



RESEARCH ARTICLE

# Detecting and describing ramification for structured ring spectra

Eva Höning<sup>1</sup> and Birgit Richter<sup>2</sup> 

<sup>1</sup>Department of Mathematics, Radboud University, Nijmegen, The Netherlands

<sup>2</sup>Fachbereich Mathematik der Universität Hamburg, Hamburg, Germany

**Corresponding author:** Birgit Richter; Email: [birgit.richter@uni-hamburg.de](mailto:birgit.richter@uni-hamburg.de)

**Received:** 4 October 2022; **Revised:** 15 March 2023; **Accepted:** 16 March 2023; **First published online:** 24 April 2023

**Keywords:** Tame ramification; wild ramification; topological André–Quillen homology; topological Hochschild homology; topological modular forms; Tate vanishing

**2020 Mathematics Subject Classification:** *Primary* - 55P43; *Secondary* - 11S15, 11R04

## Abstract

John Rognes developed a notion of Galois extension of commutative ring spectra, and this includes a criterion for identifying an extension as unramified. Ramification for commutative ring spectra can be detected by relative topological Hochschild homology and by topological André–Quillen homology. In the classical algebraic context, it is important to distinguish between tame and wild ramification. Noether’s theorem characterizes tame ramification in terms of a normal basis, and tame ramification can also be detected via the surjectivity of the trace map. For commutative ring spectra, we suggest to study the Tate construction as a suitable analog. It tells us at which integral primes there is tame or wild ramification, and we determine its homotopy type in examples in the context of topological K-theory and topological modular forms.

## 1. Introduction

Classically, ramification is studied in the setting of extensions of rings of integers in number fields. If  $K \subset L$  is an extension of number fields and if  $\mathcal{O}_K \rightarrow \mathcal{O}_L$  is the corresponding extension of rings of integers, then a prime ideal  $\mathfrak{p} \subset \mathcal{O}_K$  ramifies in  $L$ , if  $\mathfrak{p}\mathcal{O}_L = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$  in  $\mathcal{O}_L$  and  $e_i > 1$  for at least one  $1 \leq i \leq s$ . The ramification is *tame* when the ramification indices  $e_i$  are all relatively prime to the residue characteristic of  $\mathfrak{p}$ , and it is *wild* otherwise. Auslander and Buchsbaum [1] considered ramification in the setting of general noetherian rings. If  $K \subset L$  is a finite  $G$ -Galois extension, then  $\mathcal{O}_K \rightarrow \mathcal{O}_L$  is unramified, if and only if  $\mathcal{O}_K = \mathcal{O}_L^G \rightarrow \mathcal{O}_L$  is a Galois extension of commutative rings, and this in turn says that  $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \cong \prod_G \mathcal{O}_L$  (see [13, Remark 1.5 (d)], [1] or [44, Example 2.3.3] for more details).

Our main interest is to investigate notions of ramified extensions of ring spectra and to study examples.

### 1.1. Galois extensions

Rognes [44, Definition 4.1.3] introduces  $G$ -Galois extensions of ring spectra. A map  $A \rightarrow B$  of commutative ring spectra is a  $G$ -Galois extension for a finite group  $G$ , if certain cofibrancy conditions are satisfied, if  $G$  acts on  $B$  from the left through commutative  $A$ -algebra maps and if the following two conditions are satisfied:

1. The map from  $A$  to the homotopy fixed points of  $B$  with respect to the  $G$ -action,  $i: A \rightarrow B^{hG}$  is a weak equivalence.

## 2. The map

$$h : B \wedge_A B \rightarrow \prod_G B \quad (1.1)$$

is a weak equivalence.

Here,  $h$  is right adjoint to the composite map:

$$B \wedge_A B \wedge G_+ \longrightarrow B \wedge_A B \longrightarrow B,$$

induced by the  $G$ -action  $B \wedge G_+ \cong G_+ \wedge B \rightarrow B$  on  $B$  and the multiplication on  $B$ .

Condition (1) is the fixed point condition familiar from ordinary Galois theory. Condition (2) is needed to ensure that the map  $A \rightarrow B$  is *unramified*. Among other things, it implies for instance that the  $A$ -endomorphisms of  $B$ ,  $F_A(B, B)$ , correspond to the group elements in  $G$  in the sense that

$$j : B \wedge G_+ \rightarrow F_A(B, B),$$

is a weak equivalence, where  $j$  is right adjoint to the composite map

$$(B \wedge G_+) \wedge_A B \rightarrow B \wedge_A B \rightarrow B,$$

which is again induced by the  $G$ -action and the multiplication on  $B$ .

If  $A$  is the Eilenberg–MacLane spectrum  $H\mathcal{O}_K$  and  $B = H\mathcal{O}_L$  for a  $G$ -Galois extension  $K \subset L$ , then  $H\mathcal{O}_K \rightarrow H\mathcal{O}_L$  is a  $G$ -Galois extension of ring spectra if and only if  $\mathcal{O}_K \rightarrow \mathcal{O}_L$  is a  $G$ -Galois extension of commutative rings.

### 1.2. Ramification

For certain Galois extensions, Ausoni and Rognes [3] conjecture a version of Galois descent for algebraic K-theory. A descent result that covers many of the conjectured cases is established in [15]. In some cases, descent can be established even in the presence of ramification. Ausoni [2, Theorem 10.2] shows for instance that the canonical map  $K(\ell_p) \rightarrow K(ku_p)^{hC_{p-1}}$  is an equivalence after  $p$ -completion despite the fact that the inclusion of the  $p$ -completed connective Adams summand,  $\ell_p$ , into  $p$ -completed topological connective K-theory,  $ku_p$ , should be viewed as a tamely ramified extension of commutative ring spectra. In other cases that are not Galois extensions, for instance in the case of the map  $ko \rightarrow ku$  from real to complex connective topological K-theory that shows features of a wildly ramified extension, one can consider a modified version of descent [15, §5.4].

How can we detect ramification? The unramified condition from (1.1) ensures for instance that  $A \rightarrow B$  is separable [44, Definition 9.1.1], and this in turn implies that the canonical map from  $B$  to the relative topological Hochschild homology,  $\mathrm{THH}^A(B)$ , is an equivalence and that the spectrum of topological André–Quillen homology  $\mathrm{TAQ}^A(B)$  [6] is trivial. So if we know for a map of commutative ring spectra  $A \rightarrow B$  that  $B \rightarrow \mathrm{THH}^A(B)$  is not a weak equivalence or that  $\pi_* \mathrm{TAQ}^A(B) \neq 0$ , then this is an indicator for ramification. We will study examples of nonvanishing  $\mathrm{TAQ}$  in Section 2.1 and study relative topological Hochschild homology in examples related to level-2-structures on elliptic curves in Section 2.2.

### 1.3. Types of ramification

Whereas detecting ramification for structured ring spectra is rather straightforward in many cases, it is less clear whether a map  $A \rightarrow B$  is tamely or wildly ramified; it might also be that there are more types of ramification. Several methods from algebra do not carry over. One obstacle is for instance that there is no concept of ideals for commutative ring spectra that has all features of the algebraic one. Jeff Smith proposed a definition of ideals (see [25] for an available account), but this notion is not geared toward the case of *commutative* ring spectra. Determining the homotopy type of the spectrum of topological André–Quillen homology is often hard with the notable exception of suitable Thom spectra [8, Theorem 5 and Corollary]. Therefore, we did not see a way of studying its annihilator as an analog of the different.

Spectra with trivial negative homotopy groups are called connective. For a spectrum  $A$ , we denote by  $\tau_{\geq 0}A$  its connective cover. It is a connective spectrum whose homotopy groups in nonnegative degrees agree with those of  $A$ . Akhil Mathew shows in [36, Theorem 6.17] that connective Galois extensions are algebraically étale: the induced map on homotopy groups is étale in a graded sense. So, in particular, connective covers of Galois extensions are rarely Galois extensions, because several known examples such as  $KO \rightarrow KU$  and examples of Galois extensions in the context of topological modular forms are far from behaving nicely on the level of homotopy groups. We like to think of these connective covers as analog of rings of integers in number fields, but we cannot offer any systematic approach behind this interpretation. If you start with a periodic ring spectrum, then just cutting away the negative homotopy groups might not produce a good connective model. In such cases, there might be several meaningful choices for connective models and the analogy is then even less clear.

We determine relative topological Hochschild homology and the bottom nontrivial homotopy groups of the topological André–Quillen spectrum in the cases  $ko \rightarrow ku$ ,  $\ell \rightarrow ku_{(p)}$ ,  $\mathrm{tmf}_0(3)_{(2)} \rightarrow \mathrm{tmf}_1(3)_{(2)}$ ,  $\mathrm{tmf}_{(3)} \rightarrow \mathrm{tmf}_0(2)_{(3)}$ ,  $\mathrm{Tmf}_{(3)} \rightarrow \mathrm{Tmf}_0(2)_{(3)}$ , and  $\mathrm{tmf}_0(2)_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}$ . We also study a version of the discriminant map in the context of structured ring spectra and apply it to the examples  $\ell \rightarrow ku_{(p)}$  and  $ko \rightarrow ku$  in Section 2.3.

For certain finite extensions of discrete valuation rings, tame ramification is equivalent to being log-étale (see for instance [46, Example 4.32]). It is known by work of Sagave [50] that  $\ell \rightarrow ku_{(p)}$  is log-étale if one considers the log structures generated by  $v_1 \in \pi_{2p-2}\ell$  and  $u \in \pi_2(ku)$ . We show that  $ko \rightarrow ku$  is not log-étale if one considers the log structures generated by the Bott elements  $\omega \in \pi_8(ko)$  and  $u \in \pi_2(ku)$ . This might be seen as an indicator for wild ramification in this case.

#### 1.4. Tate cohomology and Tate construction

Emmy Noether shows [42, §2] that tame ramification is equivalent to the existence of a normal basis. Tame ramification can also be detected by the surjectivity of the trace map [12, Theorem 2, Chapter 1, §5]. This in turn yields a vanishing of Tate cohomology.

In stable homotopy theory, Tate cohomology is modeled by the Tate construction. If  $E$  is a spectrum with an action of a finite group  $G$ , then there is a norm map  $N : E_{hG} \rightarrow E^{hG}$  from the homotopy orbits of  $E$  with respect to  $G$ ,  $E_{hG}$ , to the homotopy fixed points,  $E^{hG}$ . Its cofiber is the Tate construction of  $E$  with respect to  $G$ ,  $E^{tG}$ . If  $E$  is an Eilenberg–MacLane spectrum  $E = HA$ , then the homotopy groups of the Tate construction agree with the Tate cohomology groups in the sense that  $\pi_*(HA^{tG}) \cong \hat{H}^{-*}(G; A)$ .

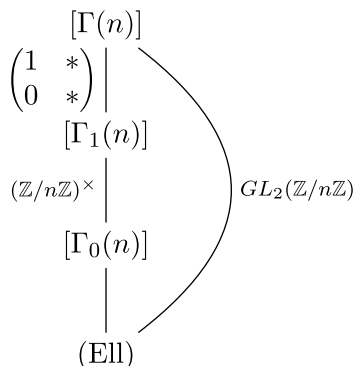
Using Tate spectra as a possible criterion for wild ramification is for instance suggested by Rognes in [47] and in [37]. Rognes also shows a version of Noether’s theorem in [45, Theorem 5.2.5]: if a spectrum with a  $G$ -action  $X$  is in the thick subcategory generated by spectra of the form  $G_+ \wedge W$ , then  $X^{tG} \simeq *$ , so in particular, if  $B$  has a normal basis,  $B \simeq G_+ \wedge A$ , then  $B^{tG} \simeq *$ .

For a finite group  $G$  and a connective spectrum  $B$ , the Tate construction  $B^{tG}$  is trivial if and only if the unit  $1 \in \pi_0 B$  is in the image of the algebraic norm map. We study examples in the context of topological K-theory, topological modular forms, and cochains on classifying spaces with coefficients in Lubin–Tate spectra (also known as Morava E-theory) whose Tate construction is nontrivial. Our hope is that the structure of the Tate construction in such cases might tell us something about the type of ramification. In the examples where we can completely determine the homotopy type of the Tate construction, however, we obtain generalized Eilenberg–Mac Lane spectra.

#### 1.5. Topological modular forms

Several of our examples use topological modular forms with level structures. The spectrum of topological modular forms,  $\mathrm{TMF}$ , arises as the global sections of a structure sheaf of  $E_\infty$ -ring spectra on the moduli stack of elliptic curves,  $\mathcal{M}_{\mathrm{ell}}$ . A variant of it,  $\mathrm{Tmf}$ , lives on a compactified version,  $\overline{\mathcal{M}}_{\mathrm{ell}}$ . Its connective version is denoted by  $\mathrm{tmf}$ . There are other variants corresponding to level structures on

elliptic curves. Recall that a  $\Gamma(n)$ -structure (or level  $n$ -structure for short) carries the datum of a chosen isomorphism between the  $n$ -torsion points of an elliptic curve and the group  $(\mathbb{Z}/n\mathbb{Z})^2$ . A  $\Gamma_1(n)$ -structure corresponds to the choice of a point of exact order  $n$ , whereas a  $\Gamma_0(n)$ -structure comes from the choice of a subgroup of order  $n$  of the  $n$ -torsion points. See [27, Chapter 3] for the precise definitions and for background. These level structures give rise to a tower of moduli problems (see [27, p. 200] and [17])



with corresponding commutative ring spectra  $\mathrm{TMF}_0(n) \rightarrow \mathrm{TMF}_1(n) \rightarrow \mathrm{TMF}(n)$  and their compactified versions  $\mathrm{Tmf}_0(n) \rightarrow \mathrm{Tmf}_1(n) \rightarrow \mathrm{Tmf}(n)$  [23, Theorem 6.1].

In [37], Mathew and Meier prove that the maps  $\mathrm{Tmf} \left[ \frac{1}{n} \right] \rightarrow \mathrm{Tmf}(n)$  are *not* Galois extensions but they satisfy Tate vanishing, which might be seen as an indication of tame ramification. In contrast, we will identify cases when  $\mathrm{tmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})}$  is nontrivial (see Theorem 3.13).

This paper is intended as a starting point for the investigation of different types of ramification for structured ring spectra. We are aware that for a deeper understanding of ramification, one probably needs to use stacks (see e.g., [36, 41]).

## 2. Detecting ramification

For connective commutative ring spectra that satisfy a mild finiteness condition, the common notions of étaleness are all equivalent [35, Corollary 3.1]: for a map  $A \rightarrow B$  of such spectra  $\mathrm{TAQ}^A(B) \simeq *$  if and only if the natural map  $B \rightarrow \mathrm{THH}^A(B)$  is an equivalence if and only if  $A \rightarrow B$  is étale in the sense of Lurie [32, Definition 7.5.1.4], in particular

$$\pi_* B \cong \pi_*(A) \otimes_{\pi_0(A)} \pi_0(B).$$

So we know that in the following examples there *is* ramification. The question is whether the invariants that are used can tell us something about the type of ramification.

### 2.1. Topological André–Quillen homology

For a map of connective commutative ring spectra  $i : A \rightarrow B$ , we use the connectivity of the map to determine the bottom homotopy group of  $\mathrm{TAQ}^A(B)$  [6].

#### 2.1.1. Algebraic cases

If  $\mathcal{O}_K \rightarrow \mathcal{O}_L$  is an extension of number rings with corresponding extension of number fields  $K \subset L$ , then of course we cannot use a connectivity argument for understanding  $\mathrm{TAQ}$ , but here, the algebraic module of Kähler differentials,  $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ , is isomorphic to the first Hochschild homology group  $\mathrm{HH}_1^{\mathcal{O}_K}(\mathcal{O}_L)$  which in turn is isomorphic to  $\pi_0 \mathrm{TAQ}^{H\mathcal{O}_K}(H\mathcal{O}_L)$ . This follows from combining [7, Theorem 2.4], which ensures

that  $\pi_0 \mathbf{TAQ}^{H\mathcal{O}_K}(H\mathcal{O}_L)$  is isomorphic to the zeroth Gamma homology group  $H\Gamma_0(\mathcal{O}_L|\mathcal{O}_K; \mathcal{O}_L)$ , with [43, Proposition 6.5], which yields  $H\Gamma_0(\mathcal{O}_L|\mathcal{O}_K; \mathcal{O}_L) \cong \Omega^1_{\mathcal{O}_L|\mathcal{O}_K}$ .

2.1.2. *The connective Adams summand*

Let  $\ell$  denote the Adams summand of connective  $p$ -localized topological complex K-theory,  $ku_{(p)}$ . Here,  $p$  is an odd prime.

The inclusion  $i : \ell \rightarrow ku_{(p)}$  induces an isomorphism on  $\pi_0$  and  $\pi_1$ . Thus, by the Hurewicz theorem for topological André–Quillen homology [6, Lemma 8.2], [4, Lemma 1.2], we get that  $\pi_2 \mathbf{TAQ}^\ell(ku_{(p)})$  is the bottom homotopy group and is isomorphic to the second homotopy group of the cone of  $i$ , and this in turn can be determined by the long exact sequence:

$$\dots \rightarrow \pi_2(\ell) = 0 \rightarrow \pi_2(ku_{(p)}) \rightarrow \pi_2(\text{cone}(i)) \rightarrow \pi_1(\ell) = 0 \rightarrow \dots$$

Hence, we have  $\pi_2 \mathbf{TAQ}^\ell(ku_{(p)}) \cong \mathbb{Z}_{(p)}$ .

We know from [19] that  $\ell \rightarrow ku_{(p)}$  shows features of a tamely ramified extension of number rings, and Sagave shows [50, Theorem 6.1] that  $\ell \rightarrow ku_{(p)}$  is log-étale.

2.1.3. *Real and complex connective topological K-theory*

The complexification map  $c : ko \rightarrow ku$  induces an isomorphism on  $\pi_0$  and an epimorphism on  $\pi_1$ , so it is a 1-equivalence. Hence, again  $\pi_2 \text{cone}(c) \cong \pi_2(\mathbf{TAQ}^{ko}(ku))$  is the bottom homotopy group, but here we obtain an extension:

$$0 \rightarrow \pi_2 ku = \mathbb{Z} \rightarrow \pi_2 \text{cone}(c) \rightarrow \pi_1(ko) = \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

In order to understand  $\pi_2 \text{cone}(c)$ , we consider the cofiber sequence:

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\delta} \Sigma^2 KO$$

and the commutative diagram on homotopy groups:

$$\begin{array}{ccccccc} \mathbb{Z}/2\mathbb{Z} = \pi_2 ko & \xrightarrow[\cong]{\pi_2(c)} & \pi_2 ku & \longrightarrow & \pi_2 \text{cone}(c) & \longrightarrow & \pi_1 ko \xrightarrow{\pi_1(c)} 0 = \pi_1 ku \\ \cong \downarrow \pi_2(\tau_{ko}) & & \cong \downarrow \pi_2(\tau_{ku}) & & \downarrow g & & \cong \downarrow \pi_1(\tau_{ko}) & & \cong \downarrow \pi_1(\tau_{ku}) \\ \mathbb{Z}/2\mathbb{Z} = \pi_2 KO & \xrightarrow[\cong]{\pi_2(c)} & \pi_2 KU & \xrightarrow[\cong]{\pi_2(\delta)} & \pi_2 \Sigma^2 KO & \xrightarrow[\cong]{\pi_2(\Sigma\eta)} & \pi_1 KO \xrightarrow{\pi_1(c)} 0 = \pi_1 KU. \end{array}$$

Here,  $\tau_e : e \rightarrow E$  denotes the map from the connective cover  $e$  of  $E$  to  $E$ . The middle vertical map  $g$  is the map induced by the cofiber sequences. By the five lemma,  $g$  is an isomorphism hence

$$\pi_2 \text{cone}(c) \cong \pi_2 \Sigma^2 KO \cong \mathbb{Z}.$$

So this group is also torsion-free. We will later see that  $ko \rightarrow ku$  is not log-étale, and we will see some other indicators for wild ramification, but the bottom homotopy group of  $\mathbf{TAQ}^{ko}(ku)$  does not detect that.

2.1.4. *Connective topological modular forms with level structure (case  $n = 3$ )*

We consider  $\mathbf{tmf}_1(3)$ . Its homotopy groups are  $\pi_*(\mathbf{tmf}_1(3)) \cong \mathbb{Z}[\frac{1}{3}][a_1, a_3]$  with  $|a_i| = 2i$ . See [23] for some background. There is a  $C_2$ -action on  $\mathbf{tmf}_1(3)$  coming from the permutation of elements of exact order three and one denotes by  $\mathbf{tmf}_0(3)$  the connective cover of the homotopy fixed points,  $\mathbf{tmf}_1(3)^{hC_2}$ . There is a homotopy fixed point spectral sequence that was studied in detail in [33] for the periodic versions. In [23, p. 407], it is explained how to adapt this calculation to the connective variants: the terms in the spectral sequence with  $s > t - s \geq 0$  can be ignored. The  $C_2$ -action on the  $a_i$ 's is given by the sign-action, so if  $\tau$  generates  $C_2$ , then  $\tau(a_i^n) = (-1)^n a_i^n$ .

This implies that only  $H^0(C_2; \pi_0(\mathbf{tmf}_1(3))) \cong \mathbb{Z}[\frac{1}{3}]$  survives to  $\pi_0(\mathbf{tmf}_0(3))$ . For  $\pi_1$ , we get a contribution from  $H^1(C_2; \pi_2(\mathbf{tmf}_1(3)))$ , giving a  $\mathbb{Z}/2\mathbb{Z}$  generated by the class of  $a_1$  (this detects an  $\eta$ ). For  $\pi_2(\mathbf{tmf}_0(3))$ , the class of  $a_1^2$  generates a copy of  $\mathbb{Z}/2\mathbb{Z}$ .

Hence, the map  $j : \mathbf{tmf}_0(3)_{(2)} \rightarrow \mathbf{tmf}_1(3)_{(2)}$  is 1-connected, so  $\pi_2 \mathbf{TAQ}^{\mathbf{tmf}_0(3)_{(2)}}(\mathbf{tmf}_1(3)_{(2)})$  is the bottom homotopy group and is isomorphic to  $\pi_2(\mathbf{cone}(j))$  which sits in an extension. We can use the commutative diagram of commutative ring spectra from [23, Theorem 6.3]

$$\begin{array}{ccc} \mathbf{tmf}_0(3) & \longrightarrow & \mathbf{tmf}_1(3) \\ \downarrow & & \downarrow \\ ko[\frac{1}{3}] & \longrightarrow & ku[\frac{1}{3}] \end{array}$$

in order to determine to  $\pi_2(\mathbf{cone}(j))$ . By [30, Theorem 1.2], there is a cofiber sequence of  $\mathbf{tmf}_1(3)_{(2)}$ -modules

$$\Sigma^6 \mathbf{tmf}_1(3)_{(2)} \xrightarrow{v_2} \mathbf{tmf}_1(3)_{(2)} \longrightarrow ku_{(2)}$$

and hence  $\pi_2(\mathbf{tmf}_1(3)_{(2)}) \cong \pi_2(ku_{(2)})$ .

The diagram

$$\begin{array}{ccccccccc} \pi_2 \mathbf{tmf}_0(3)_{(2)} & \xrightarrow{0} & \pi_2 \mathbf{tmf}_1(3)_{(2)} & \longrightarrow & \pi_2 \mathbf{cone}(j) & \longrightarrow & \pi_1 \mathbf{tmf}_0(3)_{(2)} & \longrightarrow & \pi_1 \mathbf{tmf}_1(3)_{(2)} = 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ \pi_2 ko_{(2)} & \xrightarrow{0} & \pi_2 ku_{(2)} & \longrightarrow & \pi_2 \mathbf{cone}(c) & \longrightarrow & \pi_1 ko_{(2)} & \xrightarrow{0} & \pi_1 ku_{(2)} = 0 \end{array}$$

commutes and the 5-lemma implies that  $\pi_2(\mathbf{cone}(j)) \cong \mathbb{Z}_{(2)}$ .

2.1.5. *Connective topological modular forms with level structure (case  $n = 2, p = 3$ )*

Forgetting a  $\Gamma_0(2)$ -structure yields a map  $f : \mathbf{tmf}_3 \rightarrow \mathbf{tmf}_0(2)_{(3)}$  such that  $f$  is a 3-equivalence. We will recall more details about these spectra at the beginning of Section 2.2. Again, we obtain that the bottom nontrivial homotopy group of the spectrum of topological André–Quillen homology is  $\pi_4(\mathbf{TAQ}^{\mathbf{tmf}_3}(\mathbf{tmf}_0(2)_{(3)})) \cong \pi_4(\mathbf{cone}(f))$ . There is a short exact sequence

$$0 = \pi_4 \mathbf{tmf}_3 \rightarrow \pi_4 \mathbf{tmf}_0(2)_{(3)} = \mathbb{Z}_{(3)} \rightarrow \pi_4 \mathbf{cone}(f) \rightarrow \pi_3 \mathbf{tmf}_3 \cong \mathbb{Z}/3\mathbb{Z} \rightarrow 0$$

so a priori  $\pi_4 \mathbf{cone}(f)$  could be isomorphic to  $\mathbb{Z}_{(3)}$  or to  $\mathbb{Z}_{(3)} \oplus \mathbb{Z}/3\mathbb{Z}$ .

There is an equivalence:

$$\mathbf{tmf}_3 \wedge T \simeq \mathbf{tmf}_0(2)_{(3)}$$

where  $T = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$  with  $\alpha_1$  denoting the generator of  $\pi_3^s$  at 3, [10, Lemma 2, p. 382], [36, Theorem 4.15]. Thus,  $T$  is part of a cofiber sequence:

$$S^0 \rightarrow T \rightarrow \Sigma^4 \mathbf{cone}(\alpha_1)$$

and we obtain a cofiber sequence:

$$\mathbf{tmf}_3 = \mathbf{tmf}_3 \wedge S^0 \rightarrow \mathbf{tmf}_3 \wedge T \simeq \mathbf{tmf}_0(2)_{(3)} \rightarrow \mathbf{tmf}_3 \wedge \Sigma^4 \mathbf{cone}(\alpha_1)$$

and thus

$$\pi_4(\mathbf{cone}(f)) \cong \pi_4(\mathbf{tmf}_3 \wedge \Sigma^4 \mathbf{cone}(\alpha_1)) \cong \pi_0(\mathbf{tmf}_3 \wedge \mathbf{cone}(\alpha_1)).$$

But as we have a short exact sequence:

$$0 = \pi_0(\Sigma^3 \mathbf{tmf}_3) \rightarrow \pi_0(\mathbf{tmf}_3) \cong \mathbb{Z}_{(3)} \rightarrow \pi_0(\mathbf{tmf}_3 \wedge \mathbf{cone}(\alpha_1)) \rightarrow 0$$

we obtain

$$\pi_4(\mathbf{TAQ}^{\mathbf{tmf}_3}(\mathbf{tmf}_0(2)_{(3)})) \cong \mathbb{Z}_{(3)}.$$

2.2. Relative topological Hochschild homology

In [19] (see also [20] for a correction), we show that the relative topological Hochschild homology spectra  $\mathrm{THH}^\ell(ku_{(p)})$  and  $\mathrm{THH}^{ko}(ku)$  have highly nontrivial homotopy groups. Here, we extend these results to the relative THH-spectra of  $\mathrm{tmf}_{(3)} \rightarrow \mathrm{tmf}_0(2)_{(3)}$ ,  $\mathrm{Tmf}_{(3)} \rightarrow \mathrm{Tmf}_0(2)_{(3)}$  and  $\mathrm{tmf}_0(2)_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}$ . For formulas concerning the coefficients of elliptic curves, we refer to [16].

Recall that we have  $\mathrm{tmf}_0(2)_{(3)} \simeq \tau_{\geq 0} \mathrm{tmf}(2)_{(3)}^{hC_2}$ . By [53, §7], we know that  $\pi_* \mathrm{tmf}(2)_{(3)} \cong \mathbb{Z}_{(3)}[\lambda_1, \lambda_2]$  with  $|\lambda_i| = 4$  and with  $C_2$ -action given by  $\lambda_1 \mapsto \lambda_2$  and  $\lambda_2 \mapsto \lambda_1$  [53, Lemma 7.3]. Since  $|C_2|$  is invertible in  $\pi_* \mathrm{tmf}(2)_{(3)}$ , the  $E^2$ -page of the homotopy fixed point spectral sequence is given by:

$$H^*(C_2, \pi_* \mathrm{tmf}(2)_{(3)}) = H^0(C_2, \pi_* \mathrm{tmf}(2)_{(3)}) = \pi_*(\mathrm{tmf}(2)_{(3)})^{C_2}.$$

Thus, we have

$$\pi_* \mathrm{tmf}_0(2)_{(3)} = \mathbb{Z}_{(3)}[\lambda_1 + \lambda_2, \lambda_1 \lambda_2] = \mathbb{Z}_{(3)}[a_2, a_4]$$

with  $a_2 = -(\lambda_1 + \lambda_2)$  and  $a_4 = \lambda_1 \lambda_2$ . Recall the following facts about the homotopy of  $\mathrm{tmf}_{(3)}$  (see for instance [18, p. 192]), we have

$$\pi_* \mathrm{tmf}_{(3)} = \begin{cases} \mathbb{Z}_{(3)}\{1\}, & * = 0; \\ \mathbb{Z}/3\mathbb{Z}\{\alpha_1\} & * = 3; \\ \mathbb{Z}_{(3)}\{c_4\}, & * = 8; \\ \mathbb{Z}_{(3)}\{c_6\}, & * = 12; \\ 0, & * = 4, 5, 6, 7; \end{cases}$$

where  $\alpha_1$  is the image of  $\alpha_1 \in \pi_3(S_{(3)})$  under  $\pi_3(S_{(3)}) \rightarrow \pi_3 \mathrm{tmf}_{(3)}$ . By [53, Proof of Proposition 10.3], we have that the map  $\pi_* \mathrm{tmf}_{(3)} \rightarrow \pi_* \mathrm{tmf}_0(2)_{(3)}$  satisfies  $c_4 \mapsto 16a_2^2 - 48a_4$  and  $c_6 \mapsto -64a_2^3 + 288a_2 a_4$ . (There is a discrepancy between our sign for  $c_6$  and that in [53].)

We know from personal communication with Mike Hill that there is a fiber sequence of  $\mathrm{tmf}_0(2)_{(3)}$ -modules:

$$\mathrm{Tmf}_0(2)_{(3)} \longrightarrow \mathrm{tmf}_0(2)_{(3)}[a_2^{-1}] \times \mathrm{tmf}_0(2)_{(3)}[a_4^{-1}] \xrightarrow{f} \mathrm{tmf}_0(2)_{(3)}[(a_2 a_4)^{-1}].$$

See [24, Proposition 4.24] for the analogous statement at  $p = 2$ . The kernel of  $\pi_*(f)$  has  $\mathbb{Z}_{(3)}$ -basis:

$$\{(a_2^n a_4^m, -a_2^n a_4^m) \mid n, m \in \mathbb{N}\}$$

and the cokernel has  $\mathbb{Z}_{(3)}$ -basis:

$$\left\{ \frac{1}{a_2^n a_4^m} \mid n \geq 1, m \geq 1 \right\}.$$

We get that in negative degrees  $\pi_* \mathrm{Tmf}_0(2)_{(3)}$  is given by:

$$\bigoplus_{n,m \geq 1} \mathbb{Z}_{(3)} \left\{ \frac{1}{a_2^n a_4^m} \right\},$$

where  $\frac{1}{a_2^n a_4^m}$  has degree  $-4n - 8m - 1$ . The  $\pi_* \mathrm{tmf}_0(2)_{(3)}$ -action is given by:

$$a_2 \cdot \frac{1}{a_2^n a_4^m} = \begin{cases} \frac{1}{a_2^{n-1} a_4^m}, & \text{if } n \geq 2 \\ 0, & \text{otherwise,} \end{cases}$$

and analogously for  $a_4$ .

By the gap theorem (see for instance [28]), we have  $\pi_* \mathrm{Tmf}_{(3)} \cong 0$  for  $-21 < * < 0$ .

**Lemma 2.1.**

$$\pi_*(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)}) \cong \mathbb{Z}_{(3)}[a_2, a_4, r]/r^3 + a_2r^2 + a_4r =: \mathbb{Z}_{(3)}[a_2, a_4, r]/I$$

where  $r$  has degree 4 and is mapped to 0 under the multiplication map.

*Proof.* As above, we use that we have an equivalence of  $\mathbf{tmf}_3$ -modules  $\mathbf{tmf}_3 \wedge T \simeq \mathbf{tmf}_0(2)_{(3)}$ . Here,  $T$  is defined by the cofiber sequences:

$$S^3_{(3)} \xrightarrow{\alpha_1} S^0_{(3)} \longrightarrow \text{cone}(\alpha_1) \longrightarrow S^4_{(3)}$$

and

$$S^7_{(3)} \xrightarrow{\phi} \text{cone}(\alpha_1) \longrightarrow T \longrightarrow S^8_{(3)},$$

where  $S^7_{(3)} \xrightarrow{\phi} \text{cone}(\alpha_1) \rightarrow S^4_{(3)}$  is equal to  $\alpha_1$ . We get an equivalence of left  $\mathbf{tmf}_0(2)_{(3)}$ -modules:

$$\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)} \simeq \mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} (\mathbf{tmf}_3 \wedge T) \simeq \mathbf{tmf}_0(2)_{(3)} \wedge T.$$

Smashing the above cofiber sequences with  $\mathbf{tmf}_0(2)_{(3)}$  gives cofiber sequences of  $\mathbf{tmf}_0(2)_{(3)}$ -modules:

$$\Sigma^3 \mathbf{tmf}_0(2)_{(3)} \xrightarrow{\bar{\alpha}_1} \mathbf{tmf}_0(2)_{(3)} \longrightarrow \mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1) \xrightarrow{\delta} \Sigma^4 \mathbf{tmf}_0(2)_{(3)}$$

and

$$\Sigma^7 \mathbf{tmf}_0(2)_{(3)} \xrightarrow{\bar{\phi}} \mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1) \longrightarrow \mathbf{tmf}_0(2)_{(3)} \wedge T \xrightarrow{\Delta} \Sigma^8 \mathbf{tmf}_0(2)_{(3)}.$$

The map  $\bar{\alpha}_1$  is zero in the derived category of  $\mathbf{tmf}_0(2)_{(3)}$ -modules, because  $\pi_*(\mathbf{tmf}_0(2)_{(3)})$  is concentrated in even degrees. We therefore get an equivalence of  $\mathbf{tmf}_0(2)_{(3)}$ -modules:

$$\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1) \simeq \mathbf{tmf}_0(2)_{(3)} \vee \Sigma^4 \mathbf{tmf}_0(2)_{(3)}.$$

This implies that  $\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1)$  has nontrivial homotopy groups only in even degrees, and therefore that  $\bar{\phi}$  is zero in the derived category of  $\mathbf{tmf}_0(2)_{(3)}$ -modules. We get an equivalence of  $\mathbf{tmf}_0(2)_{(3)}$ -modules:

$$\mathbf{tmf}_0(2)_{(3)} \wedge T \simeq \mathbf{tmf}_0(2)_{(3)} \vee \Sigma^4 \mathbf{tmf}_0(2)_{(3)} \vee \Sigma^8 \mathbf{tmf}_0(2)_{(3)}.$$

We can assume that the map  $\mathbf{tmf}_3 \rightarrow \mathbf{tmf}_0(2)_{(3)}$  factors in the derived category of  $\mathbf{tmf}_3$ -modules as:

$$\mathbf{tmf}_3 \longrightarrow \mathbf{tmf}_3 \wedge \text{cone}(\alpha_1) \longrightarrow \mathbf{tmf}_3 \wedge T \xrightarrow{\simeq} \mathbf{tmf}_0(2)_{(3)}$$

This implies that the inclusion in the first smash factor:

$$\eta_L : \mathbf{tmf}_0(2)_{(3)} \rightarrow \mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)}$$

is given by:

$$\mathbf{tmf}_0(2)_{(3)} \longrightarrow \mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1) \longrightarrow \mathbf{tmf}_0(2)_{(3)} \wedge T \xrightarrow{\simeq} \mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)}.$$

We obtain that the map:

$$\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1) \longrightarrow \mathbf{tmf}_0(2)_{(3)} \wedge T \simeq \mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)} \longrightarrow \mathbf{tmf}_0(2)_{(3)}$$

is a left inverse for  $\mathbf{tmf}_0(2)_{(3)} \rightarrow \mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1)$ . It is also clear that the inclusion in the second smash factor  $\eta_R : \mathbf{tmf}_0(2)_{(3)} \rightarrow \mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)}$  is given by:

$$\mathbf{tmf}_0(2)_{(3)} \xrightarrow{\simeq} \mathbf{tmf}_3 \wedge T \longrightarrow \mathbf{tmf}_0(2)_{(3)} \wedge T \xrightarrow{\simeq} \mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)}.$$

We claim that

$$\eta_R(a_2) \in \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_3} \mathbf{tmf}_0(2)_{(3)}) \cong \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge T) \cong \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1))$$

maps to three times a unit under:

$$\delta_4 : \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1)) \rightarrow \pi_4(\Sigma^4 \mathbf{tmf}_0(2)_{(3)}) \cong \mathbb{Z}_{(3)}.$$



By commutativity of the diagram:

$$\begin{array}{ccccc}
 \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge T) & \xleftarrow{\cong} & \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1)) & \xrightarrow{\delta_4} & \pi_4(\Sigma^4 \mathbf{tmf}_0(2)_{(3)}) \\
 \eta_R \uparrow & & \uparrow & & \uparrow \\
 \pi_4(\mathbf{tmf}_{(3)} \wedge T) & \xleftarrow{\cong} & \pi_4(\mathbf{tmf}_{(3)} \wedge \text{cone}(\alpha_1)) & \longrightarrow & \pi_4(\Sigma^4 \mathbf{tmf}_{(3)})
 \end{array}$$

it suffices to show that  $a_2 \in \pi_4(\mathbf{tmf}_{(3)} \wedge T)$  maps to three times a unit under the bottom map. This follows by the exact sequence:

$$\begin{array}{ccccccc}
 \pi_4(\mathbf{tmf}_{(3)}) & \longrightarrow & \pi_4(\mathbf{tmf}_{(3)} \wedge \text{cone}(\alpha_1)) & \longrightarrow & \pi_4(\Sigma^4 \mathbf{tmf}_{(3)}) & \longrightarrow & \pi_4(\Sigma \mathbf{tmf}_{(3)}) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \mathbb{Z}_{(3)} & \longrightarrow & \mathbb{Z}_{(3)} & \longrightarrow & \mathbb{Z}/3\mathbb{Z}\{\alpha_1\}.
 \end{array}$$

We define  $r$  to be the unique element in  $\pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1))$  that maps to that unit under  $\delta_4$  and that is in the kernel of the composition of  $\pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1)) \cong \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge T)$  and the multiplication map

$$\pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge T) \cong \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_{(3)}} \mathbf{tmf}_0(2)_{(3)}) \rightarrow \pi_4(\mathbf{tmf}_0(2)_{(3)}).$$

We have that  $3r - \eta_R(a_2)$  is in the image of  $\pi_4(\mathbf{tmf}_0(2)_{(3)}) \rightarrow \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1))$  and thus can be written as  $3r - \eta_R(a_2) = n \cdot a_2$  for an  $n \in \mathbb{Z}_{(3)}$ . Applying the map

$$\pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge \text{cone}(\alpha_1)) \cong \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge T) \cong \pi_4(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_{(3)}} \mathbf{tmf}_0(2)_{(3)}) \rightarrow \pi_4(\mathbf{tmf}_0(2)_{(3)})$$

gives  $n = -1$ .

We claim that  $\eta_R(a_4) \in \pi_8(\mathbf{tmf}_0(2)_{(3)} \wedge T)$  maps to three times a unit under

$$\Delta_8 : \pi_8(\mathbf{tmf}_0(2)_{(3)} \wedge T) \rightarrow \pi_8(\Sigma^8 \mathbf{tmf}_0(2)_{(3)}).$$

As above one sees that it suffices to show that  $a_4$  maps to three times a unit under the map  $\pi_8(\mathbf{tmf}_{(3)} \wedge T) \rightarrow \pi_8(\Sigma^8 \mathbf{tmf}_{(3)})$ . For this, we consider the exact sequence:

$$\begin{array}{ccccccc}
 \pi_8(\mathbf{tmf}_{(3)} \wedge \text{cone}(\alpha_1)) & \longrightarrow & \pi_8(\mathbf{tmf}_{(3)} \wedge T) & \longrightarrow & \pi_8(\Sigma^8 \mathbf{tmf}_{(3)}) & \longrightarrow & \pi_8(\Sigma \mathbf{tmf}_{(3)} \wedge \text{cone}(\alpha_1)). \\
 & & \cong \uparrow & & \cong \uparrow & & \\
 & & \mathbb{Z}_{(3)}\{a_2^2\} \oplus \mathbb{Z}_{(3)}\{a_4\} & & \mathbb{Z}_{(3)} & & 
 \end{array}$$

Using that  $\pi_4(\mathbf{tmf}_{(3)}) = 0 = \pi_5(\mathbf{tmf}_{(3)})$ , one gets that  $\pi_8(\mathbf{tmf}_{(3)}) \cong \pi_8(\mathbf{tmf}_{(3)} \wedge \text{cone}(\alpha_1))$ , and under this isomorphism the first map in the exact sequence identifies with

$$\pi_8(\mathbf{tmf}_{(3)}) \cong \mathbb{Z}_{(3)}\{c_4\} \rightarrow \pi_8(\mathbf{tmf}_0(2)_{(3)}) \cong \mathbb{Z}_{(3)}\{a_2^2\} \oplus \mathbb{Z}_{(3)}\{a_4\}, \quad c_4 \mapsto 16a_2^2 - 48a_4.$$

As  $\pi_6(\mathbf{tmf}_{(3)}) = 0 = \pi_7(\mathbf{tmf}_{(3)})$ , one gets that  $\pi_7(\mathbf{tmf}_{(3)} \wedge \text{cone}(\alpha_1)) \cong \pi_7(\Sigma^4 \mathbf{tmf}_{(3)})$ , and under this isomorphism the third map in the exact sequence identifies with

$$\pi_8(\Sigma^8 \mathbf{tmf}_{(3)}) \cong \mathbb{Z}_{(3)} \rightarrow \pi_3(\mathbf{tmf}_{(3)}) \cong \mathbb{Z}/3\mathbb{Z}\{\alpha_1\}, \quad 1 \mapsto \alpha_1.$$

One obtains that the second map in the exact sequence maps  $a_4$  to  $3 \cdot m$  and  $a_2^2$  to  $9 \cdot m$  for a unit  $m \in \mathbb{Z}_{(3)}$ .

Since the map  $\pi_*(\mathbf{tmf}_{(3)}) \rightarrow \pi_*(\mathbf{tmf}_0(2)_{(3)})$  maps  $c_4$  to  $16a_2^2 - 48a_4$ , we have the equation:

$$16 \cdot a_2^2 - 48 \cdot a_4 = 16 \cdot \eta_R(a_2)^2 - 48 \cdot \eta_R(a_4)$$

in  $\pi_*(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_{(3)}} \mathbf{tmf}_0(2)_{(3)})$ . Replacing  $\eta_R(a_2)$  by  $3r + a_2$  and using torsion-freeness, one gets the equation:

$$\eta_R(a_4) = a_4 + 3r^2 + 2a_2r.$$

We apply the map  $\Delta_8 : \pi_8(\mathbf{tmf}_0(2)_{(3)} \wedge T) \rightarrow \pi_8(\Sigma^8 \mathbf{tmf}_0(2)_{(3)})$  to this equation and obtain by torsion-freeness of  $\pi_*(\mathbf{tmf}_0(2)_{(3)})$ :

$$\Delta_8(r^2) = m.$$

We thus have an isomorphism of left  $\pi_*(\mathbf{tmf}_0(2)_{(3)})$ -modules:

$$\pi_*(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_0(2)_{(3)}} \mathbf{tmf}_0(2)_{(3)}) \cong \pi_*(\mathbf{tmf}_0(2)_{(3)}) \oplus \pi_*(\mathbf{tmf}_0(2)_{(3)})\{r\} \oplus \pi_*(\mathbf{tmf}_0(2)_{(3)})\{r^2\}.$$

Since the map  $\pi_*(\mathbf{tmf}_0(2)_{(3)}) \rightarrow \pi_*(\mathbf{tmf}_0(2)_{(3)})$  maps  $c_6$  to  $-64a_2^3 + 288a_2a_4$ , we have

$$-64 \cdot a_2^3 + 288 \cdot a_2 \cdot a_4 = -64 \cdot \eta_R(a_2)^3 + 288 \cdot \eta_R(a_2) \cdot \eta_R(a_4)$$

in  $\pi_*(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_0(2)_{(3)}} \mathbf{tmf}_0(2)_{(3)})$ . Replacing  $\eta_R(a_2)$  by  $3r + a_2$  and  $\eta_R(a_4)$  by  $a_4 + 3r^2 + 2a_2r$  and using torsion-freeness, one gets

$$r^3 + a_2r^2 + a_4r = 0.$$

This implies the lemma. □

**Remark 2.2.** One can give a different proof of Lemma 2.1 using the perspective of Hopf algebroids and associated stacks (see computations in [9, Section 5]).

**Theorem 2.3.** The canonical map  $\mathbf{tmf}_0(2)_{(3)} \rightarrow \mathrm{THH}_*^{\mathbf{tmf}_0(2)_{(3)}}(\mathbf{tmf}_0(2)_{(3)})$  is far from being an equivalence. More precisely,

$$\begin{aligned} \mathrm{THH}_*^{\mathbf{tmf}_0(2)_{(3)}}(\mathbf{tmf}_0(2)_{(3)}) &\cong \mathbb{Z}_{(3)}[a_2, a_4] \oplus \bigoplus_{i \geq 0} \Sigma^{14i+5} \mathbb{Z}_{(3)}[a_2] \\ &\cong \pi_* \mathbf{tmf}_0(2)_{(3)} \oplus \bigoplus_{i \geq 0} \Sigma^{14i+5} \pi_* \mathbf{tmf}_0(2)_{(3)}/(a_4). \end{aligned}$$

*Proof.* We use the Tor spectral sequence

$$E_{**}^2 = \mathrm{Tor}_{**}^{\pi_*(\mathbf{tmf}_0(2)_{(3)} \wedge_{\mathbf{tmf}_0(2)_{(3)}} \mathbf{tmf}_0(2)_{(3)})}(\pi_* \mathbf{tmf}_0(2)_{(3)}, \pi_* \mathbf{tmf}_0(2)_{(3)}) \Rightarrow \pi_* \mathrm{THH}_*^{\mathbf{tmf}_0(2)_{(3)}}(\mathbf{tmf}_0(2)_{(3)})$$

in order to calculate relative topological Hochschild homology. For determining

$$\mathrm{Tor}_{**}^{\mathbb{Z}_{(3)}[a_2, a_4, r]/I}(\mathbb{Z}_{(3)}[a_2, a_4], \mathbb{Z}_{(3)}[a_2, a_4])$$

we consider the free resolution of  $\mathbb{Z}_{(3)}[a_2, a_4]$  as a  $\mathbb{Z}_{(3)}[a_2, a_4, r]/I$ -module:

$$\dots \longrightarrow \Sigma^{12} \mathbb{Z}_{(3)}[a_2, a_4, r]/I \xrightarrow{r^2 + a_2r + a_4} \Sigma^4 \mathbb{Z}_{(3)}[a_2, a_4, r]/I \xrightarrow{r} \mathbb{Z}_{(3)}[a_2, a_4, r]/I.$$

Applying  $(-)\otimes_{\mathbb{Z}_{(3)}[a_2, a_4, r]/I} \mathbb{Z}_{(3)}[a_2, a_4]$  yields

$$\dots \longrightarrow \Sigma^{12} \mathbb{Z}_{(3)}[a_2, a_4] \xrightarrow{a_4} \Sigma^4 \mathbb{Z}_{(3)}[a_2, a_4] \xrightarrow{0} \mathbb{Z}_{(3)}[a_2, a_4]$$

and hence we get

$$E_{n,*}^2 = \begin{cases} \pi_*(\mathbf{tmf}_0(2)_{(3)}), & n = 0; \\ \Sigma^{4+12k} \mathbb{Z}_{(3)}[a_2], & n = 2k + 1, k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

We note that all nontrivial classes in positive filtration degree have an odd total degree. Since the edge morphism  $\pi_*(\mathbf{tmf}_0(2)_{(3)}) \rightarrow \mathrm{THH}_*^{\mathbf{tmf}_0(2)_{(3)}}(\mathbf{tmf}_0(2)_{(3)})$  is the unit, the classes in filtration degree zero cannot

be hit by a differential and the spectral sequence collapses at the  $E^2$ -page. Since  $E_{n,m}^2 = E_{n,m}^\infty$  is a free  $\mathbb{Z}_{(3)}$ -module for all  $n, m$ , there are no additive extensions.  $\square$

As for the connective covers, we have an equivalence of  $\mathbf{Tmf}_{(3)}$ -modules  $\mathbf{Tmf}_{(3)} \wedge T \simeq \mathbf{Tmf}_0(2)$  [36, §4.6] such that the map  $\mathbf{Tmf}_{(3)} \rightarrow \mathbf{Tmf}_0(2)_{(3)}$  factors in the derived category of  $\mathbf{Tmf}_{(3)}$ -modules as:

$$\mathbf{Tmf}_{(3)} \rightarrow \mathbf{Tmf}_{(3)} \wedge \text{cone}(\alpha_1) \rightarrow \mathbf{Tmf}_{(3)} \wedge T \simeq \mathbf{Tmf}_0(2)_{(3)}.$$

Using the gap theorem, one can argue analogously to the proof of Lemma 2.1 to show that

$$\pi_*(\mathbf{Tmf}_0(2)_{(3)} \wedge_{\mathbf{Tmf}_{(3)}} \mathbf{Tmf}_0(2)_{(3)}) \cong \pi_*\mathbf{Tmf}_0(2)_{(3)}[r]/(r^3 + a_2r^2 + a_4r).$$

**Theorem 2.4.** *There is an additive isomorphism:*

$$\text{THH}^{\mathbf{Tmf}_{(3)}}(\mathbf{Tmf}_0(2)_{(3)}) \cong \pi_*\mathbf{Tmf}_0(2)_{(3)} \oplus \bigoplus_{i \geq 0} \Sigma^{14i+5} \mathbb{Z}_{(3)}[a_2] \oplus \bigoplus_{i \geq 1} \Sigma^{14i} \mathbb{Z}_{(3)} \left\{ \frac{1}{a_2^i a_4} \right\}.$$

*Proof.* As above we have the following free resolution of  $\pi_*(\mathbf{Tmf}_0(2)_{(3)})$  as a module over

$$C_* = \pi_*(\mathbf{Tmf}_0(2)_{(3)} \wedge_{\mathbf{Tmf}_{(3)}} \mathbf{Tmf}_0(2)_{(3)}):$$

$$\dots \xrightarrow{r} \Sigma^{12} C_* \xrightarrow{r^2 + a_2r + a_4} \Sigma^4 C_* \xrightarrow{r} C_* \longrightarrow \pi_*\mathbf{Tmf}_0(2)_{(3)} \longrightarrow 0.$$

We get that the  $E^2$ -page of the Tor spectral sequence:

$$E_{*,*}^2 = \text{Tor}_{*,*}^{C_*}(\pi_*\mathbf{Tmf}_0(2)_{(3)}, \pi_*\mathbf{Tmf}_0(2)_{(3)}) \longrightarrow \pi_*\text{THH}^{\mathbf{Tmf}_{(3)}}(\mathbf{Tmf}_0(2)_{(3)})$$

is given by:

$$E_{n,*}^2 = \begin{cases} \pi_*\mathbf{Tmf}_0(2)_{(3)}, & n = 0; \\ \Sigma^{4+12k} \pi_*\mathbf{Tmf}_0(2)_{(3)}/a_4, & n = 2k + 1, k \geq 0; \\ \ker(\Sigma^{12k} \pi_*\mathbf{Tmf}_0(2)_{(3)} \xrightarrow{\cdot a_4} \Sigma^{12(k-1)+4} \pi_*\mathbf{Tmf}_0(2)_{(3)}), & n = 2k, k > 0 \end{cases}$$

$$= \begin{cases} \pi_*\mathbf{Tmf}_0(2)_{(3)}, & n = 0; \\ \Sigma^{4+12k} \mathbb{Z}_{(3)}[a_2], & n = 2k + 1, k \geq 0; \\ \Sigma^{12k} \bigoplus_{n \geq 1} \mathbb{Z}_{(3)} \left\{ \frac{1}{a_2^n a_4} \right\}, & n = 2k, k > 0. \end{cases}$$

Since all nontrivial classes in positive filtration have an odd total degree, the spectral sequence collapses at the  $E^2$ -page. There are no additive extensions, because the  $E^\infty = E^2$ -page is a free  $\mathbb{Z}_{(3)}$ -module.  $\square$

**Theorem 2.5.** *We have an additive isomorphism:*

$$\pi_*\text{THH}^{\mathbf{tmf}_0(2)_{(3)}}(\mathbf{tmf}(2)_{(3)}) \cong \mathbb{Z}_{(3)}[\lambda_1, \lambda_2] \oplus \bigoplus_{i \geq 0} \Sigma^{10i+5} \mathbb{Z}_{(3)}[\lambda_1].$$

*Proof.* The map  $\pi_*\mathbf{tmf}_0(2)_{(3)} \rightarrow \pi_*\mathbf{tmf}(2)_{(3)}$  is given by  $\mathbb{Z}_{(3)}[\lambda_1 + \lambda_2, \lambda_1\lambda_2] \rightarrow \mathbb{Z}_{(3)}[\lambda_1, \lambda_2]$ . One easily sees that

$$\mathbb{Z}_{(3)}[\lambda_1, \lambda_2] \cong \mathbb{Z}_{(3)}[\lambda_1 + \lambda_1, \lambda_1\lambda_2] \oplus \mathbb{Z}_{(3)}[\lambda_1 + \lambda_2, \lambda_1\lambda_2]\lambda_1,$$

so  $\pi_*\mathbf{tmf}(2)_{(3)}$  is a free  $\pi_*\mathbf{tmf}_0(2)_{(3)}$ -module. We get

$$\begin{aligned} \pi_*(\mathbf{tmf}(2)_{(3)} \wedge_{\mathbf{tmf}_0(2)_{(3)}} \mathbf{tmf}(2)_{(3)}) &= \pi_*\mathbf{tmf}(2)_{(3)} \otimes_{\pi_*\mathbf{tmf}_0(2)_{(3)}} \pi_*\mathbf{tmf}(2)_{(3)} \\ &= \mathbb{Z}_{(3)}[\lambda_1, \lambda_2, a]/a^2 + \lambda_1a - \lambda_2a, \end{aligned}$$

where  $a = \eta_R(\lambda_1) - \lambda_1$ . Let  $C_* = \mathbb{Z}_{(3)}[\lambda_1, \lambda_2, a]/a^2 + \lambda_1 a - \lambda_2 a$ . We have the following free resolution of  $\pi_* \mathbf{tmf}(2)_{(3)}$  as a  $C_*$ -module:

$$\dots \xrightarrow{\cdot a} \Sigma^8 C_* \xrightarrow{\cdot (a+\lambda_1-\lambda_2)} \Sigma^4 C_* \xrightarrow{\cdot a} C_* \longrightarrow \pi_* \mathbf{tmf}(2)_{(3)} \longrightarrow 0$$

Thus, the  $E^2$ -page of the Tor spectral sequence:

$$E_{*,*}^2 = \mathrm{Tor}_{*,*}^{C_*}(\pi_* \mathbf{tmf}(2)_{(3)}, \pi_* \mathbf{tmf}(2)_{(3)}) \longrightarrow \pi_* \mathrm{T HH}^{\mathrm{tmf}_0(2)_{(3)}}(\mathbf{tmf}(2)_{(3)})$$

is given by:

$$E_{n,*}^2 = \begin{cases} \pi_* \mathbf{tmf}(2)_{(3)}, & n = 0; \\ \Sigma^{8k+4} \pi_* \mathbf{tmf}(2)_{(3)} / (\lambda_1 - \lambda_2), & n = 2k + 1, k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since the nontrivial classes in positive filtration have odd total degree, the spectral sequence collapses at the  $E^2$ -page. There are no additive extensions, because the  $E^2 = E^\infty$ -page is a free  $\mathbb{Z}_{(3)}$ -module.  $\square$

### 2.3. The discriminant map

Let  $A \rightarrow B$  be a map of commutative ring spectra and  $G$  be a finite group that acts on  $B$  through  $A$ -algebra maps such that  $A \rightarrow B^{hG}$  is an equivalence. One then can define the discriminant map  $\mathfrak{d}_{B|A} : B \rightarrow F_A(B, A)$  [44, Definition 6.4.5]. The map  $\mathfrak{d}_{B|A}$  is right adjoint to the trace pairing:

$$B \wedge_A B \xrightarrow{\mu} B \xrightarrow{tr} A.$$

Here,  $\mu$  is the multiplication of  $B$ , and  $tr$  is defined to be the composite:

$$B \longrightarrow B_{hG} \xrightarrow{N} B^{hG} \xleftarrow{\simeq} A,$$

where  $N$  is the norm map. One has that  $(A \rightarrow B) \circ tr$  is homotopic to  $\sum_{g \in G} g$ . If  $A \rightarrow B$  is a  $G$ -Galois extension, then  $\mathfrak{d}_{B|A}$  is a weak equivalence [44, Proposition 6.4.7]. Rognes proposes that the deviation of  $\mathfrak{d}_{B|A}$  from being a weak equivalence might be used for measuring ramification. Note that if  $A$  and  $B$  are connective, then  $tr$  and  $\mathfrak{d}_{B|A}$  are defined even if only  $A \simeq \tau_{\geq 0} B^{hG}$ . We show in the examples of  $\ell_p \rightarrow ku_p$  and  $ko \rightarrow ku$  that  $\mathfrak{d}$  does notice the ramification, but it does not give any information about the type of ramification.

**Proposition 2.6.** *There is a cofiber sequence:*

$$ku_p \xrightarrow{\mathfrak{d}_{ku_p|\ell_p}} F_{\ell_p}(ku_p, \ell_p) \longrightarrow \bigvee_{i=1}^{p-2} \Sigma^{-2p+2i+2} H\mathbb{Z}_p.$$

*Proof.* We know that  $F_{\ell_p}(ku_p, \ell_p)$  can be decomposed as:

$$F_{\ell_p}(ku_p, \ell_p) \simeq F_{\ell_p} \left( \bigvee_{i=0}^{p-2} \Sigma^{2i} \ell_p, \ell_p \right) \cong \prod_{i=0}^{p-2} \Sigma^{-2i} \ell_p \simeq \bigvee_{i=0}^{p-2} \Sigma^{-2i} \ell_p$$

and  $\mathfrak{d}_{ku_p|\ell_p}$  can be identified with a map:

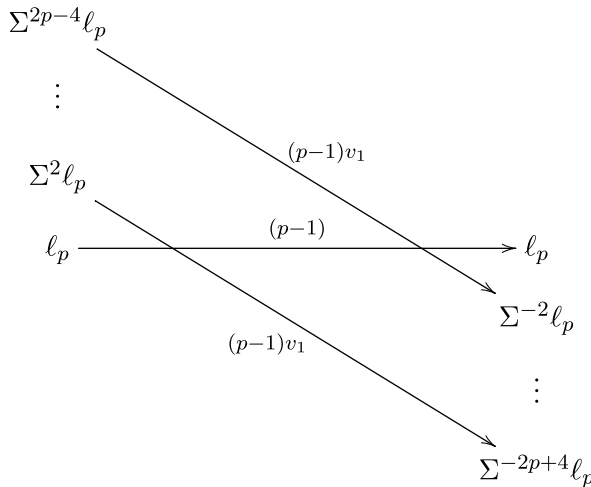
$$\bigvee_{i=0}^{p-2} \Sigma^{2i} \ell_p \rightarrow \bigvee_{i=0}^{p-2} \Sigma^{-2i} \ell_p.$$

As  $\pi_* ku_p$  is a free graded  $\pi_* \ell_p$ -module, we can calculate the effect of  $\mathfrak{d}_{ku_p|\ell_p}$  algebraically via the trace pairing: the element  $\Sigma^{2i} 1 \in \pi_* \Sigma^{2i} \ell_p$  corresponds to  $u^i$ , and it maps an element  $u^j$  to  $tr(u^j \cdot u^i)$ . Since

$(\ell_p \rightarrow ku_p) \circ tr$  is  $\sum_{g \in C_{p-1}} g$ , the sum of  $(p - 1)$   $\ell_p$ -algebra maps, we have

$$tr(u^{i+j}) = \begin{cases} 0, & (p - 1) \nmid i + j, \\ (p - 1)u^{i+j}, & (p - 1) \mid i + j. \end{cases}$$

Hence on the level of homotopy groups,  $\mathfrak{d}_{ku|\ell_p}$  maps  $1 \in \pi_0 \Sigma^0 \ell_p$  to  $(p - 1) \cdot 1 \in \pi_* \Sigma^0 \ell_p = \pi_* \ell_p$  and for  $0 < i \leq p - 2$  it maps  $\Sigma^{2i} 1 \in \pi_* \Sigma^{2i} \ell_p$  via multiplication with  $(p - 1)v_1 = (p - 1)u^{p-1}$  to  $\pi_* \Sigma^{-2p+2i+2} \ell_p$ . On the summands  $\Sigma^{2i} \ell_p$ , we get the following maps:



As  $(p - 1)$  is a unit in  $\pi_0(\ell_p)$  the cofiber of

$$\Sigma^{2i} \ell_p \xrightarrow{(p-1)v_1} \Sigma^{-2p+2i+2} \ell_p$$

is  $\Sigma^{-2p+2i+2} H\mathbb{Z}_p$ . □

Note that  $ko \simeq \tau_{\geq 0} ku^{hC_2}$ , but as the trace map  $tr : ku \rightarrow ku^{hC_2}$  has the connective spectrum  $ku$  as a source, it factors through  $\tau_{\geq 0} ku^{hC_2} \simeq ko$ , and we obtain a discriminant  $\mathfrak{d}_{ku|ko} : ku \rightarrow F_{ko}(ku, ko)$ . We fix notation for  $\pi_* ko$  as:

$$\pi_* ko = \mathbb{Z}[\eta, y, \omega]/(2\eta, \eta^3, \eta y, y^2 - 4\omega)$$

with  $|\eta| = 1$ ,  $|y| = 4$ , and  $|\omega| = 8$ .

**Proposition 2.7.** *There is a cofiber sequence  $ku \xrightarrow{\mathfrak{d}_{ku|ko}} F_{ko}(ku, ko) \rightarrow \Sigma^{-2} H\mathbb{Z}$ .*

*Proof.* The cofiber sequence  $\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{\delta} \Sigma^2 ko \xrightarrow{\eta} \Sigma ko$  induces a cofiber sequence:

$$F_{ko}(\Sigma ko, ko) \xrightarrow{\eta} F_{ko}(\Sigma^2 ko, ko) \xrightarrow{\delta} F_{ko}(ku, ko) \xrightarrow{c} F_{ko}(ko, ko) \xrightarrow{\eta} F_{ko}(\Sigma ko, ko)$$

which is equivalent to

$$\Sigma^{-1} ko \xrightarrow{\eta} \Sigma^{-2} ko \xrightarrow{\delta} F_{ko}(ku, ko) \xrightarrow{-c} ko \xrightarrow{-\eta} \Sigma^{-1} ko.$$

This is the twofold desuspension of the cofiber sequence of and hence,

$$F_{ko}(ku, ko) \simeq \Sigma^{-2} ku.$$

We consider the composition  $c_* \circ \mathfrak{d}_{ku|ko} : ku \rightarrow F_{ko}(ku, ko) \rightarrow F_{ko}(ku, ku)$ . As  $c_*$  is part of the cofiber sequence

$$F_{ko}(ku, \Sigma ko) \xrightarrow{\eta_*} F_{ko}(ku, ko) \xrightarrow{c_*} F_{ko}(ku, ku)$$

and as  $\eta$  is trivial on  $ku$ , we know that  $c_*$  induces a monomorphism on the level of homotopy groups.

As  $\mathfrak{d}_{ku|ko}$  is adjoint to the trace pairing, the composite

$$\pi_* ku \longrightarrow \pi_* F_{ko}(ku, ko) \longrightarrow \pi_* F_{ko}(ku, ku)$$

can be identified with

$$\pi_* ku \longrightarrow \pi_* F_{ko}(ku, ku) \xrightarrow{(\text{id}+t)_*} F_{ko}(ku, ku)$$

where  $t$  denotes the generator of  $C_2$ , and the first map is adjoint to the multiplication  $ku \wedge_{ko} ku \rightarrow ku$ .

The target of  $c_*$  is  $F_{ko}(ku, ku) \simeq F_{ku}(ku \wedge_{ko} ku, ku)$ , and we know by work of the first author, documented in [20, Proof of Lemma 0.1] that

$$\pi_* F_{ku}(ku \wedge_{ko} ku, ku) \cong \text{Hom}_{ku_*}(ku_*[s]/(s^2 - su), \Sigma^{-*}ku_*),$$

so we can control the effect of  $c_* \circ \mathfrak{d}_{ku|ko}$  on homotopy groups.

Note that  $t$  induces a  $ku$ -linear map  $t_* : ku \rightarrow t^*ku$ , where  $t^*ku$  is the  $ku$ -module given by restriction of scalars along  $t$ .

As  $t^2 = \text{id}$ , we therefore obtain

$$F_{ko}(ku, ku) \xrightarrow{t_*} F_{ko}(ku, t^*ku)$$

and a commutative diagram

$$\begin{array}{ccc} F_{ko}(ku, ku) & \xrightarrow{t} & F_{ko}(ku, ku) \\ \simeq \downarrow & & \downarrow \simeq \\ F_{ku}(ku \wedge_{ko} ku, ku) & \xrightarrow{\beta} & F_{ku}(ku \wedge_{ko} ku, ku) \end{array}$$

Here,  $\beta$  induces the map on  $\pi_*$  that sends an  $f : (ku \wedge_{ko} ku)_* \rightarrow \Sigma^{-i}ku_*$  to

$$(ku \wedge_{ko} ku)_* \xrightarrow{(t \wedge \text{id})_*} (ku \wedge_{ko} ku)_* \xrightarrow{f} \Sigma^{-i}ku_* \xrightarrow{t} \Sigma^{-i}ku_*.$$

If we denote the right unit  $\eta_R : ku \rightarrow ku \wedge_{ko} ku$  applied to  $u$  by  $u_r$ , then we have the relation  $2s + u_r = u$ . As  $(t \wedge \text{id})_*(u) = -u$  and  $(t \wedge \text{id})_*(u_r) = u_r$ , this implies that

$$(t \wedge \text{id})_*(2s) = 2s - 2u.$$

Torsion-freeness then yields  $(t \wedge \text{id})_*(s) = s - u$ .

The adjoint of the multiplication map  $\pi_* ku \rightarrow \pi_* F_{ko}(ku, ku)$  maps  $u^i$  to the map that sends 1 to  $\Sigma^{-2i}u^i$  and  $s$  to zero. Therefore, the composite  $c_* \circ \mathfrak{d}_{ku|ko}$  maps  $u^i$  to the map with values  $1 \mapsto \Sigma^{-2i}(u^i + (-1)^i u^i)$  and

$$s \mapsto (s - u)u^i \mapsto -t(u^{i+1}) = (-1)^i u^{i+1}.$$

In order to understand the effect of  $\mathfrak{d}_{ku|ko}$ , we consider the diagram

$$\begin{array}{ccc} \pi_* ku & \xrightarrow{(\mathfrak{d}_{ku|ko})_*} & \pi_* F_{ko}(ku, ko) \cong \pi_{*+2}(ku) \xrightarrow{c_*} \pi_* F_{ko}(ku, ku) \cong \pi_*(\Sigma^{-2}ku \vee ku) \\ & & \downarrow c^* \qquad \qquad \qquad \downarrow c^* \\ & & \pi_* F_{ko}(ko, ko) \cong \pi_*(ko) \xrightarrow{c_*} \pi_* F_{ko}(ko, ku) \cong \pi_*(ku) \end{array}$$

where we can identify  $c^* : \pi_* F_{ko}(ku, ko) \cong \pi_{*+2}(ku) \rightarrow \pi_*(ko)$  with  $\pi_* \Sigma^{-2}\delta$ .

The application of  $c^*$  gives the restriction to the unit  $c : ko \rightarrow ku$ . Say  $(\mathfrak{d}_{ku|ko})_*(u^2) = x \in \pi_6(ku)$ . Then  $\pi_* \Sigma^{-2}\delta(x) = \lambda y$ , and as  $c_*(y) = 2u^2$ , we obtain that  $c^*(\mathfrak{d}_{ku|ko})_*(u^2) = \Sigma^{-4}y$  and therefore  $(\mathfrak{d}_{ku|ko})_*(u^2) = u^3$ .

Similarly  $c^*(\partial_{ku|ko})_*(u^4) = \Sigma^{-8}2\omega$  and  $(\partial_{ku|ko})_*(u^4) = u^5$ . By  $\pi_*(ko)$ -linearity, these calculations yield

$$(\partial_{ku|ko})_*(u^{2i}) = u^{2i+1}$$

for all  $i \geq 0$ .

Restriction to the unit of the odd powers of  $u$  gives zero.

All the  $u^i$  send  $s$  to  $\pm u^{i+1}$  under  $c_* \circ (\partial_{ku|ko})_*$ , so also the odd powers of  $u$  have to hit a generator under  $(\partial_{ku|ko})_*$ , so as a map from  $ku$  to  $\Sigma^{-2}ku$  the map  $\partial_{ku|ko}$  has cofiber  $\Sigma^{-2}H\mathbb{Z}$ .  $\square$

**Remark 2.8.** To prove Proposition 2.7, one can alternatively use that the map  $\partial_{ku|ko} : ku \rightarrow F_{ko}(ku, ko) \simeq \Sigma^{-2}ku$  is in fact  $ku$ -linear [44, Lemma 6.4.6] and that it becomes an equivalence after applying  $-\wedge_{ko} KO \simeq -\wedge_{ku} KU$ , because  $KO \rightarrow KU$  is a Galois extension.

### 3. Describing ramification

#### 3.1. Log-étaleness

It is shown in [49] and [50] that  $\ell \rightarrow ku_{(p)}$  is log-étale with respect to the log structures that are generated by  $v_1$  and by  $u$ . We will use the class  $u \in \pi_2 ku_{(2)}$  in order to define a pre-log structure for  $ko_{(2)} \rightarrow ku_{(2)}$  and show that  $ko_{(2)} \rightarrow ku_{(2)}$  is not log-étale with respect to this pre-log structure. This indicates that the map is not tamely ramified. We use the notation from [50].

Let  $\omega$  denote the Bott element  $\omega \in \pi_8 ko_{(2)}$ . The complexification map sends  $\omega$  to  $u^4$ .

By [50, Lemma 6.2], we have an exact sequence

$$\begin{array}{c} \xrightarrow{\pi_1 \text{TAQ}^C(ku_{(2)})} \\ \text{---} \xrightarrow{\pi_0 \, ku_{(2)} \wedge \gamma(D(u))/\gamma(D(w))} \longrightarrow \pi_0 \text{TAQ}^{(ko_{(2)}, D(w))}(ku_{(2)}, D(u)) \longrightarrow \pi_0 \text{TAQ}^C(ku_{(2)}) \end{array}$$

Here,  $D(u)$  and  $D(\omega)$  are the pre-log structures for the elements  $u$  and  $\omega$  as in [50, Construction 4.2], and  $\text{TAQ}^{(-,-)}$  is log topological André–Quillen homology [50, Definition 5.20]. The commutative ring spectrum  $C$  is given by  $ko_{(2)} \wedge_{S^{\mathcal{J}}D(w)} S^{\mathcal{J}}D(u)$ , where  $S^{\mathcal{J}}$  is the functor from commutative  $\mathcal{J}$ -space monoids to commutative ring spectra defined in [52, p. 2139]. For the definition of the functor  $\gamma(-)$ , see [51, Section 3] and [50, p. 457]. Using [50, Lemma 4.6], it follows that  $\gamma(D(w))$  and  $\gamma(D(u))$  have the homotopy type of the sphere and that  $\gamma(D(w)) \rightarrow \gamma(D(u))$  is multiplication by 4. Therefore, we get

$$\pi_0(ku_{(2)} \wedge \gamma(D(u))/\gamma(D(w))) = \mathbb{Z}/4\mathbb{Z}.$$

We want to show that  $\pi_1 \text{TAQ}^C(ku_{(2)}) = 0 = \pi_0 \text{TAQ}^C(ku_{(2)})$ . By [6, Lemma 8.2], it suffices to show that  $C \rightarrow ku_{(2)}$  is an 1-equivalence. Since  $\pi_1(ku_{(2)}) = 0$ , it is enough to show that the map is an isomorphism on  $\pi_0$ . Since  $S^{\mathcal{J}}D(w)$  and  $S^{\mathcal{J}}D(u)$  are concentrated in nonnegative  $\mathcal{J}$ -space degrees by [48, Example 6.8], they are connective. Thus, it is enough to show that  $S^{\mathcal{J}}D(w) \rightarrow S^{\mathcal{J}}D(u)$  induces an isomorphism on  $\pi_0$ . For this, we only have to prove that  $H_0(S^{\mathcal{J}}D(w), \mathbb{Z}) \rightarrow H_0(S^{\mathcal{J}}D(u), \mathbb{Z})$  is an isomorphism. Since this map is a ring map, we only need to know that both sides are  $\mathbb{Z}$ . This follows from [49, Proposition 5.2, Corollary 5.3]. Hence, we obtain the following result:

**Theorem 3.1.** *The map  $(ko_{(2)}, D(\omega)) \rightarrow (ku_{(2)}, D(u))$  is not log-étale.*

One could try to distinguish between tame and wild ramification by testing for log-étaleness. In many examples, however, it is less obvious what a suitable log structure would be. Calculations with log structures that are generated by more than one element are challenging because the methods above do not work. For a thorough investigation of log-étaleness and for related calculations, see Lundemo [31].

3.2. Ramification and Tate cohomology

In the algebraic context of Galois extensions of number fields and corresponding extension of number rings, tame ramification yields a normal basis and a surjective trace map. Both facts are actually also sufficient in order to distinguish tame from wild ramification. For structured ring spectra, it does not work to impose these properties on the level of homotopy groups, because even for finite faithful Galois extensions these would not hold. Instead, we propose to use the Tate construction in order to understand ramification.

**Remark 3.2.** Let  $G$  be a finite group. Usually one calls a  $G$ -module  $M$  cohomologically trivial, if  $\hat{H}^i(H; M) = 0$ , for all  $i \in \mathbb{Z}$  and all  $H < G$ . If  $M$  is a commutative ring  $S$ , however, it suffices to require  $\hat{H}^i(G; S) = 0$  for all  $i \in \mathbb{Z}$ : In particular,  $\hat{H}^0(G; S) = 0$ , and hence the norm map  $N_G : S_G \rightarrow S^G$  (resp. the trace map  $tr_G : S \rightarrow S^G$ ) is surjective. Thus,  $1_{S^G}$  is in the image of the norm, say  $N_G[x] = 1_{S^G}$  for  $[x] \in S_G$ . If  $H < G$ , then we consider the diagram:

$$\begin{array}{ccc} H^0(G; S) = S^G & \xrightarrow{i^*} & H^0(H; S) = S^H \\ N_G \uparrow & & \uparrow N_H \\ H_0(G; S) = S_G & \xrightarrow{tr_H^G} & H_0(H; S) = S_H \end{array}$$

and therefore we can express  $1_{S^H}$  as:

$$1_{S^H} = i^*(1_{S^G}) = i^*N_G[x] = N_H tr_H^G[x],$$

so  $1_{S^H}$  is in the image of  $N_H$  and  $\hat{H}^0(H; S) = 0$ . But  $\hat{H}^*(H; S)$  is a graded commutative ring with unit  $[1_{S^H}] = 0$ , and thus  $\hat{H}^*(H; S) = 0$ .

The same argument shows that the surjectivity of the trace map suffices for being cohomologically trivial.

In particular, if the Tate cohomology is nontrivial, then the trace map is not surjective and this indicates wild ramification. In the following, we transfer this relationship to structured ring spectra.

We need the following generalization of Tate cohomology; for background, see [21]. If  $E$  is a spectrum with an action of a finite group  $G$ , then there is a norm map  $N : E_{hG} \rightarrow E^{hG}$  from the homotopy coinvariants of  $E$  with respect to  $G$ ,  $E_{hG}$ , to the homotopy fixed points,  $E^{hG}$ . Its cofiber is the Tate construction of  $E$  with respect to  $G$ ,  $E^{tG}$ . If  $E$  is an Eilenberg–MacLane spectrum  $E = HD$  for some abelian group  $D$ , then  $\pi_*((HD)^{tG}) \cong \hat{H}^{-*}(G; D)$ .

Even if  $A \rightarrow B$  is a  $G$ -Galois extension of ring spectra in the sense of Rognes [44, Definition 4.1.3], it is not true that this implies that  $B$  is faithful as an  $A$ -module [44, Definition 4.3.1]. An example due to Wieland is the  $C_2$ -Galois extension  $F((BC_2)_+, H\mathbb{F}_2) \rightarrow F((EC_2)_+, H\mathbb{F}_2) \simeq H\mathbb{F}_2$  which is not faithful: the  $F((BC_2)_+, H\mathbb{F}_2)$ -module spectrum  $(H\mathbb{F}_2)^{tC_2}$  is not trivial, but  $H\mathbb{F}_2 \wedge_{F((BC_2)_+, H\mathbb{F}_2)} (H\mathbb{F}_2)^{tC_2} \sim *$ . In fact, a  $G$ -Galois extension  $A \rightarrow B$  is faithful if and only if  $B^{tG}$  is contractible [44, Proposition 6.3.3].

In the following, we denote by  $\tau_{\geq 0}X$  the connective cover of a spectrum  $X$ . Note that for a map  $A \rightarrow B$  between connective commutative ring spectra with a finite group  $G$  acting on  $B$  via commutative  $A$ -algebra maps it makes sense to replace the usual homotopy fixed point condition by the condition that  $A$  is weakly equivalent to  $\tau_{\geq 0}B^{hG}$ . In many examples,  $B^{hG}$  won't be connective. The map  $A \rightarrow B$  factors through  $A \rightarrow B^{hG} \rightarrow B$ , but as  $A$  is connective, we can consider the induced map on connective covers and obtain a map of commutative ring spectra:

$$\tau_{\geq 0}A = A \rightarrow \tau_{\geq 0}B^{hG} \rightarrow \tau_{\geq 0}B = B,$$

that turns  $\tau_{\geq 0}B^{hG}$  into a commutative  $A$ -algebra spectrum.

For any spectrum  $X$ , we denote by  $\tau_{< 0}X$  the cofiber of the map  $\tau_{\geq 0}X \rightarrow X$ .



**Lemma 3.3.** *Let  $G$  be a finite group and let  $e$  be a naive connective  $G$ -spectrum. Then,*

$$\tau_{\geq 0}e^{hG} \rightarrow e^{hG} \rightarrow \tau_{< 0}e^{tG}$$

*is a cofiber sequence and in particular,  $\tau_{< 0}e^{tG} \simeq \tau_{< 0}e^{hG}$ .*

*Proof.* We consider the norm sequence:

$$e_{hG} \xrightarrow{N} e^{hG} \longrightarrow e^{tG}.$$

As  $e_{hG}$  is a connective spectrum, we have that  $\pi_{-1}e_{hG} = 0$ . Hence, applying  $\tau_{\geq 0}$  still gives rise to a cofiber sequence:

$$\tau_{\geq 0}e_{hG} = e_{hG} \xrightarrow{\tau_{\geq 0}N} \tau_{\geq 0}e^{hG} \longrightarrow \tau_{\geq 0}e^{tG}$$

We combine the norm cofiber sequences with the defining cofiber sequence of  $\tau_{< 0}$  and obtain

$$\begin{array}{ccccc} \tau_{\geq 0}e_{hG} & \xlongequal{\quad} & e_{hG} & \longrightarrow & * \\ \tau_{\geq 0}N \downarrow & & N \downarrow & & \downarrow \\ \tau_{\geq 0}e^{hG} & \longrightarrow & e^{hG} & \longrightarrow & \tau_{< 0}e^{hG} \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{\geq 0}e^{tG} & \longrightarrow & e^{tG} & \longrightarrow & \tau_{< 0}e^{tG} \end{array}$$

Thus,  $\tau_{< 0}e^{hG} \simeq \tau_{< 0}e^{tG}$  and the cofiber sequence in the second row then yields the claim. □

**Remark 3.4.** *In many cases, if  $B^{tG} \not\cong *$ , then  $\pi_*(B^{tG})$  is actually periodic. As the canonical Künneth map:*

$$\pi_*(B^{tG}) \otimes_{\pi_*(B^{hG})} \pi_*(B) \rightarrow \pi_*(B^{tG} \wedge_{B^{hG}} B)$$

*is a map of graded commutative rings and as  $\pi_*(B^{tG}) \cong \pi_*(B^{hG})$  in negative degrees, a periodicity generator in a negative degree would map to zero in  $\pi_*B$  for connective  $B$  and hence  $\pi_*(B^{tG}) \otimes_{\pi_*(B^{hG})} \pi_*(B)$  is the zero ring. But then also  $\pi_*(B^{tG} \wedge_{B^{hG}} B) \cong 0$  and*

$$B^{tG} \wedge_{B^{hG}} B \simeq *.$$

*Therefore,  $B$  would not be a faithful  $B^{hG}$ -module in these cases. This emphasizes the importance of replacing the condition that  $A$  be weakly equivalent to  $B^{hG}$  by the requirement that  $A \simeq \tau_{\geq 0}(B^{hG})$ .*

*From Lemma 3.3, we also know that in order to show that  $B^{tG} \not\cong *$  for connective  $B$  it is sufficient to show that  $\tau_{< 0}B^{hG}$  is not trivial.*

We recall the following result from [44]:

**Proposition 3.5.** [44, Proposition 6.3.3] *Assume that  $G$  is a finite group,  $B$  is a cofibrant commutative  $A$ -algebra on which  $G$  acts via maps of commutative  $A$ -algebras. If  $B$  is dualizable and faithful as an  $A$ -module and if*

$$h : B \wedge_A B \xrightarrow{\sim} F(G_+, B),$$

*then  $B^{tG} \simeq *$ .*

Rognes assumes that  $A \simeq B^{hG}$ , but that assumption is not needed. A referee actually noted that it follows from the remaining assumptions in the Proposition: smashing the map  $A \rightarrow B^{hG}$

with  $B$  over  $A$  yields  $B \rightarrow B \wedge_A B^{hG}$ . Dualizability of  $B$  as an  $A$ -module identifies the latter with  $(B \wedge_A B)^{hG} \simeq (F(G_+, B))^{hG} \simeq B$ . As  $B$  is faithful as an  $A$ -module, this implies  $A \simeq B^{hG}$ .

**Remark 3.6.** Assume that  $G$  is a finite group and that  $B$  is a cofibrant commutative  $A$ -algebra on which  $G$  acts via maps of commutative  $A$ -algebras. If  $B$  is dualizable and faithful as an  $A$ -module and if  $B^{tG} \not\simeq *$ , then we know that  $h : B \wedge_A B \rightarrow F(G_+, B)$  cannot be a weak equivalence, that is, that  $A \rightarrow B$  is ramified.

In the following, we study the Tate constructions in several examples. To compute the homotopy of  $B^{tG}$ , we use the Tate spectral sequence:

$$E_{n,m}^2 = \hat{H}^{-n}(G; \pi_m(B)) \implies \pi_{n+m} B^{tG}$$

which is of standard homological type, multiplicative, and conditionally convergent. In particular by [11, Theorem 8.2], it converges strongly if it collapses at a finite stage.

If the spectrum  $B$  is connective, then the vanishing of  $\pi_* B^{tG}$  is equivalent to the fact that  $1 \in \pi_0 B$  is in the image of the norm map  $N : \pi_0(B_{hG}) \rightarrow \pi_0 B$ . We learned a proof of this fact from one of the referees of an earlier version: one direction follows directly because the Postnikov section  $B \rightarrow H(\pi_0(B))$  is a  $G$ -equivariant map of commutative ring spectra. For the reverse note that the vanishing of  $\hat{H}^*(G; \pi_0(B))$  implies the vanishing of  $\hat{H}^*(G; \pi_n(B))$  for all  $n$ . Thus, the  $E^2$ -page of the Tate spectral sequence is trivial and hence  $B^{tG} \simeq *$ .

**Remark 3.7.** Fix a prime  $p$ . In the connective examples below, if  $\hat{H}^*(G; \pi_0(B_{(p)})) = 0$ , then we might say that  $B$  has tame ramification at  $p$  and for primes  $p$  with  $\hat{H}^*(G; \pi_0(B_{(p)})) \neq 0$  we might say that  $B$  has wild ramification at the prime  $p$ .

Work by Greenlees, Hovey, Kuhn, and Sadofsky [22, 26, 29] shows that for any finite group and any  $K(n)$ -local spectrum (or  $T(n)$ -local spectrum), the Tate construction is trivial  $K(n)$ -locally (or  $T(n)$ -locally). See also [14]. Note that in our examples,  $K(n)$ -localization for large  $n$  is actually trivial, so the information about ramification is concentrated at small  $n$ .

We will now investigate the Tate construction in examples. First, we establish faithfulness:

**Lemma 3.8.** The map  $\mathrm{tmf}_0(2)_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}$  identifies  $\mathrm{tmf}(2)_{(3)}$  as a faithful  $\mathrm{tmf}_0(2)_{(3)}$ -module.

*Proof.* For the map  $\mathrm{tmf}_0(2)_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}$ , we know that  $C_2$  acts on  $\mathrm{tmf}(2)_{(3)}$  via commutative  $\mathrm{tmf}_0(2)_{(3)}$ -algebra maps and that  $\mathrm{tmf}_0(2)_{(3)} \simeq \tau_{\geq 0}(\mathrm{tmf}(2)_{(3)}^{hC_2})$ . The trace map  $tr : \mathrm{tmf}(2)_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}^{hC_2}$  factors through  $\tau_{\geq 0}(\mathrm{tmf}(2)_{(3)}^{hC_2}) \simeq \mathrm{tmf}_0(2)_{(3)}$ , because  $\mathrm{tmf}(2)_{(3)}$  is connective. As in [44, Lemma 6.4.3], one can show that the composite:

$$\mathrm{tmf}_0(2)_{(3)} \simeq \tau_{\geq 0}(\mathrm{tmf}(2)_{(3)}^{hC_2}) \longrightarrow \mathrm{tmf}(2)_{(3)} \xrightarrow{(0,3)tr} \tau_{\geq 0}(\mathrm{tmf}(2)_{(3)}^{hC_2}) \simeq \mathrm{tmf}_0(2)_{(3)}$$

is homotopic to the map that is the multiplication by  $|C_2| = 2$ . As 2 is invertible in  $\pi_0 \mathrm{tmf}_0(2)_{(3)}$ , the trace map  $tr : \mathrm{tmf}(2)_{(3)} \rightarrow \mathrm{tmf}_0(2)_{(3)}$  is a split surjective map of  $\mathrm{tmf}_0(2)_{(3)}$ -modules and hence  $\mathrm{tmf}_0(2)_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}$  is faithful.

Alternatively, faithfulness also follows from the fact that  $\pi_* \mathrm{tmf}(2)_{(3)}$  is a free  $\pi_* \mathrm{tmf}_0(2)_{(3)}$ -module (see the proof of Theorem 2.5). □

**Lemma 3.9.** The spectrum  $\mathrm{tmf}_0(2)_{(3)}$  is faithful as a  $\mathrm{tmf}_{(3)}$ -module spectrum.

This result also follows from [40, Proposition 4.15].

*Proof.* We already mentioned the identification  $\mathrm{tmf}_0(2)_{(3)} \simeq \mathrm{tmf}_{(3)} \wedge T$  where  $T = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$  with  $\alpha_1 \in (\pi_3 S)_{(3)}$ , [10, Lemma 2, p. 382], [36, Theorem 4.15]. Note that  $\alpha_1$  is nilpotent of order 2 because  $(\pi_6 S)_{(3)} = 0$ .

Assume that  $M$  is a  $\mathbf{tmf}_{(3)}$ -module with

$$* \simeq M \wedge_{\mathbf{tmf}_{(3)}} \mathbf{tmf}_0(2)_{(3)} \simeq M \wedge T.$$

Then, the cofiber sequences

$$S^0 \longrightarrow T \longrightarrow \Sigma^4 \text{cone}(\alpha_1) \text{ and } \text{cone}(\alpha_1) \longrightarrow T \longrightarrow S^8$$

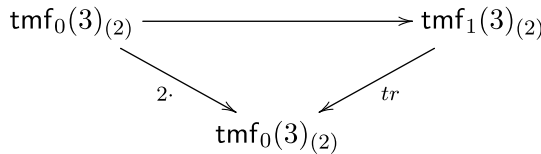
imply that  $\Sigma^4 \text{cone}(\alpha_1) \wedge M \simeq \Sigma M$  and  $\Sigma^8 M \simeq \Sigma \text{cone}(\alpha_1) \wedge M$  and therefore,

$$\Sigma^{10} M \simeq M.$$

The equivalence is induced by a class in  $\pi_{10} S_{(3)} \cong \mathbb{Z}/3\mathbb{Z}\{\beta_1\}$ . As this is nilpotent, we get that  $M \simeq *$ . □

**Remark 3.10.** *It is known that  $ko \rightarrow ku$  is faithful [44, Proposition 5.3.1] and dualizable, and it is clear that  $\ell \rightarrow ku_{(p)}$  is faithful and dualizable as the inclusion of a summand. As  $\mathbf{tmf}_1(3)_{(2)}$  can be identified with  $\mathbf{tmf}_{(2)} \wedge DA(1)$  as a  $\mathbf{tmf}_{(2)}$ -module [36, Theorem 4.12], where  $DA(1)$  is a finite cell complex realizing the double of  $A(1) = \langle Sq^1, Sq^2 \rangle$ , it is dualizable. An argument as in [44, Proof of Proposition 5.4.5] shows that  $\mathbf{tmf}_{(2)} \rightarrow \mathbf{tmf}_1(3)_{(2)}$  is faithful.*

At the moment we don't know whether  $\mathbf{tmf}_0(3)_{(2)} \rightarrow \mathbf{tmf}_1(3)_{(2)}$  is faithful. The diagram



commutes, so if  $M$  is a  $\mathbf{tmf}_0(3)_{(2)}$ -module spectrum with  $M \wedge_{\mathbf{tmf}_0(3)_{(2)}} \mathbf{tmf}_1(3)_{(2)} \simeq *$ , then multiplication by 2 is a trivial self-map on  $M$ . Meier shows [40, Proposition 4.13] that  $\mathbf{tmf}_1(3)$  is not perfect as a  $\mathbf{tmf}_0(3)$ -module; hence,  $\mathbf{tmf}_1(3)$  is not a dualizable  $\mathbf{tmf}_0(3)$ -module.

Meier also proves that  $\mathbf{tmf} \left[ \frac{1}{n} \right] \rightarrow \mathbf{tmf}(n)$  is dualizable and faithful for all  $n$  [40, Theorem 4.4, Proposition 4.15]; thus,  $\mathbf{tmf}(2)_{(3)}$  is dualizable and faithful as a  $\mathbf{tmf}_{(3)}$ -module.

We show that the extensions  $\mathbf{tmf}_0(3)_{(2)} \rightarrow \mathbf{tmf}_1(3)_{(2)}$  and  $\mathbf{tmf}_{(3)} \rightarrow \mathbf{tmf}(2)_{(3)}$  have nontrivial Tate spectra. For  $ku$ , the Tate spectrum with respect to the complex conjugation  $C_2$ -action satisfies

$$ku^{C_2} \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^{4i} H\mathbb{Z}/2\mathbb{Z}.$$

This result is due to Rognes (compare [44, §5.3]). As  $ku_{(p)}^{C_2} \simeq *$  for odd primes  $p$ , 2 is the only wildly ramified prime.

**Theorem 3.11.** *For  $\mathbf{tmf}_1(3)_{(2)}$  with its  $C_2$ -action, we obtain an equivalence of spectra:*

$$\mathbf{tmf}_1(3)_{(2)}^{C_2} \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^{8i} H\mathbb{Z}/2\mathbb{Z}.$$

*Proof.* We use the calculations in [33]. They compute the homotopy fixed point spectral sequence:

$$E_{n,m}^2 = H^{-n}(C_2; \pi_m \mathbf{TMF}_1(3)_{(2)}) \longrightarrow \pi_{n+m} \mathbf{TMF}_0(3)_{(2)},$$

where  $\pi_* \mathbf{TMF}_1(3)_{(2)} = \mathbb{Z}_{(2)}[a_1, a_3][[\Delta^{-1}]]$  with  $\Delta = a_3^3(a_1^3 - 27a_3)$ . From their computations, we deduce the following behavior of the Tate spectral sequence:

$$E_{n,m}^2 = \hat{H}^{-n}(C_2; \pi_m \mathbf{TMF}_1(3)_{(2)}) \longrightarrow \pi_{n+m} \mathbf{TMF}_1(3)_{(2)}^{C_2}. \tag{3.1}$$

Let  $R_{n,m}$  be the bigraded ring  $\mathbb{Z}/2[a_1, a_3][[\Delta^{-1}]][\zeta^{\pm}]$  with  $|\zeta| = (-1, 0)$ . If we assign odd weight to  $a_1, a_3$ , and  $\zeta$ , then the  $E^2$ -page of the Tate spectral sequence is the even part of  $R_{n,m}$ . Alternatively, it is

given by:

$$E_{*,*}^2 = S_*[\Delta^{-1}][x^\pm],$$

where  $S_*$  is the subalgebra of  $\mathbb{Z}/2\mathbb{Z}[a_1, a_3]$  generated by  $a_1^2, a_1a_3, a_3^2$ , and where  $x = \zeta a_3^3 \in E_{-1,18}$ . Note that  $a_3^2$  is invertible in this ring with  $a_3^{-2} = ((a_1a_3)a_1^2 - 27a_3^2)\Delta^{-1}$ . By Mahowald–Rezk’s computations, the first nontrivial differential is  $d^3$  and we have

$$\begin{aligned} d^3(a_1^2) &= (x(a_1a_3)a_3^{-4})^3, & d^3(a_1a_3) &= 0, & d^3(a_3^2) &= x^3(a_1a_3)a_3^{-8}, \\ d^3(x) &= 0, & d^3(\Delta^{-1}) &= 0. \end{aligned}$$

Using the Leibniz rule, we get that the class  $c_{n,m,k,l,i} = (a_1^2)^n(a_1a_3)^m(a_3^2)^k\Delta^{-l}x^i$  with  $n, m, k, l \in \mathbb{N}$  and  $i \in \mathbb{Z}$  has differential

$$d^3(c_{n,m,k,l,i}) = (n+k)x^3(a_1a_3)a_3^{-10}c_{n,m,k,l,i}.$$

It follows that  $\ker d^3$  is generated as an  $\mathbb{F}_2$ -vector space by the classes  $c_{n,m,k,l,i}$  with  $n+k=0$  in  $\mathbb{F}_2$ . We claim that

$$E_{*,*}^4 \cong \mathbb{F}_2[x^\pm, \Delta^\pm].$$

To see this, note the following: If  $n+k=0$  in  $\mathbb{F}_2$  and  $m > 0$ , then  $c_{n,m,k,l,i}$  is zero in  $E_{*,*}^4$  because

$$d^3(c_{n,m-1,k+5,l,i-3}) = c_{n,m,k,l,i}.$$

If  $n+k=0$  in  $\mathbb{F}_2$  and  $n, k > 0$ , then we have  $c_{n,0,k,l,i} = c_{n-1,2,k-1,l,i}$ . This is in the image of  $d^3$ , because  $n-1+k-1=0$  in  $\mathbb{F}_2$  and  $2 > 0$ . If  $n=0$  in  $\mathbb{F}_2$  and  $n > 0$ , then

$$\begin{aligned} c_{n,0,0,l,i} &= (a_1^2)^n\Delta^{-l}x^i \\ &= (a_1a_3)^2(a_1^2)^{n-1}a_3^{-2}\Delta^{-l}x^i \\ &= (a_1a_3)^2(a_1^2)^{n-1}((a_1a_3)a_1^2 + a_3^2)\Delta^{-1}\Delta^{-l}x^i \\ &= c_{n,3,0,l+1,i} + c_{n-1,2,1,l+1,i}, \end{aligned}$$

and both of these summands are in the image of  $d^3$ . Furthermore, note that in  $E_{*,*}^4$  we have

$$\Delta = (a_1a_3)^3 + a_3^4 = c_{0,3,0,0,0} + a_3^4 = a_3^4.$$

This implies that for  $k=0$  in  $\mathbb{F}_2$  we have

$$c_{0,0,k,l,i} = (a_3^4)^{\frac{k}{2}}\Delta^{-l}x^i \equiv \Delta^{-l+\frac{k}{2}}x^i$$

in  $E_{*,*}^4$ . We thus get a surjective map  $\mathbb{F}_2[x^\pm, \Delta^\pm] \rightarrow E_{*,*}^4$ , which is injective, because the classes  $\Delta^l x^i$  for  $l, i \in \mathbb{Z}$  are not divisible by  $(a_1a_3)$  in  $S_*[\Delta^{-1}][x^\pm]$ .

From Mahowald–Rezk’s computations, we get that the next nontrivial differential is  $d^7$  and that we have

$$d^7(x) = 0 \text{ and } d^7(\Delta) = x^7\Delta^{-4}.$$

This gives  $E_{*,*}^8 = 0$ .

We now want to determine the behavior of the Tate spectral sequence:

$$E_{n,m}^2 = \hat{H}^{-n}(C_2; \pi_m \mathbf{tmf}_1(3)_{(2)}) \implies \pi_{n+m} \mathbf{tmf}_1(3)_{(2)}^{tC_2}. \tag{3.2}$$

If we assign again odd weight to  $a_1, a_3$ , and  $\zeta$ , then the  $E^2$ -page is the even part of

$$\mathbb{Z}/2\mathbb{Z}[a_1, a_3][\zeta^\pm],$$

and one sees that the map of spectral sequences from (3.2) to (3.1) is injective. We get that  $d^3$  is the first nontrivial differential in (3.2) and that we have

$$\begin{aligned} d^3(a_1a_3) &= 0, & d^3(a_1^2) &= (a_1\zeta)^3, \\ d^3(a_3^2) &= a_1a_3^2\zeta^3, & d^3(a_3\zeta) &= (a_1a_3)\zeta^4, \\ d^3(a_1\zeta) &= 0, & d^3(\zeta^2) &= a_1\zeta^5. \end{aligned}$$

Note that an  $\mathbb{F}_2$ -basis of the  $E^3$ -page is given by the classes:

$$\begin{aligned} d_{n,m,i} &= (a_1^2)^n (a_3^2)^m (\zeta^2)^i, \\ e_{n,m,i} &= (a_1^2)^n (a_1a_3) (a_3^2)^m (\zeta^2)^i, \\ f_{n,m,i} &= (a_1^2)^n (a_3^2)^m (a_1\zeta) (\zeta^2)^i, \\ g_{n,m,i} &= (a_1^2)^n (a_3^2)^m (a_3\zeta) (\zeta^2)^i, \end{aligned}$$

for  $n, m \in \mathbb{N}$ , and  $i \in \mathbb{Z}$ .

The  $d^3$ -differential on these classes is given by:

$$\begin{aligned} d^3(d_{n,m,i}) &= (n + m + i) \cdot f_{n,m,i+1}, \\ d^3(e_{n,m,i}) &= (n + m + i) \cdot g_{n+1,m,i+1}, \\ d^3(f_{n,m,i}) &= (n + m + i) \cdot d_{n+1,m,i+2}, \\ d^3(g_{n,m,i}) &= (n + m + i + 1) \cdot e_{n,m,i+2}. \end{aligned}$$

We get

$$E_{*,*}^4 = \bigoplus_{\substack{m \in \mathbb{N}, i \in \mathbb{Z} \\ m+i=0 \text{ in } \mathbb{F}_2}} \mathbb{F}_2\{d_{0,m,i}\} \oplus \bigoplus_{\substack{m \in \mathbb{N}, i \in \mathbb{Z} \\ m+i+1=0 \text{ in } \mathbb{F}_2}} \mathbb{F}_2\{g_{0,m,i}\}.$$

The map of spectral sequences from (3.2) to (3.1) satisfies

$$d_{0,m,i} \mapsto \Delta^{\frac{m-3i}{2}} x^{2i}, \quad g_{0,m,i} \mapsto \Delta^{\frac{m-3i-1}{2}} x^{2i+1}.$$

In particular, one sees that it is injective on  $E^4$ -pages. We conclude that the next nontrivial differential in spectral sequence (3.2) is  $d^7$  and that we have

$$d^7(d_{0,m,i}) = \frac{m-3i}{2} g_{0,m,i+3}, \quad d^7(g_{0,m,i}) = \frac{m-3i-1}{2} d_{0,m+1,i+4}.$$

We obtain that

$$E_{*,*}^8 = \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_2\{d_{0,0,4i}\} = \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_2\{\zeta^{8i}\}.$$

Since the  $E^8$ -page is concentrated in the zeroth row, the spectral sequence collapses at this stage. This gives the answer on the level of homotopy groups. As  $\text{tmf}_1(3)^{C_2}$  is an  $E_\infty$ -ring spectrum [39], it is in particular an  $E_2$ -ring spectrum and therefore a result by Hopkins–Mahowald (see [38, Theorem 4.18]) implies that  $\text{tmf}_1(3)^{C_2}$  receives a map from  $H\mathbb{F}_2$  and therefore is a generalized Eilenberg–MacLane spectrum of the claimed form. □

**Theorem 3.12.** *The  $\Sigma_3$ -action on  $\text{tmf}(2)_{(3)}$  yields*

$$\text{tmf}(2)_{(3)}^{\Sigma_3} \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^{12i} H\mathbb{Z}/3\mathbb{Z}.$$

*Proof.* We use the calculation of [53]. She proves that  $\mathrm{Tmf}(2)_{(3)}^{\Sigma_3} \simeq *$  via the Tate spectral sequence:

$$E_{n,m}^2 = \hat{H}^{-n}(\Sigma_3; \pi_m(\mathrm{Tmf}(2)_{(3)})) \implies \pi_{n+m}(\mathrm{Tmf}(2)_{(3)}^{\Sigma_3}).$$

The  $E^2$ -page is given by:

$$\mathbb{Z}/3\mathbb{Z}[\alpha, \beta^\pm, \Delta^\pm]/\alpha^2$$

with  $|\alpha| = (-1, 4)$ ,  $|\beta| = (-2, 12)$ , and  $|\Delta| = (0, 24)$ , and the differentials are determined by:

$$d^5(\Delta) = \alpha\beta^2 \quad \text{and} \quad d^9(\alpha\Delta^2) = \beta^5.$$

Since  $\mathrm{tmf}(2)_{(3)}$  is the connective cover of  $\mathrm{Tmf}(2)_{(3)}$ , the  $E^2$ -page of the Tate spectral sequence:

$$\bar{E}_{n,m}^2 = \hat{H}^{-n}(\Sigma_3; \pi_m(\mathrm{tmf}(2)_{(3)})) \implies \pi_{n+m}(\mathrm{tmf}(2)_{(3)}^{\Sigma_3})$$

is the  $\mathbb{Z}/3\mathbb{Z}$ -module

$$\bigoplus_{\substack{k,l \in \mathbb{Z} \\ k+2l \geq 0}} \mathbb{Z}/3\mathbb{Z}\{\beta^k \Delta^l\} \oplus \bigoplus_{\substack{k,l \in \mathbb{Z} \\ 1+3k+6l \geq 0}} \mathbb{Z}/3\mathbb{Z}\{\alpha\beta^k \Delta^l\}.$$

Using the map of Tate spectral sequences  $\bar{E}_{*,*}^* \rightarrow E_{*,*}^*$  one sees that

$$\bar{E}_{*,*}^6 = \bigoplus_{\substack{k,l \in \mathbb{Z} \\ k+6l \geq 0}} \mathbb{Z}/3\mathbb{Z}\{\beta^k (\Delta^3)^l\} \oplus \bigoplus_{\substack{k,l \in \mathbb{Z} \\ 13+3k+18l \geq 0}} \mathbb{Z}/3\mathbb{Z}\{(\alpha\Delta^2)\beta^k (\Delta^3)^l\}.$$

Since  $E_{*,*}^6 = \mathbb{Z}/3\mathbb{Z}[\alpha\Delta^2, \beta^\pm, \Delta^\pm]/(\alpha\Delta^2)^2$  the map  $\bar{E}_{*,*}^6 \rightarrow E_{*,*}^6$  is injective. Thus,  $\bar{d}^9$  is determined by  $d^9$  and one gets

$$\bar{E}_{*,*}^{10} = \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}/3\mathbb{Z}\{(\beta^{-6} \Delta^3)^k\}.$$

The class  $\beta^{-6} \Delta^3$  has bidegree  $(12, 0)$ , and so  $\bar{E}_{*,*}^{10}$  is concentrated in line zero and the spectral sequence collapses at this stage. □

So the extensions  $ko_{(2)} \rightarrow ku_{(2)}$ ,  $\mathrm{tmf}_{(3)} \rightarrow \mathrm{tmf}(2)_{(3)}$ , and  $\mathrm{tmf}_0(3)_{(2)} \rightarrow \mathrm{tmf}_1(3)_{(2)}$  have nontrivial Tate constructions.

In contrast,  $KO \rightarrow KU$  is a faithful  $C_2$ -Galois [44, §5], and  $\mathrm{TMF}_0(3) \rightarrow \mathrm{TMF}_1(3)$  and  $\mathrm{Tmf}_0(3) \rightarrow \mathrm{Tmf}_1(3)$  are both faithful  $C_2$ -Galois extensions [37, Theorem 7.12]. In general,  $\mathrm{TMF}[1/n] \rightarrow \mathrm{TMF}(n)$  is a faithful  $GL_2(\mathbb{Z}/n\mathbb{Z})$ -Galois extension [37, Theorem 7.6] and the Tate spectrum  $\mathrm{Tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})}$  is contractible [37, Theorem 7.11].

For general  $n > 1$ , constructions of  $\mathrm{tmf}_1(n)$  and  $\mathrm{tmf}_0(n)$  are tricky: for some large  $n$ ,  $\pi_1 \mathrm{Tmf}_1(n)$  is nontrivial. Lennart Meier constructs a connective version of  $\mathrm{Tmf}_1(n)$  with trivial  $\pi_1$  as an  $E_\infty$ -ring spectrum in [40, Theorem 1.1] so that there are  $E_\infty$ -models of  $\mathrm{tmf}_1(n)$  for all  $n$ .

We cannot determine the homotopy type of the  $GL_2(\mathbb{Z}/n\mathbb{Z})$ -Tate construction of  $\mathrm{tmf}(n)$  for arbitrary  $n > 1$ , but we can identify cases where it is nontrivial.

**Theorem 3.13.** *Assume that for  $n \geq 2$  we have that  $\pi_1 \mathrm{Tmf}_1(n) = 0$ . Then  $\mathrm{tmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})} \simeq *$  if and only if the order of  $SL_2(\mathbb{Z}/n\mathbb{Z})$ , or equivalently the order of  $GL_2(\mathbb{Z}/n\mathbb{Z})$ , is a unit in  $\mathbb{Z}[\frac{1}{n}]$ .*

*In particular, if  $n \geq 2$  with  $\pi_1 \mathrm{Tmf}_1(n) = 0$  and  $2 \nmid n$  or if  $n = 2^k$  for  $k \geq 1$ , then  $\mathrm{tmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})} \not\simeq *$ .*

*Proof.* Since  $\mathrm{tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})}$  is connective, the defining cofiber sequence of  $\mathrm{tmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})}$  gives an exact sequence:

$$\dots \longrightarrow \pi_0 \mathrm{tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})} \xrightarrow{N} \pi_0 \mathrm{tmf}(n)^{hGL_2(\mathbb{Z}/n\mathbb{Z})} \longrightarrow \pi_0 \mathrm{tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})} \longrightarrow 0.$$

We have that  $\pi_0(\mathbf{tmf}(n)) \cong \mathbb{Z}[\frac{1}{n}, \zeta_n]$ , where  $\zeta_n$  is a primitive  $n$ th root of unity. Consider the commutative diagram

$$\begin{CD} \pi_0(\mathbf{tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})}) @>N>> \pi_0(\mathbf{tmf}(n)^{hGL_2(\mathbb{Z}/n\mathbb{Z})}) \\ @VVV @VVV \\ \pi_0(H\mathbb{Z}[\frac{1}{n}, \zeta_n]_{hGL_2(\mathbb{Z}/n\mathbb{Z})}) @>N>> \pi_0(H\mathbb{Z}[\frac{1}{n}, \zeta_n]^{hGL_2(\mathbb{Z}/n\mathbb{Z})}) = \mathbb{Z}[\frac{1}{n}, \zeta_n]^{GL_2(\mathbb{Z}/n\mathbb{Z})} \end{CD}$$

By the homotopy orbit spectral sequence, we have that the left-hand vertical map is an isomorphism. By [27, p. 282], an element  $A \in GL_2(\mathbb{Z}/n\mathbb{Z})$  acts on  $\zeta_n^i$  as:

$$A\zeta_n^i = \zeta_n^{\det A \cdot i}.$$

This implies that the ring in the lower right corner is  $\mathbb{Z}[\frac{1}{n}]$ . Since we also have

$$\pi_0 \mathbf{tmf}(n)^{hGL_2(\mathbb{Z}/n\mathbb{Z})} \cong \pi_0 \mathbf{tmf} \left[ \frac{1}{n} \right] = \mathbb{Z} \left[ \frac{1}{n} \right]$$

and since the right-hand vertical map in the diagram is a map of rings, it follows that it is an isomorphism. We thus have to compute the cokernel of the algebraic norm map:

$$N : \mathbb{Z} \left[ \frac{1}{n}, \zeta_n \right]_{GL_2(\mathbb{Z}/n\mathbb{Z})} \longrightarrow \mathbb{Z} \left[ \frac{1}{n}, \zeta_n \right]^{GL_2(\mathbb{Z}/n\mathbb{Z})} = \mathbb{Z} \left[ \frac{1}{n} \right].$$

We claim that its image is  $|SL_2(\mathbb{Z}/n\mathbb{Z})| \mathbb{Z}[\frac{1}{n}]$  so that  $\pi_0(\mathbf{tmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})}) \cong \mathbb{Z}[\frac{1}{n}] / |SL_2(\mathbb{Z}/n\mathbb{Z})|$ .

Let  $\varphi(-)$  denote the Euler  $\varphi$ -function and let  $\mu(-)$  denote the Möbius function. If  $d$  is the order of a power  $\zeta_n^i$ , then the norm map  $N$  sends  $\zeta_n^i$  to

$$\begin{aligned} \sum_{A \in GL_2(\mathbb{Z}/n\mathbb{Z})} \zeta_n^{i \det(A)} &= |SL_2(\mathbb{Z}/n\mathbb{Z})| \cdot \sum_{r \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^{ir} \\ &= |SL_2(\mathbb{Z}/n\mathbb{Z})| \cdot \frac{\varphi(n)}{\varphi(d)} \cdot \mu(d) \end{aligned}$$

For the second equality note that the canonical map  $(\mathbb{Z}/n\mathbb{Z})^\times \rightarrow (\mathbb{Z}/d\mathbb{Z})^\times$  is a surjection whose kernel has order  $\frac{\varphi(n)}{\varphi(d)}$ . Now, let  $d$  be the maximal number which is square-free and divides  $n$ . Then, since  $d$  is square-free, we have  $\mu(d) \in \{1, -1\}$ . If

$$n = \prod_{\substack{p|n \\ p \text{ prime}}} p^{k_p},$$

we have

$$\varphi(n) = \prod_{\substack{p|n \\ p \text{ prime}}} p^{k_p-1}(p-1)$$

and

$$\frac{\varphi(n)}{\varphi(d)} = \prod_{\substack{p|n \\ p \text{ prime}}} p^{k_p-1}.$$

Since the latter is a unit in  $\mathbb{Z}[\frac{1}{n}]$ , we get that the image of the norm is  $|SL_2(\mathbb{Z}/n\mathbb{Z})| \mathbb{Z}[\frac{1}{n}]$ . Therefore, we have  $\mathbf{tmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})} \simeq *$  if and only if  $|SL_2(\mathbb{Z}/n\mathbb{Z})|$  is a unit in  $\mathbb{Z}[\frac{1}{n}]$ . Since

$$|SL_2(\mathbb{Z}/n\mathbb{Z})| = n^3 \prod_{\substack{p|n \\ p \text{ prime}}} \frac{1}{p^2} (p+1)(p-1)$$

and  $|GL_2(\mathbb{Z}/n\mathbb{Z})| = \varphi(n) \cdot |SL_2(\mathbb{Z}/n\mathbb{Z})|$ , we see that  $|SL_2(\mathbb{Z}/n\mathbb{Z})|$  is invertible in  $\mathbb{Z}[\frac{1}{n}]$  if and only if  $|GL_2(\mathbb{Z}/n\mathbb{Z})|$  is invertible in  $\mathbb{Z}[\frac{1}{n}]$ .

If  $n \geq 2$  and  $2 \nmid n$ , then  $|GL_2(\mathbb{Z}/n\mathbb{Z})|$  and  $|SL_2(\mathbb{Z}/n\mathbb{Z})|$  are not units in  $\mathbb{Z}[\frac{1}{n}]$ : let  $q$  be an odd prime factor of  $n$ . Then in

$$n^3 \prod_{\substack{p|n \\ p \text{ prime}}} \frac{1}{p^2} (p+1)(p-1)$$

we have a factor of  $q-1$  and this is even, but 2 is not invertible in  $\mathbb{Z}[\frac{1}{n}]$ .

If  $n = 2^k$  for some  $k \geq 1$ , we obtain

$$|SL_2(\mathbb{Z}/n\mathbb{Z})| = 2^{3k} \frac{3}{4}$$

which contains 3 as a non-invertible factor. □

Note that Meier shows [40, Theorem 4.4 and Proposition 4.15] that  $\text{trmf}(n)$  is dualizable and faithful as a  $\text{trmf}[1/n]$ -module.

**Remark 3.14.** For many  $n$ , the Tate construction  $\text{trmf}(n)^{GL_2(\mathbb{Z}/n\mathbb{Z})}$  is actually trivial. If  $n = 2^k 3^\ell$  with  $k, \ell \geq 1$  for instance, the order of  $GL_2(\mathbb{Z}/n\mathbb{Z})$  is invertible in  $\mathbb{Z}[\frac{1}{n}]$ . Similarly, if  $n = p_1 \cdot \dots \cdot p_r$  for primes  $p_i$ , then  $|GL_2(\mathbb{Z}/n\mathbb{Z})|$  is invertible in  $\mathbb{Z}[\frac{1}{n}]$  if for all  $p_i$  the numbers  $p_i - 1$  and  $p_i + 1$  are invertible in  $\mathbb{Z}[\frac{1}{n}]$ . This is for instance the case if  $n = 2 \cdot 3 \cdot \dots \cdot p_m$  is the product of the first  $m$  prime numbers for any  $m \geq 2$  or for  $n = 2 \cdot 3 \cdot 7 = 42$  but not for  $n = 2 \cdot 3 \cdot 11$ .

We close with a periodic example. For a fixed prime  $p$ , let  $E_n$  denote the Lubin-Tate spectrum whose coefficient ring is

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]],$$

where  $u$  is an element of degree 2 and the  $u_i$ s have degree 0. For a perfect field  $k$ ,  $W(k)$  denotes the ring of Witt vectors of  $k$ . The ring  $W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] = \pi_0(E_n)$  represents deformations of the height  $n$  Honda formal group law over  $\mathbb{F}_{p^n}$ . The quotient  $E_n/(p, u_1, \dots, u_{n-1}) = K_n$  is a 2-periodic version of Morava K-theory whose coefficient ring is the graded field  $\pi_*(K_n) = \mathbb{F}_{p^n}[[u^{\pm 1}]]$ .

For any finite group  $G$ ,  $F(BG_+, E_n) \rightarrow F(EG_+, E_n) \simeq E_n$  is faithful in the  $K_n$ -local category [5, Theorem 4.4]. At the moment, we don't know whether  $E_n$  is a dualizable  $F(BG_+, E_n)$ -module for any finite group  $G$ .

In [5, Theorem 5.1], it is shown that  $F((BC_{p^r})_+, E_n) \rightarrow E_n$  is ramified and one can also consider more general groups than  $C_{p^r}$ . The corresponding Tate constructions are not trivial:

**Lemma 3.15.** For all  $r \geq 1$  and  $n \geq 1$

$$E_n^{tC_{p^r}} \not\cong *$$

*Proof.* The Tate spectral sequence

$$E_2^{s,t} = \hat{H}^{-s}(C_{p^r}; \pi_t E_n) \Rightarrow \pi_{s+t}(E_n^{tC_{p^r}})$$

has as  $E^2$ -term

$$\hat{H}^{-s}(C_{p^r}; \pi_t E_n) \cong \begin{cases} \pi_t E_n^{tC_{p^r}} / p^r = \pi_t E_n / p^r, & \text{for } s \text{ even,} \\ \ker(N) / \text{im}(t-1) = 0, & \text{for } s \text{ odd.} \end{cases}$$



As  $\pi_*(E_n)$  is concentrated in even degrees, the whole  $E_2$ -term is concentrated in bidegrees  $(s, t)$  where  $s$  and  $t$  are even. Therefore, all differentials have to be trivial and  $E_2 = E_\infty$ . Thus,  $\pi_*(E_n^{tC_p})$  is highly nontrivial.  $\square$

**Theorem 3.16.** *Assume that  $G$  is a finite group with  $p \mid |G|$ . Then,  $E_n^G$  is nontrivial when  $E_n$  is the Lubin-Tate spectrum at the prime  $p$ .*

*Proof.* The assumption implies that  $G$  has  $C_p$  as a subgroup. The restriction map induces a map on Tate constructions  $E_n^G \rightarrow E_n^{tC_p}$ . For the remainder of the proof, we use the notation from [21], denoting the Tate construction  $E_n^G$  by the  $G$ -fixed points  $t((E_n)_G)^G$  of a  $G$ -spectrum  $t((E_n)_G)$ . McClure [39] shows that the  $E_\infty$ -structure on Tate constructions  $t((E_n)_G)^G$  is compatible with inclusions of subgroups and Greenlees–May show [21, Proposition 3.7] that for any subgroup  $H < G$  the  $H$ -spectrum  $t((E_n)_G)$  is equivalent to  $t((E_n)_H)$ . Therefore, the inclusion of fixed points  $t((E_n)_G)^G \rightarrow t((E_n)_G)^H$  is a map of  $E_\infty$ -ring spectra. As we know that  $E_n^{tC_p} = t((E_n)_G)^{C_p}$  is nontrivial by Lemma 3.15,  $E_n^G$  cannot be trivial, either.  $\square$

**Acknowledgments.** We thank Lennart Meier for sharing a draft version of [40] with us, and we thank him and Mike Hill for patient explanations about tmf and friends. We are grateful to John Rognes and Mark Behrens for several helpful comments. We thank the anonymous referees for several useful remarks. During this project, the first named author was a postdoc at the *Universität Hamburg* funded by the DFG priority program SPP 1786 *Homotopy Theory and Algebraic Geometry*. The last named author would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program *Homotopy Harnessing Higher Structures* when work on this paper was undertaken. This work was supported by EPSRC grant number EP/R014604/1.

**Competing interests.** The authors declare none.

## References

- [1] M. Auslander and D. A. Buchsbaum, On ramification theory in noetherian rings, *Am. J. Math.* **81** (1959), 749–765.
- [2] C. Ausoni, Topological Hochschild homology of connective complex K-theory, *Am. J. Math.* **127**(6) (2005), 1261–1313.
- [3] C. Ausoni and J. Rognes, The chromatic red-shift in algebraic K-theory, *Enseign. Math.* **54**(2) (2008), 9–11.
- [4] A. Baker, H. Gilmour and P. Reinhard, Topological André-Quillen homology for cellular commutative S-algebras *Abh. Math. Semin. Univ. Hambg.* **78**(1) (2008), 27–50.
- [5] A. Baker and B. Richter, Galois theory and Lubin-Tate cochains on classifying spaces, *Cent. Eur. J. Math.* **9**(5) (2011), 1074–1087.
- [6] M. Basterra, André-Quillen cohomology of commutative S-algebras, *J. Pure Appl. Algebra* **144**(2) (1999), 111–143.
- [7] Basterra M. and Richter B., (Co-)homology theories for commutative (S-)algebras, in *Structured Ring Spectra* (Cambridge University Press, Cambridge, 2004), 115–131.
- [8] M. Basterra and M. A. Mandell, Homology and cohomology of E1 ring spectra, *Math. Z.* **249**(4) (2005), 903–944.
- [9] T. Bauer, *Computation of the homotopy of the spectrum tmf* (Groups, Homotopy and Configuration Spaces, Tokyo, 2008), 11–40.
- [10] M. Behrens, A modular description of the K(2)-local sphere at the prime 3, *Topology* **45**(2) (2006), 343–402.
- [11] J. M. Boardman, Conditionally convergent spectral sequences, in *Homotopy invariant algebraic structures* (Baltimore, MD, 1998) (1999), 49–84.
- [12] Cassels J. W. S. and Fröhlich A., Algebraic number theory, in *Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union* (Academic Press, London; Thompson Book Co., Inc., Washington, D.C., 1967).
- [13] S. U. Chase, D. K. Harrison and A. Rosenberg, Galois theory and Galois cohomology of commutative rings, *Mem. Am. Math. Soc.* **52** (1965), 15–33.
- [14] D. Clausen and A. Mathew, A short proof of telescopic Tate vanishing, *Proc. Am. Math. Soc.* **145**(12) (2017), 5413–5417.
- [15] D. Clausen, A. Mathew, N. Naumann and J. Noel, Descent in algebraic K-theory and a conjecture of Ausoni-Rognes, *J. Eur. Math. Soc. (JEMS)* **22**(4) (2020), 1149–1200.
- [16] P. Deligne, Courbes elliptiques: formulaire d’après J. Tate, in *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math.*, vol. 476 (Springer, Berlin/Heidelberg, 1975), 53–73.
- [17] P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, in *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math.*, vol. 349 (Springer, Berlin/Heidelberg, 1973), 143–316.

- [18] C. L. Douglas, J. Francis, A. G. Henriques and A. Michael Hill, *Topological modular forms, Mathematical surveys and monographs*, vol. 201 (American Mathematical Society, Providence, RI, 2014).
- [19] B. I. Dundas, A. Lindenstrauss and B. Richter, Towards an understanding of ramified extensions of structured ring spectra, *Math. Proc. Camb. Philos. Soc.* **168**(3) (2020), 435–454.
- [20] B. I. Dundas, A. Lindenstrauss and B. Richter, Corrigendum: towards an understanding of ramified extensions of structured ring spectra, *Math. Proc. Camb. Philos. Soc.* **172**(1) (2022), 247–248.
- [21] J. P. C. Greenlees and J. P. May, Generalized Tate cohomology, *Mem. Am. Math. Soc.* **113**(543) (1995) viii+178.
- [22] J. P. C. Greenlees and H. Sadofsky, The Tate spectrum of vn-periodic complex oriented theories, *Math. Z.* **222**(3) (1996), 391–405.
- [23] M. A. Hill and T. Lawson, Topological modular forms with level structure, *Invent. Math.* **203**(2) (2016), 359–416.
- [24] M. A. Hill and L. Meier, The C2-spectrum  $Tmf_1(3)$  and its invertible modules, *Algebr. Geom. Topol.* **17**(4) (2017), 1953–2011.
- [25] M. Hovey, Smith ideals of structured ring spectra. arXiv: 1401.2850.
- [26] M. Hovey and H. Sadofsky, Tate cohomology lowers chromatic Bousfield classes, *Proc. Am. Math. Soc.* **124**(11) (1996), 3579–3585.
- [27] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves, Annals of mathematics studies*, vol. 108 (Princeton University Press, Princeton, NJ, 1985).
- [28] J. Konter, The homotopy groups of the spectrum  $Tmf$ . arXiv: 1212.3656.
- [29] N. J. Kuhn, Tate cohomology and periodic localization of polynomial functors, *Invent. Math.* **157**(2) (2004), 345–370.
- [30] T. Lawson and N. Naumann, Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2, *Int. Math. Res. Not.* **10** (2014), 2773–2813.
- [31] T. Lundemo, On the relationship between logarithmic TAQ and logarithmic THH, *Doc. Math.* **26** (2021), 1187–1236.
- [32] J. Lurie, Higher Algebra. Available at <http://www.math.harvard.edu/lurie/>
- [33] M. Mahowald and C. Rezk, Topological modular forms of level 3, *Pure Appl. Math. Q.* **5**(2) (2009), 853–872. Special Issue: In honor of Friedrich Hirzebruch. Part 1.
- [34] A. Mathew, The Galois group of a stable homotopy theory, *Adv. Math.* **291** (2016), 403–541.
- [35] A. Mathew, THH and base-change for Galois extensions of ring spectra, *Algebr. Geom. Topol.* **17**(2) (2017), 693–704.
- [36] A. Mathew, The homology of  $tmf$ , *Homology Homotopy Appl.* **18**(2) (2016), 1–29.
- [37] A. Mathew and L. Meier, Affineness and chromatic homotopy theory, *J. Topol.* **8**(2) (2015), 476–528.
- [38] A. Mathew, N. Naumann and J. Noel, On a nilpotence conjecture of J. P. May, *J. Topol.* **8**(4) (2015), 917–932.
- [39] J. E. McClure, E1-ring structures for Tate spectra, *Proc. Am. Math. Soc.* **124**(6) (1996), 1917–1922.
- [40] L. Meier, Connective Models for Topological Modular Forms of Level  $n$ . arXiv: 2104.12649, to appear in *Algebr. Geom. Topol.*
- [41] L. Meier and V. Ozornova, Rings of modular forms and a splitting of  $TMF_0(7)$ , *Selecta Math. (N.S.)* **26**(1) (2020), Paper No. 7, 73.
- [42] E. Noether, Normalbasis bei Körpern ohne höhere Verzweigung, *J. Reine Angew. Math.* **167** (1932), 147–152.
- [43] A. Robinson and S. Whitehouse, Operads and  $\Gamma$ -homology of commutative rings, *Math. Proc. Camb. Philos. Soc.* **132**(2) (2002), 197–234.
- [44] J. Rognes, Galois extensions of structured ring spectra, *Mem. Am. Math. Soc.* **192**(898) (2008), viii–98.
- [45] J. Rognes, Stably dualizable groups, *Mem. Am. Math. Soc.* **192**(898) (2008), 99–137.
- [46] J. Rognes, Topological logarithmic structures, in *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)* (2009), 401–544.
- [47] J. Rognes, Algebraic K-theory of strict ring spectra, in *Proceedings of the International Congress of Mathematicians—Seoul 2014*, vol. II (2014), 1259–1283.
- [48] J. Rognes, S. Sagave and C. Schlichtkrull, Localization sequences for logarithmic topological Hochschild homology, *Math. Ann.* **363**(3–4) (2015), 1349–1398.
- [49] J. Rognes, S. Sagave and C. Schlichtkrull, Logarithmic topological Hochschild homology of topological K-theory spectra, *J. Eur. Math. Soc. (JEMS)* **20**(2) (2018), 489–527.
- [50] S. Sagave, Logarithmic structures on topological K-theory spectra, *Geom. Topol.* **18**(1) (2014), 447–490.
- [51] S. Sagave, Spectra of units for periodic ring spectra and group completion of graded E1 spaces, *Algebr. Geom. Topol.* **16**(2) (2016), 1203–1251.
- [52] S. Sagave and C. Schlichtkrull, Diagram spaces and symmetric spectra, *Adv. Math.* **231**(3–4) (2012), 21160–22193.
- [53] V. Stojanoska, Duality for topological modular forms, *Doc. Math.* **17** (2012), 271–311.