0_{ij} = 0$. This proves the theorem.

Remark. In the notation of (1.16) we have just shown that for the iterated integrals (2.4)

$$t_{g_0}(v_0^n) = 1/n!\,, \qquad t_{g_1}(v_0^n) = 0 \tag{2.18}$$

$$t_{g_0}(v_0^{k-1}v_1v_0^{n-k}) = 0 \tag{2.19}$$

$$t_{g_1}(v^{k-1}v_1v_0^{n-k}) = \begin{cases} 1 & k = n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

§ 3. Shuffle relations

Until now we have kept fixed the ordering of $\omega_1 \cdots \omega_n$. We shall also need to consider simultaneously all the iterated integrals of the form $\int \omega_{i_1} \cdots \omega_{i_s}$ where $I = (i, \cdots, i_s)$ is a finite sequence of the numbers $1, \cdots, n$. For $I = (i_1, \cdots, i_s)$ and $J = (j_1, \cdots, j_t)$ two such sequences a *shuffle* of I and J is a sequence $K = (k_1, \cdots, k_{s+t})$ such that for s places $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_s \leqq s+t$ we have $k_{\nu_l} = i_l$, $l = 1, \cdots, s$, and for the remaining t places $1 \leqq \mu_1 \leqq \cdots \leqq \mu_t \leqq s+t$, $k_{\mu_m} = j_m$, $m = 1, \cdots, t$. Following R. Ree [13] we shall say that a collection of complex numbers $\{c(I)\}$, $I = (i_1, \cdots, i_s)$, satisfies the shuffle relations if

$$c(i_1, \cdots, i_s)c(j_1, \cdots, j_t) = \sum_K c(k_1, \cdots, k_{s+t}) \tag{3.1}$$

where $K = (k_1, \cdots, k_{s+t})$ runs through all shuffles of I and J. Such a collection corresponds to a formal power series

$$C(X) = \sum_I c(i_1, \cdots, i_t)X_{i_1}X_{i_2} \cdots X_{i_s} \qquad (c_\phi = 1) \tag{3.2}$$

in the non-commuting variables $X_1, \cdots, X_n$, and (3.1) is satisfied if and only if $\log C(X)$ is a Lie-element (Ree [13, Theorem 2.5]). Also let recall (Ree [13, Theorem 2.4]) that the set of such power series is a group under the usual multiplication

$$E(X) = C(X)D(X) \tag{3.3}$$

where

$$e(i_i, \cdots, i_s) = \sum_{\nu=0}^{s} c(i_1, \cdots, i_\nu)d(i_{\nu+1}, \cdots, i_s) . \tag{3.4}$$

Finally the following observation easily follows by induction:

PROPOSITION 3.2. *Suppose the complex numbers $\{c(I)\}$ satisfy the shuffle relations* (3.1). *Then the iterated itegrals $\int \omega_{i_1} \cdots \omega_{i_s}$, $I = (i_i, \cdots, i_s)$, with initial conditions $\int^{z_0} \omega_{i_1} \cdots \omega_{i_s} = c(i_1, \cdots, i_s)$ also satisfy the shuffle relations, that is,*

$$\left(\int \omega_{i_1} \cdots \omega_{i_s}\right)\left(\int \omega_{j_1} \cdots \omega_{j_t}\right) = \sum_K \int \omega_{k_1} \cdots \omega_{k_{s+t}} \tag{3.3}$$

where $K = (k_1, \cdots, k_{s+t})$ runs through all shuffles of I and J.

Remark 1. For $C(X) = \sum_I c(I)X_I$ we thus obtain a formal power series

$$A(z)(X) = \sum_{i_1 \cdots i_s}\left(\int^z \omega_i \cdots \omega_{i_s}\right)X_{i_1} \cdots X_{i_s}$$

where the coefficients are holomorphic functions on M with $A(z_0)(X) = C(X)$. For g a covering transformation of $\tilde{M}$ we have by (1.16):

$$A(gz)(X) = T_g(X) \cdot A(z)(X)$$

where

$$T_g(X) = \sum_{i_1 \cdots i_s} t_g(\omega_{i_1}, \cdots, \omega_{i_s})X_{i_1} \cdots X_{i_s} .$$

Hence we conclude that also the collection $\{t_g(\omega_{i_1} \cdots \omega_{i_s})\}$ satisfies the shuffle relations, i.e.

$$t_g(\omega_{i_1}\cdots\omega_{i_s})t_g(\omega_{j_1}\cdots\omega_{j_t}) = \sum_K t_g(\omega_{k_1}\cdots\omega_{k_{s+t}}). \tag{3.4}$$

Remark 2. If $\{c(I)\}$ satisfy (3.1) then they also satisfy the similar relation with more factors

$$c(i_1,\cdots,i_s)c(j_1,\cdots,j_t)\cdots c(l_1,\cdots,l_v) = \sum_K c(k_1,\cdots,k_{s+t+\cdots+v}) \tag{3.5}$$

with the obvious definition of shuffles of $I, J, \cdots, L$.

Remark 3. In Proposition 3.2 we considered $\int\omega_{i_1}\cdots\omega_{i_s}$ defined for all sequences $I=(i_1,\cdots,i_s)$ of numbers between 1 and n. In our application we shall only specify the integrals $\int\omega_{i_1}\cdots\omega_{i_s}$ for sequences $I=(i_1,\cdots,i_s)$ with no repetitions. In this case the relations (3.3) and (3.4) only make sense (and are true) for I and J disjoint. For example in the case considered in Section 2 where $\omega_1=\upsilon_1$ and $\omega_i=\upsilon_0$, $i=2,\cdots,n$, we have for $I=(i_1,\cdots,i_n)$ a sequence without repetitions $\int\omega_{i_1}\cdots\omega_{i_n}=l_n^k$ for some k. Again (3.3) is shown by induction for disjoint sequences I and J using (2.6) and it follows that also (3.4) holds in this case.

We also need the following formula, which is valid just $c(I)$ is defined for sequences $I=(i_1,\cdots,i_s)$ with no repetitions.

PROPOSITION 3.6. *Suppose $\{c(I)\}$ satisfy the shuffle relations, then*

$$\sum_{s=1}^{n}(-1)^{n-s}\sum_{1\leqq j_1<j_2<\cdots<j_s\leqq n} c(j_s+1,\cdots,j_1)c(j_1+1,\cdots,j_2)\cdots c(j_{s-1}+1,\cdots,j_s)=0 \tag{3.7}$$

for $n>1$. (Here $1,\cdots,n$ are taken in cyclic order).

Proof. We expand each term using (3.5) and want to determine the coefficient of $c(i_1,\cdots,i_n)$ in (3.7) for each permutation $I=(i_1,\cdots,i_n)$ of $(1,\cdots,n)$. Now a product

$$c(j_s+1,\cdots,j_1)c(j_1+1,\cdots,j_2)\cdots c(j_{s-1}+1,\cdots,j_s)$$

contributes to this coefficient if and only if I is a shuffle for the s sequences $(j_s+1,\cdots,j_1),\cdots,(j_{s-1}+1,\cdots,j_s)$. We will call the s-tuple $[j_1<\cdots<j_s]$ a *subdivision* of $(1,\cdots,n)$ and say that I is a shuffle for $[j_1<\cdots<j_s]$. The subdivisions are clearly ordered by inclusion, and it is easy to see that a given permutation $I=(i_1,\cdots,i_n)$ determines a *minimal*

subdivision for which it is a shuffle. Hence if the minimal one has cardinality k then I is a shuffle for $\binom{n-k}{s}$ subdivisions of length $k+s$, $s = 0, \cdots, n-k$. It follows that the coefficient of $c(i_1, \cdots, i_n)$ is

$$\sum_{s=0}^{n-k} (-1)^{k+s}\binom{n-k}{s} = \begin{cases} 0 & \text{if } k < n \\ (-1)^n & \text{if } k = n. \end{cases}$$

But if $n > 1$ it is easy to see that the minimal subdivision for any permutation I has length at most $n-1$ so that the coefficient of $c(I)$ is zero.

§ 4. Tensor valued functions

We now assume that the iterated integrals $\int \omega_{i_1} \cdots \omega_{i_s}$ satisfy Proposition 3.2 and furthermore that the monodromy constants $t_g(\omega_{i_1} \cdots \omega_{i_s})$ are all rational (again it is enough to assume this for sequences $(i_1, \cdots, i_s)$ with no repetitions). In particular the expression

$$\bar{\Phi} = \bar{\Phi}(\omega_1, \cdots, \omega_n) = \int \omega_1 \otimes \cdots \otimes \int \omega_n \tag{4.1}$$

gives a well-defined map

$$\bar{\Phi}\colon M \longrightarrow \mathbb{C}/\mathbb{Q} \otimes \cdots \otimes \mathbb{C}/\mathbb{Q} \qquad (n \text{ copies})$$

where the tensor product is taken over $\mathbb{Q}$.

THEOREM 4.2. *Suppose* $n > 1$. *Then there is a well-defined lift* Φ *in the diagram*

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[dr, "\bar{\Phi}"'] & \mathbb{C} \otimes \cdots \otimes \mathbb{C} \arrow[d] \\ & \mathbb{C}/\mathbb{Q} \otimes \cdots \otimes \mathbb{C}/\mathbb{Q} \end{tikzcd}$$

given by the expression

$$\begin{aligned} \Phi = \Phi(\omega_1, \cdots, \omega_n) = \sum_{s=1}^{n} (-1)^{n-s} \sum_{1 \le j_1 < \cdots < j_s \le n} & \underbrace{1 \otimes \cdots \otimes 1}_{j_1 - 1} \otimes \int \omega_{j_s+1} \cdots \omega_{j_1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_2 - j_1 - 1} \otimes \int \omega_{j_1+1} \cdots \omega_{j_2} \otimes \cdots \\ & \cdots \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_s - j_{s-1} - 1} \otimes \int \omega_{j_{s-1}+1} \cdots \omega_{j_s} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n - j_s} \end{aligned} \tag{4.3}$$

(*where* $\omega_1, \cdots, \omega_n$ *are taken in cyclic order*).

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Proof. We must show that for g a covering transformation of $\tilde{M}$

$$\Phi(gz) = \Phi(z), \qquad \text{for all } z \in \tilde{M}.$$

By (1.16)

$$(4.4) \qquad \Phi(gz) = \sum_{s=1}^{n} (1)^{n-s} \sum_{1 \leqq j_1 < j_2 < \cdots < j_s \leqq n} \sum_{i_1=j_1}^{j_2} \sum_{i_2=j_2}^{j_3} \cdots \sum_{i_s=j_s}^{j_1}$$
$$t_g(\omega_{j_1+1} \cdots \omega_{i_1}) \cdots t_g(\omega_{j_{s-1}+1} \cdots \omega_{i_{s-1}}) t_g(\omega_{j_s+1} \cdots \omega_{i_s})$$
$$\cdot \underbrace{1 \otimes \cdots \otimes 1}_{j_1-1} \otimes \int^z \omega_{i_s+1} \cdots \omega_{j_1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_2-j_1-1} \otimes \int^z \omega_{i_1+1} \cdots \omega_{j_2} \otimes \cdots$$
$$\cdots \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_s-j_{s-1}-1} \otimes \int^z \omega_{i_{s-1}} \cdots \omega_{j_s} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j_s},$$

where again $\omega_1 \cdots \omega_n$ are taken in cyclic order. Thus each term in (4.4) corresponds to a sequence of i's and j's such that either

$$(4.5) \qquad 1 \leqq j_1 \leqq i_1 \leqq j_2 \leqq i_2 \cdots \leqq j_s \leqq i_s \leqq n$$

or

$$(4.6) \qquad 1 \leqq i_s \leqq j_1 \leqq i_1 \leqq j_2 \cdots \leqq i_{s-1} \leqq j_s \leqq n$$

with strict inequality among the j's. The terms in which $j_p = i_p$ for all $p = 1, \cdots, s$, are exactly the terms of $\Phi(z)$, so in the remaining terms we have $j_p < i_p$ for some p. In this case we have either

a) $j_p < i_p = j_{p+1}$, or

b) $j_p < i_p < j_{p+1}$.

Now any term of type b) cancels with one of type a). In fact if a sequence as in say (4.5) satisfies b) then the corresponding term cancels with the term corresponding to $j'_1, i'_1, \cdots, j'_{s+1}, i'_{s+1}$ where

$$i'_\nu = \begin{cases} i_\nu, & \text{for } \nu \leqq p, \\ i_{\nu-1}, & \text{for } \nu > p, \end{cases} \qquad j'_\nu = \begin{cases} j_\nu, & \text{for } \nu \leqq p, \\ i_p, & \text{for } \nu = p, \\ j_{\nu-1}, & \text{for } \nu > p+1. \end{cases}$$

It follows that the only non-cancelling terms of $\Phi(gz) - \Phi(z)$ are the terms in which

$$\cdots j_p < i_p = j_{p+1} < i_{p+1} = j_{p+2} \cdots$$

and since this is true in cyclic order we obtain $i_p = j_{p+1}$ for all $p = 1, \cdots, s-1$, $i_s = j_1$. Hence

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$$\begin{aligned}\Phi(gz) - \Phi(z) &= \sum_{s=1}^{n} (-1)^{n-s} \sum_{1 \leqq j_1 < \cdots < j_s \leqq n} t_g(\omega_{j_s+1} \cdots \omega_{j_1}) t_g(\omega_{j_1+1} \cdots \omega_{j_2}) \cdots \\ &\qquad \cdots t_g(\omega_{j_{s-1}+1} \cdots \omega_{j_s}) \cdot \underbrace{1 \otimes \cdots \otimes 1}_{n} \\ &= 0\end{aligned}$$

by Proposition 3.6. This proves the theorem.

In particular in the case of polylogarithms we use the notation l_n^k of Section 2, and obtain a lift of the function L_n^* defined by (0.3) in the introduction:

COROLLARY 4.7. *Suppose $n > 1$. Then there is well-defined lift L_n in the diagram*

$$\begin{tikzcd} \mathbb{C} - \{0, 1\} \arrow[r, "L_n"] \arrow[dr, "L_n^*"'] & \mathbb{C} \otimes \cdots \otimes \mathbb{C} \arrow[d, "e"] \\ & \mathbb{C}^* \otimes \cdots \otimes \mathbb{C}^* \end{tikzcd}$$

given by

$$(4.8) \qquad \begin{aligned} L_n = \sum_{s=1}^{n} (-1)^{n-s} \sum_{1 \leqq j_1 < \cdots < j_s \leqq n} & \underbrace{1 \otimes \cdots \otimes 1}_{j_1 - 1} \otimes l_{n-j_s+j_1}^{n-j_s+1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_2 - j_1 - 1} \otimes \\ & \otimes l_{j_2 - j_1}^0 \otimes \cdots \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_s - j_{s-1} - 1} \otimes l_{j_s - j_{s-1}}^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n - j_s} . \end{aligned}$$

Remark 1. Let σ denote the cyclic permutation of $\mathbb{C} \otimes \cdots \otimes \mathbb{C}$ given by $\sigma(a_1 \otimes \cdots \otimes a_n) = a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}$. Then it is easy to check that

$$\sigma \circ \Phi(\omega_1, \cdots, \omega_n) = \Phi(\omega_n, \omega_1, \cdots, \omega_{n-1})$$

and hence by iteration

$$\sigma^k \circ \Phi(\omega_1, \cdots, \omega_n) = \Phi(\omega_{n-k+1}, \cdots, \omega_n, \omega_1, \cdots, \omega_{n-k})$$

for $k = 0, \cdots, n-1$. In particular

$$(4.9) \qquad L_n^k = \sigma^{k-1} \circ L_n = \Phi(\upsilon_0, \cdots, \upsilon_1, \cdots, \upsilon_0), \qquad k = 1, \cdots, n.$$

with υ_1 on the k'th place corresponding to the "k'th case" of Section 2. Clearly $\sigma^k \circ L_n$ is a lift of

$$(4.10) \qquad L_n^{k*}(z) = z \otimes \cdots \otimes (1-z) \otimes \cdots \otimes z \in \mathbb{C}^* \otimes \cdots \otimes \mathbb{C}^*$$

with $(1-z)$ on the k'th place. In the next section we shall also use the notation

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$$L_n^0 = \Phi(v_0, \cdots, v_0) = \sum_{s=1}^{n} (-1)^{n-s} \sum_{1 \leq j_1 < \cdots < j_s \leq n} \underbrace{1 \otimes \cdots \otimes 1}_{j_1 - 1} \otimes l^0_{n-j_s+j_1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_2 - j_1 - 1} \otimes l^0_{j_2 - j_1 - 1} \otimes \cdots \cdots \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_s - j_{s-1} - 1} \otimes l^0_{j_s - j_{s-1}} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n - j_s}. \tag{4.11}$$

which is the lift of

$$L_n^{0*}(z) = z \otimes \cdots \otimes z \in \mathbb{C}^* \otimes \cdots \mathbb{C}^*. \tag{4.12}$$

Remark 2. By Proposition 3.6 it clearly follows that $\Phi(\omega_1, \cdots, \omega_n)$ is in the kernel of the natural homomorphism $\mu\colon \mathbb{C} \otimes \cdots \otimes \mathbb{C} \to \mathbb{C}$ given by multiplication:

$$\mu(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_n. \tag{4.13}$$

§ 5. Relations

We will finally study relations among tensor valued functions $\Phi(\omega_1, \cdots, \omega_n)$ as in (4.3) for different choices of ω_i's. More precisely consider $\Psi\colon M \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}$ of the form

$$\Psi = \sum_{\nu=1}^{m} \Phi(\omega_1^\nu, \cdots, \omega_n^\nu) \tag{5.1}$$

and we want to find conditions for Ψ to be constant. (Relations in more than one variable can of course be reduced to this case by keeping some variables fixed). A necessary condition is of course that

$$\bar{\Psi} = \sum_{\nu=1}^{m} \bar{\Phi}(\omega_1^\nu, \cdots, \omega_n^\nu) = \sum_{\nu=1}^{m} \int \omega_1^\nu \otimes \cdots \otimes \int \omega_n^\nu \tag{5.2}$$

is constant in $\mathbb{C}/\mathbb{Q} \otimes \cdots \otimes \mathbb{C}/\mathbb{Q}$. As we shall see it is possible successively to work backwards from (5.2) to (5.1). First let

$$r_k\colon \mathbb{C} \otimes \cdots \otimes \mathbb{C} \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}/\mathbb{Q} \otimes \cdots \otimes \mathbb{C}$$

be the natural reduction mod $\mathbb{Q}$ on the k'th factor.

PROPOSITION 5.3. *Let $\Psi\colon M \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}$ be given by* (5.1). *Then Ψ is constant if and only if $r_k \circ \Psi$ is constant for all $k = 1, \cdots, n$.*

Proof. Choose $z_0 \in M$ and put $\alpha = \Psi(z_0)$. Then given $z \in M$,

$$r_k(\Psi(z) - \alpha) = 0, \qquad \text{for all } k = 1, \cdots, n. \tag{5.4}$$

Hence

$$\Psi(z) - \alpha \in \mathbb{Q} \otimes \cdots \otimes \mathbb{Q} \xrightarrow[\cong]{\mu} \mathbb{Q}$$

where $\mu\colon \mathbb{C} \otimes \cdots \otimes \mathbb{C} \to \mathbb{C}$ is given in (4.13). Now $\mu \circ (\Psi - \alpha)\colon M \to \mathbb{C}$ is locally given by a holomorphic function and so, since it takes only rational values, must be constant. Therefore also $\Psi - \alpha$ is constant and hence 0, so $\Psi = \alpha$.

Remark. By the last remark of Section 4 if $\Psi = \alpha$ is constant, then α is in the kernel of $\mu\colon \mathbb{C} \otimes \cdots \otimes \mathbb{C} \to \mathbb{C}$ and hence is uniquely determined by its reductions $r_k\alpha$.

Thus Proposition 5.3 reduces the verification of a relation among $\Phi(\omega_1^\nu, \cdots, \omega_n^\nu)$ to a relation among $r_k\Phi(\omega_1^\nu, \cdots, \omega_n^\nu)$ for each $k = 1, \cdots, n$. Now, if we put

$$\hat{\Phi}(\omega_1, \cdots, \omega_n) = r_n\Phi(\omega_1, \cdots, \omega_n) \in \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes \mathbb{C}/\mathbb{Q} \tag{5.5}$$

then by the Remark 1 following Corollary 4.7

$$\sigma^{n-k} \circ r_k\Phi(\omega_1, \cdots, \omega_n) = \hat{\Phi}(\omega_{k+1}, \cdots, \omega_n, \omega_1, \cdots, \omega_k)\,.$$

Hence we conclude that the conditions of Proposition 5.3 are equivalent to

$$\sum_{\nu=1}^{m} \hat{\Phi}(\omega_{k+1}^\nu, \cdots, \omega_n^\nu, \omega_1^\nu, \cdots, \omega_k^\nu) = \text{const.}, \qquad k = 0, 1, \cdots, n-1. \tag{5.6}$$

Here $\hat{\Phi}(\omega_1, \cdots, \omega_n)$ is given by

$$\begin{aligned} \hat{\Phi}(\omega_1, \cdots, \omega_n) = \sum_{s=0}^{n-1} (-1)^{n-s+1} & \sum_{1 \leq j_1 < \cdots < j_s \leq n-1} \\ & \underbrace{1 \otimes \cdots \otimes 1}_{j_1 - 1} \otimes \int \omega_1 \cdots \omega_{j_1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_2 - j_1 - 1} \otimes \cdots \otimes \\ & \underbrace{1 \otimes \cdots \otimes 1}_{n - j_s - 1} \otimes \int \omega_{j_s + 1} \cdots \omega_n \end{aligned} \tag{5.7}$$

where the term corresponding to $s = 0$ is just

$$1 \otimes \cdots \otimes 1 \otimes \int \omega_1 \cdots \omega_n\,.$$

Again let

$$r_k\colon \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes \mathbb{C}/\mathbb{Q} \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}/\mathbb{Q} \otimes \cdots \otimes \mathbb{C} \otimes \mathbb{C}/\mathbb{Q} \tag{5.8}$$

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denote the reduction mod $\mathbb{Q}$ on the k'th factor. Then we have (cf. Bloch [2, Corollary 6.2.3]).

THEOREM 5.9. *Let* $\hat{\Psi}: M \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}/\mathbb{Q}$ *be given by*

$$\hat{\Psi} = \sum_{\nu=1}^{m} \hat{\Phi}(\omega_1^\nu, \cdots, \omega_n^\nu) . \tag{5.10}$$

Then $\hat{\Psi}$ *is constant if and only if* $r_k \circ \hat{\Psi}$ *is constant for all* $k = 1, \cdots, n-1$.

Proof. Choose $z_0 \in M$ and put $\hat{\alpha} = \hat{\Psi}(z)$. We want to show that $\hat{\Psi} - \hat{\alpha}$ is constant in a neighborhood of z_0. Now by assumption for each $z \in M$

$$\hat{\Psi}(z) - \hat{\alpha} = 1 \otimes \cdots \otimes 1 \otimes a(z) \tag{5.11}$$

and we claim that a can be chosen complex analytic in a neighborhood U of z_0. In fact by construction $\hat{\Psi} - \hat{\alpha}$ is represented in a suitable U by such functions, i.e. we can represent

$$(\hat{\Psi} - \hat{\alpha})|_U \in A \otimes \cdots \otimes A$$

where A is the ring of analytic functions in U. Now if we write

$$(\hat{\Psi} - \hat{\alpha})|_U = \sum q_{i_1 \cdots i_n} a_{i_1} \otimes \cdots \otimes a_{i_n} \tag{5.12}$$

where $q_{i_1 \cdots i_n} \in \mathbb{Q}$ and $a_1, \cdots, a_N \in A$ are linearly independent over $\mathbb{Q}$ then a cardinality argument shows that for some $z \in U$ also $a_1(z), \cdots, a_N(z)$ are linearly independent over $\mathbb{Q}$ from which it follows together with (5.11) that (5.12) must have the form

$$(\hat{\Psi} - \hat{\alpha})|_U = 1 \otimes \cdots \otimes 1 \otimes a + \beta \otimes 1$$

thus proving the claim. To show that a is constant we shall prove that $(\hat{\Psi} - \hat{\alpha})|_U$ is in the kernel of the map

$$D: \underbrace{A \otimes \cdots \otimes A}_{n} \longrightarrow \underbrace{A \otimes \cdots \otimes A}_{n-2} \otimes \Omega_{A/\mathbb{C}}$$

given by

$$D(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{n-1} da_n$$

where $d: A \to \Omega_{A/\mathbb{C}}$ is the usual differential. This will then show that a in (5.11) is in the kernel of the composite

$$A \xrightarrow{1 \otimes \cdots \otimes 1 \otimes \mathrm{id}} \underbrace{A \otimes \cdots \otimes A}_{n-1} \xrightarrow{\mathrm{id} \otimes d} \underbrace{A \otimes \cdots \otimes A}_{n-2} \otimes \Omega_{A/\mathbb{C}}$$

and since this kernel clearly consists of the constants it follows that a is constant. Thus it remains to show

$$D\hat{\Phi}(\omega_1, \cdots, \omega_n) = 0 \tag{5.13}$$

where $\hat{\Phi}(\omega_1, \cdots, \omega_n)$ is given by (5.7). Now using

$$d\int \omega_{j_s+1} \cdots \omega_n = \Big(\int \omega_{j_s+1} \cdots \omega_{n-1}\Big)\omega_n \quad \text{and} \quad d\int \omega_n = \omega_n$$

we obtain

$$\begin{aligned} D\hat{\Phi} = {} & \sum_{s=0}^{n-2} (-1)^{n-s+1} \sum_{1 \leq j_1 < \cdots < j_s < n-1} \\ & \underbrace{1 \otimes \cdots \otimes 1}_{j_1-1} \otimes \int \omega_1 \otimes \cdots \otimes \omega_{j_1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j_2-j_1-1} \otimes \cdots \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j_s-2} \\ & \qquad \otimes \Big(\int \omega_{j_s+1} \cdots \omega_{n-1}\Big)\omega_n \\ & + \sum_{s=1}^{n-1} (-1)^{n-s+1} \sum_{1 \leq j_1 < \cdots < j_{s-1} < j_s = n-1} \\ & \underbrace{1 \otimes \cdots \otimes 1}_{j_1-1} \otimes \int \omega_1 \cdots \omega_{j_1} \otimes \cdots \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j_{s-1}-1} \otimes \Big(\int \omega_{j_{s-1}+1} \cdots \omega_{n-1}\Big)\omega_n \\ = {} & 0 \end{aligned}$$

which proves (5.13) and ends the proof of the theorem.

Remark. Notice that

$$r_k\hat{\Phi}(\omega_1, \cdots, \omega_n) = \hat{\Phi}(\omega_1, \cdots, \omega_k) \otimes \hat{\Phi}(\omega_{k+1}, \cdots, \omega_n)\,. \tag{5.14}$$

It follows that relations among functions of the form $\hat{\Phi}(\omega_1, \cdots, \omega_n)$ are reduced by Theorem 5.9 to relations among functions of the form $\hat{\Phi}(\omega, \cdots, \omega_k)$, $k < n$.

This remark together with Proposition 5.3 means that any relation of the form

$$\sum_{\nu=1}^{m} \Phi(\omega_1^\nu, \cdots, \omega_n^\nu) = \text{const.}$$

eventually can be deduced from relations among the functions $\int \omega_1^\nu$ with values in $\mathbf{C}/\mathbf{Q}$. In particular relations among the polylogarithmic functions L_n^k, $k = 0, \cdots, n$, defined in Section 4 can all be deduced from the defining relation for the ordinary logarithm

$$\log(z + w) = \log z + \log w \qquad \text{mod } 2\pi i.$$

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As an illustration of this principle let us mention the following relations in one variable among the trilogarithmic functions $L^i = L_3^i$, $i = 0, \cdots, 3$, defined by (4.8), (4.9), (4.11) and with values in $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$:

COROLLARY 5.15. *For $z \in \mathbb{C} \setminus \{0\}$ we have*

i) $L^1(z^{-1}) - L^1(z) + L^0(z) = 0$

ii) $$L^1(1-z) + L^1\left(1 - \frac{1}{z}\right) = L^2(z) + L^3(z) + L^0(z) + 1 \otimes 1 \otimes c - 2(1 \otimes c \otimes 1) + c \otimes 1 \otimes 1$$

where $c = -\frac{1}{(2\pi i)^3} \mathrm{Li}_3(1) = -\frac{1}{(2\pi i)^3} \sum_{k=1}^{\infty} 1/k^3$.

These identities are analogous to the identities in Lewin's book [9, A.2.6, (5)–(9)]. We will not give the proof which is straight forward by successive applications of Theorem 5.9, the remark (5.14), and Proposition 5.3. The constant in ii) is determined by evaluation on $\sqrt[3]{-1} = e^{\pi i/3}$.

From these identities one easily deduce

COROLLARY 5.16. *For $z \in \mathbb{C} - \{0\}$ put $R(z) = L^1(z) - L^2(z)$: Then*

i) $R(z^{-1}) - R(z) = 0$

ii) $R(1-z) + R\left(1 - \frac{1}{z}\right) + R(z) = 3(1 \otimes 1 \otimes c - 1 \otimes c \otimes 1)$

It is interesting to note that relations of exactly this form naturally occur in the homology of the chain complex of configurations in the complex projective plane. We shall return to this subject and its connection with the homology of the discrete group $\mathrm{Gl}(n, \mathbb{C})$ in a paper with Sah [6] (see also Sah [14, Section 4]).

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