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Complexity for modules over the classical Lie superalgebra $\mathfrak{gl}(m|n)$

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Abstract

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and let $\mathcal{F}$ be the category of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$. In [B. D. Boe, J. R. Kujawa and D. K. Nakano, *Complexity and module varieties for classical Lie superalgebras*, Int. Math. Res. Not. IMRN (2011), 696–724], we demonstrated that for any module M in $\mathcal{F}$ the rate of growth of the minimal projective resolution (i.e. the complexity of M) is bounded by the dimension of $\mathfrak{g}_{\bar{1}}$. In this paper we compute the complexity of the simple modules and the Kac modules for the Lie superalgebra $\mathfrak{gl}(m|n)$. In both cases we show that the complexity is related to the atypicality of the block containing the module.

1. Introduction

1.1 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra over the complex numbers $\mathbb{C}$. For classical Lie superalgebras, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. An important category of $\mathfrak{g}$-supermodules is the category $\mathcal{F} := \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over $\mathfrak{g}_{\bar{0}}$. The category $\mathcal{F}$ has enough projectives and is in general not semisimple. In [BKN11] we showed that (i) $\mathcal{F}$ is a self-injective category (meaning that a module being projective is equivalent to the module being injective) and (ii) every module in $\mathcal{F}$ admits a projective resolution which has polynomial rate of growth. For a module $M \in \mathcal{F}$, the complexity $c_{\mathcal{F}}(M)$ is the rate of growth of the minimal projective resolution of M. In [BKN11], it was proved by constructing an explicit Koszul-type resolution that $c_{\mathcal{F}}(M) \leqslant \dim \mathfrak{g}_{\bar{1}}$ for all $M \in \mathcal{F}$.

It is well known that if G is a finite group scheme, then the category of rational modules for G satisfies the same properties (i) and (ii) as described above. In this context, the complexity of a module was first introduced by Alperin in 1977; see [Alp77]. By using the fact that the cohomology ring for G is finitely generated (cf. [FS97]), one can construct the (cohomological) support variety $\mathcal{V}_G(M)$ of a module M, whose dimension coincides with the complexity $c_G(M)$. This realization allows one to use geometric methods to compute the complexities of important classes of modules (see [DNP12, HN02, NPV02, UGA07]).

The elusive ingredient for the superalgebra category $\mathcal{F}$ is a 'support variety' theory which would enable one to compute the complexity of modules in $\mathcal{F}$. In [BKN11], it was shown that there is a formula in terms of rates of growth of cohomology groups which realizes the complexity. The main goal of this paper will be to show how to compute the complexity for important classes

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of modules for $\mathfrak{gl}(m|n)$. In particular, we will show for $\mathfrak{gl}(m|n)$ that

$$c_{\mathcal{F}}(K(\lambda)) = (m+n)\,\mathrm{atyp}(\lambda) - \mathrm{atyp}(\lambda)^2, \tag{1.1.1}$$

which equals the dimension of the variety of $m \times n$ matrices of rank at most $\mathrm{atyp}(\lambda)$, and that

$$c_{\mathcal{F}}(L(\lambda)) = (m+n)\,\mathrm{atyp}(\lambda) - \mathrm{atyp}(\lambda)^2 + \mathrm{atyp}(\lambda). \tag{1.1.2}$$

Here $K(\lambda)$ (respectively, $L(\lambda)$) is the Kac (respectively, simple) module of highest weight λ, and $\mathrm{atyp}(\lambda)$ is the atypicality of the weight λ as defined by Kac and Wakimoto. Our calculations show that complexity depends on m and n. In particular, unlike the case with finite group schemes, complexity is not invariant under category equivalences. The proofs demonstrating these calculations employ a myriad of deep results, both known and new, about the category $\mathcal{F}$. It is also worth noting that the formulas given in (1.1.1) and (1.1.2) have the following remarkable geometric interpretation. For a $\mathfrak{gl}(m|n)$-module M let $\mathcal{X}_M$ denote the associated variety defined by Duflo and Serganova [DS05], and let $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ be the support variety of [BKN10]. Then if $X(\lambda)$ is a Kac, dual Kac, or simple $\mathfrak{gl}(m|n)$-module, we have

$$c_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{X}_{X(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(X(\lambda)).$$

It is natural to wonder whether this formula holds in general.

The paper is organized as follows. In § 2, we set up the basic conventions for classical Lie superalgebras and, in particular, the Lie superalgebra $\mathfrak{gl}(m|n)$. We then introduce a support variety theory and relate this to the calculation of complexity for modules over the 'parabolic' subalgebras of $\mathfrak{gl}(m|n)$ which, in turn, are later used (in § 6) with the Gruson–Serganova equivalences between blocks of $\mathcal{F}$ to determine a lower bound on the complexity of Kac modules. In order to establish the upper bound, we invoke results on the dimensions of projective modules in $\mathcal{F}$ developed in § 5. In § 6, we establish the aforementioned formula on the complexity of Kac (and dual Kac) modules.

The remainder of the paper is devoted to computing the complexity of simple modules in $\mathcal{F}$. This computation is much more complicated because there is no known support variety theory for modules which measures complexity for modules in $\mathcal{F}$. In § 7, we begin by establishing a lower bound on the dimension of projective indecomposable modules by using Ehrhart's theorem on counting lattice points in a polytope. The establishment of this bound also uses a combinatorial bijection on highest weights introduced by Su and Zhang. Next, in § 8, Serganova's recent verification of the generalized Kac–Wakimoto conjecture for $\mathfrak{gl}(m|n)$ is employed to reduce to a specific simple $\mathfrak{gl}(m|n)$-module of atypicality k. We then apply Brundan's deep results on the characters and extensions of simple modules in $\mathcal{F}$, which show that $\mathcal{F}$ is a highest weight category having a Kazhdan–Lusztig theory, to estimate the upper and lower bounds for the complexity of simple modules via properties of Kazhdan–Lusztig polynomials. These results, in conjunction with the results established in § 7, allow us to complete the calculation. Finally, in § 9, we introduce a new numerical invariant of $\mathcal{F}$ which remains the same under equivalence of categories. We also give evidence that this invariant is closely related to a detecting subalgebra of $\mathfrak{gl}(m|n)$ previously introduced by the authors.

2. Preliminaries

2.1 Classical Lie superalgebras

We will use the notation and conventions developed in [BKN09, BKN10]. For more details we refer the reader to [BKN09, § 2.1].

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We will work over the complex numbers $\mathbb{C}$ throughout this paper. Let $\mathfrak{g}$ be a *Lie superalgebra*, that is, a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bracket operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the usual Lie bracket axioms. The subspace $\mathfrak{g}_{\bar{0}}$ is a Lie algebra under the bracket and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. A finite-dimensional Lie superalgebra $\mathfrak{g}$ is said to be *classical* if there exist a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.[1] If $\mathfrak{g}$ is classical, then $\mathfrak{g}_{\bar{1}}$ is semisimple as a $\mathfrak{g}_{\bar{0}}$-module. A *basic classical* Lie superalgebra is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form (cf. [Kac77]).

Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. The objects of the category of $\mathfrak{g}$-supermodules are all $\mathbb{Z}_2$-graded left $U(\mathfrak{g})$-modules. To describe the morphisms, we first recall that if M and N are $\mathbb{Z}_2$-graded, then $\mathrm{Hom}_{\mathbb{C}}(M, N)$ is naturally $\mathbb{Z}_2$-graded by setting $\overline{f} = r \in \mathbb{Z}_2$ if $f(M_i) \subseteq N_{i+r}$ for $i \in \mathbb{Z}_2$. Here and elsewhere we write $\overline{v} \in \mathbb{Z}_2$ for the degree of a homogeneous element v of a $\mathbb{Z}_2$-graded vector space. We use the convention that we only state conditions for a homogenous element, with the general case given by linearity. For $\mathfrak{g}$-supermodules M and N, a homogeneous $\mathfrak{g}$-morphism $f : M \to N$ is a homogeneous linear map which satisfies

$$f(xm) = (-1)^{\overline{f}\,\overline{x}} x f(m)$$

for all homogeneous $x \in \mathfrak{g}$. Given $\mathfrak{g}$-supermodules M and N, one can use the antipode and coproduct of $U(\mathfrak{g})$ to define a $\mathfrak{g}$-supermodule structure on the contragradient dual M^* and the tensor product $M \otimes N$.

A supermodule is *finitely semisimple* if it decomposes into a direct sum of finite-dimensional simple supermodules. We write $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ for the full subcategory of all finite-dimensional $\mathfrak{g}$-supermodules which are finitely semisimple when viewed as $\mathfrak{g}_{\bar{0}}$-supermodules by restriction. As only supermodules will be considered in this paper, we will from now on use the term 'module' with the understanding that the prefix 'super' is implicit.

2.2 Complexity

Let $\{V_t \mid t \in \mathbb{N}\} = \{V_\bullet\}$ be a sequence of finite-dimensional $\mathbb{C}$-vector spaces. The *rate of growth* of $V_\bullet$, denoted by $r(V_\bullet)$, is the smallest nonnegative integer c such that there exists a constant $C > 0$ with $\dim V_t \leqslant C \cdot t^{c-1}$ for all t. If no such integer exists, then $V_\bullet$ is said to have *infinite rate of growth*. Let $M \in \mathcal{F}$ and let $P_\bullet \twoheadrightarrow M$ be a minimal projective resolution for M. Following Alperin [Alp77], we define the *complexity* of M to be $c_{\mathcal{F}}(M) := r(P_\bullet)$. The following theorem was proved by the authors in [BKN11, Proposition 2.8.1] and provides a characterization of the complexity via rates of growth of extension groups in $\mathcal{F}$. This characterization will be important for our computational purposes.

PROPOSITION 2.2.1. *Let $\mathfrak{g}$ be a classical Lie superalgebra, and let M be an object in $\mathcal{F}$. Then*

$$c_{\mathcal{F}}(M) = r\Big(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}\Big(M, \bigoplus S^{\dim P(S)}\Big)\Big),$$

where the sum is over all simple modules S in $\mathcal{F}$ and $P(S)$ is the projective cover of S.

[1] Unlike in Kac's original definition [Kac77], we do *not* require a classical Lie superalgebra to be simple.

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Note that $\operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar 0})}(M,N)$ denotes relative cohomology for the pair $(\mathfrak{g},\mathfrak{g}_{\bar 0})$. When both M and N are objects of $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar 0})}$, then by [BKN10, Theorem 2.5.1] we have

$$\operatorname{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_{\bar 0})}(M,N) \cong \operatorname{Ext}^{d}_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar 0})}}(M,N) \tag{2.2.1}$$

for all $d \geqslant 0$.

2.3 Type I Lie superalgebras

A Lie superalgebra is said to be of *type I* if it admits a $\mathbb{Z}$-grading $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ concentrated in degrees -1, 0 and 1 with $\mathfrak{g}_{\bar 0}=\mathfrak{g}_0$ and $\mathfrak{g}_{\bar 1}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_1$ and if the bracket respects this grading. Otherwise, $\mathfrak{g}$ is of type II. Examples of type I Lie superalgebras include $\mathfrak{gl}(m|n)$ and the simple Lie superalgebras of types $A(m,n)$, $C(n)$ and $P(n)$.

The simple modules for $\mathfrak{g}$, a type I classical Lie superalgebra, can be constructed in the following way. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}_0$ and $X_0^+\subseteq\mathfrak{t}^*$ the set of dominant integral weights for $\mathfrak{g}_0$ (with respect to a fixed Borel subalgebra of $\mathfrak{g}_0$). For $\lambda\in X_0^+$, let $L_0(\lambda)$ be the simple finite-dimensional $\mathfrak{g}_0$-module of highest weight λ. Set

$$\mathfrak{p}^+=\mathfrak{g}_0\oplus\mathfrak{g}_1 \quad\text{and}\quad \mathfrak{p}^-=\mathfrak{g}_0\oplus\mathfrak{g}_{-1}.$$

Since $\mathfrak{g}$ is a type I Lie superalgebra, $\mathfrak{g}_{\pm 1}$ is an abelian ideal of $\mathfrak{p}^{\pm}$. We can therefore view $L_0(\lambda)$ as a simple $\mathfrak{p}^{\pm}$-module via inflation. In this way we obtain a complete set of finite-dimensional simple modules for $\mathfrak{p}^{\pm}$.

For each $\lambda\in X_0^+$, we construct the *Kac module* $K(\lambda)$ and the *dual Kac module* $K^-(\lambda)$ by using the tensor product and the Hom-space in the following way:

$$K^+(\lambda):=K(\lambda):=U(\mathfrak{g})\otimes_{U(\mathfrak{p}^+)}L_0(\lambda) \quad\text{and}\quad K^-(\lambda):=\operatorname{Hom}_{U(\mathfrak{p}^-)}(U(\mathfrak{g}),L_0(\lambda)).$$

The module $K(\lambda)$ has a unique maximal submodule. The head of $K(\lambda)$ is the simple finite-dimensional $\mathfrak{g}$-module $L(\lambda)$. Then $\{L(\lambda)\mid\lambda\in X_0^+\}$ is a complete set of non-isomorphic simple modules in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$. Let $P(\lambda)$ (respectively, $I(\lambda)$) denote the projective cover (respectively, injective hull) in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ for the simple $\mathfrak{g}$-module $L(\lambda)$. These are all finite-dimensional. Moreover, the projective covers admit filtrations with sections being Kac modules, and the injective hulls have filtrations whose sections are dual Kac modules. These filtrations also respect the dominance ordering on weights, and thus $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ is a highest-weight category (cf. [BKN11, § 3]) as defined in [CPS88].

2.4 The Lie superalgebra $\mathfrak{gl}(m|n)$

The standard example of a type I classical Lie superalgebra is $\mathfrak{g}=\mathfrak{gl}(m|n)$. As a vector space, $\mathfrak{g}$ is the set of $(m+n)\times(m+n)$ matrices, and one may take the matrix units $E_{i,j}$, where $1\leqslant i,j\leqslant m+n$, as a basis. The even component $\mathfrak{g}_{\bar 0}$ is the span of $E_{i,j}$ where $1\leqslant i,j\leqslant m$ or $m+1\leqslant i,j\leqslant m+n$. A basis for $\mathfrak{g}_{\bar 1}$ is given by the $E_{i,j}$ such that $m+1\leqslant i\leqslant m+n$ and $1\leqslant j\leqslant n$ or $1\leqslant i\leqslant m$ and $m+1\leqslant j\leqslant m+n$. As a Lie algebra, $\mathfrak{g}_{\bar 0}\cong\mathfrak{gl}(m)\times\mathfrak{gl}(n)$, and the corresponding reductive group is $G_{\bar 0}\cong \mathrm{GL}(m)\times\mathrm{GL}(n)$. Note that $G_{\bar 0}$ acts on $\mathfrak{g}_{\bar 1}$ via the adjoint representation. As $\mathfrak{gl}(m|n)\cong\mathfrak{gl}(n|m)$ and $\mathfrak{gl}(m|0)=\mathfrak{gl}(m|0)_{\bar 0}$, we may assume without loss of generality that $m\geqslant n\geqslant 1$.

Observe that $\mathfrak{g}$ has a $\mathbb{Z}$-grading given by setting $\mathfrak{g}_0=\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{-1}$ (respectively, $\mathfrak{g}_1$) equal to the lower triangular matrices (respectively, upper triangular matrices) in $\mathfrak{g}_{\bar 1}$. In particular, $\mathfrak{g}_{\bar 1}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_1$. Furthermore, note that the bracket respects the $\mathbb{Z}$-grading.

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We will now establish some basic notation involving the root datum of $\mathfrak{g}$ which will be used later. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices, and let $\mathfrak{b}$ be the Borel subalgebra of all upper triangular matrices. Then $\mathfrak{t}$ is a Cartan subalgebra and $\mathfrak{b}_0 := \mathfrak{b} \cap \mathfrak{g}_0$ is a Borel subalgebra for $\mathfrak{g}_0$. With respect to these choices, we can make the root system and X_0^+ explicit as follows. For $i = 1, \ldots, m+n$, let $\varepsilon_i : \mathfrak{t} \to \mathbb{C}$ be the linear functional which picks out the ith diagonal entry. With respect to this basis, we define a bilinear form on $\mathfrak{t}$ by

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} \delta_{i,j} & \text{for } 1 \leqslant i, j \leqslant m, \\ -\delta_{i,j} & \text{for } m+1 \leqslant i, j \leqslant m+n, \\ 0 & \text{otherwise.} \end{cases}$$

As with $\mathfrak{gl}(m+n)$, the set

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant m+n,\ i \neq j\}$$

is the set of roots for $\mathfrak{g}$, and

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant m+n\}$$

is the set of positive roots. The set

$$\Phi_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i \neq j \leqslant m \text{ or } m+1 \leqslant i \neq j \leqslant m+n\}$$

is the set of *even* roots. The *odd* roots are then $\Phi_{\bar{1}} = \Phi \backslash \Phi_{\bar{0}}$. We set $\Phi_{\bar{0}}^+ = \Phi^+ \cap \Phi_{\bar{0}}$ and $\Phi_{\bar{1}}^+ = \Phi^+ \cap \Phi_{\bar{1}}$. With respect to our choices,

$$X_0^+ = \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \;\middle|\; \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geqslant 0} \text{ for } i \neq m, m+n \right\}.$$

As we discuss below, without loss of generality we can and will assume that $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, m+n$.

Let

$$\Phi_m^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant m\}$$

be the set of positive roots for the subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{gl}(m)$, Similarly, let

$$\Phi_n^+ = \{\varepsilon_i - \varepsilon_j \mid m+1 \leqslant i < j \leqslant m+n\}$$

be the set of positive roots for the subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{gl}(n)$.

Now define

$$\begin{aligned} \rho &= m\varepsilon_1 + (m-1)\varepsilon_2 + \cdots + \varepsilon_m - \varepsilon_{m+1} - 2\varepsilon_{m+2} - \cdots - n\varepsilon_{m+n}, \\ \rho_m &= m\varepsilon_1 + (m-1)\varepsilon_2 + \cdots + \varepsilon_m, \\ \rho_n &= -\varepsilon_{m+1} - 2\varepsilon_{m+2} - \cdots - n\varepsilon_{m+n}. \end{aligned}$$

Then $\rho = \rho_m + \rho_n$ and the elements ρ, ρ_m, ρ_n are each a constant shift of the analogous elements defined via half sums of positive roots. This shift has no effect on the contents of this paper, so we choose to use the more convenient elements defined above.

Given $\lambda \in X_0^+$, we define the *atypicality* of λ, $\operatorname{atyp}(\lambda)$, to be the maximal number of pairwise orthogonal elements of $\Phi_{\bar{1}}^+$ which are also orthogonal to $\lambda + \rho$. The atypicality is an integer in the range $0, \ldots, \min(m, n)$. If $L(\lambda)$ is a simple $\mathfrak{g}$-module of highest weight λ, then we define $\operatorname{atyp}(L(\lambda)) := \operatorname{atyp}(\lambda)$. It is known that the atypicality of a simple module is independent of the choice of Cartan and Borel subalgebras and, furthermore, is the same for all simple modules in a

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given block (see [KW94, Corollary 3.1] and [Ser96, Lemma 1.12], respectively). Hence, it makes sense to refer to the atypicality of a block.

If $\lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X_0^+$ has atypicality zero, then by [Kac78, Theorem 1] we have $P(\lambda) = K(\lambda) = L(\lambda)$ and, in particular, $K(\lambda)$ and $L(\lambda)$ have complexity zero, which is consistent with (1.1.1) and (1.1.2). If λ has atypicality greater than zero, then since $\mathfrak{gl}(m|n)$ has the one-dimensional representation given by the supertrace of weight $\varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$, we may tensor by a suitable one-dimensional representation (doing so clearly preserves complexity) and assume that $\lambda_1, \ldots, \lambda_{m+n}$ are integers. Therefore, without loss of generality we shall always assume $\operatorname{atyp}(\lambda) \geqslant 1$ and that elements of X_0^+ have integral coefficients.

2.5 Let $\mathcal{F} = \mathcal{F}_{(\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)_0)}$. Serganova provided a convenient combinatorial description of the blocks of $\mathcal{F}$, which we now recall. Given a simple $\mathfrak{g}$-module $L(\lambda)$ with $\operatorname{atyp}(\lambda) = k$, there exists a (unique) set of k positive odd pairwise orthogonal roots

$$\Omega := \{\varepsilon_{i_t} - \varepsilon_{j_t} \mid t = 1, \ldots, k\} \tag{2.5.1}$$

such that $(\lambda + \rho, \varepsilon_{i_t} - \varepsilon_{j_t}) = 0$ for all $t = 1, \ldots, k$ and where $1 \leqslant i_1, \ldots, i_k \leqslant m$ and $m + 1 \leqslant j_1, \ldots, j_k \leqslant m + n$. The *core* of λ is the pair of multisets

$$\begin{aligned}(\{(\lambda + \rho, \varepsilon_s) \mid s \in \{1, \ldots, m\} \backslash \{i_1, \ldots, i_k\}\},\\ \{(\lambda + \rho, \varepsilon_s) \mid s \in \{m+1, \ldots, m+n\} \backslash \{j_1, \ldots, j_k\}\}).\end{aligned} \tag{2.5.2}$$

We then have the following description of the blocks of $\mathcal{F}$.

Proposition 2.5.1 [GS10]. *If $L(\lambda)$ and $L(\mu)$ are two simple modules in $\mathcal{F}$, then $L(\lambda)$ and $L(\mu)$ lie in the same block if and only if $\operatorname{atyp}(\lambda) = \operatorname{atyp}(\mu)$ and the core of λ equals the core of μ.*

Given a block $\mathcal{B}$ of $\mathcal{F}$, we will abuse notation slightly by writing $\lambda \in \mathcal{B}$ to mean that the simple module $L(\lambda)$ lies in the block $\mathcal{B}$. For example, we write $\mathcal{B}_0$ for the principal block of $\mathcal{F}$ and so, by definition, $0 \in \mathcal{B}_0$.

Given $\lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \in X_0^+$ with $\operatorname{atyp}(\lambda) = k$, we set $\lambda^+ = \sum_{i=1}^{m} \lambda_i \varepsilon_i$ and define two length functions as follows. The 'naive' length function is given by

$$|\lambda| = \lambda_1 + \cdots + \lambda_m.$$

The other length function is given by

$$l(\lambda) = k(k+1)/2 + \sum_{\alpha \in \Omega} (\lambda^+ + \rho_n, \alpha).$$

By [SZ07, Remark 3.4], l defines a length function in the sense of [Bru03]. Note that if $\lambda \in \mathcal{B}_0$ for $\mathfrak{gl}(k|k)$, then $l(\lambda) = |\lambda|$.

For $\lambda, \mu \in \mathcal{B}$, we write $\lambda \leqslant \mu$ if $\mu - \lambda$ is a sum of positive roots (i.e. the usual dominance order). We write $\lambda \preccurlyeq \mu$ for the Bruhat order of [Bru03]. Note that if $\lambda = \sum \lambda_i \varepsilon_i \in \mathcal{B}$ and $\mu = \sum \mu_i \varepsilon_i \in \mathcal{B}_0$ for $\mathfrak{gl}(k|k)$, then $\lambda \preccurlyeq \mu$ if and only if $\lambda_i \leqslant \mu_i$ for $i = 1, \ldots, m$.[2]

Given a highest weight λ with $\operatorname{atyp}(\lambda) = k$, let $i_1, \ldots, i_k, j_1, \ldots, j_k$ be as in (2.5.1). We partition the elements of Φ_m^+ into three sets as follows. Let

$$A_m = \{\alpha = \varepsilon_s - \varepsilon_t \in \Phi_m^+ : |\{s, t\} \cap \{i_1, \ldots, i_k\}| = 0\},$$

[2] Note that the bilinear form used by Brundan is the negative of the one used here.

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$$B_m = \{\alpha = \varepsilon_s - \varepsilon_t \in \Phi_m^+ : |\{s,t\} \cap \{i_1, \ldots, i_k\}| = 1\},$$
$$C_m = \{\alpha = \varepsilon_s - \varepsilon_t \in \Phi_m^+ : |\{s,t\} \cap \{i_1, \ldots, i_k\}| = 2\}.$$

These sets obviously depend on λ. When appropriate, we write $A_m(\lambda)$, $B_m(\lambda)$ etc. to remind the reader of these dependencies. Observe that $\Phi_m^+ = A_m \sqcup B_m \sqcup C_m$. Define A_n, B_n and C_n analogously by replacing Φ_m^+ with Φ_n^+ and $\{i_1, \ldots, i_k\}$ with $\{j_1, \ldots, j_k\}$.

2.6 As it will be needed in what follows, we briefly review the geometric structure of $\mathfrak{g}_1$ for $\mathfrak{gl}(m|n)$. Note that G_0 acts on the variety $\mathfrak{g}_1$ via the adjoint action. Specifically, in the matrix realization of $\mathfrak{g}$ given in § 2.4, the action of $G_0 \cong \mathrm{GL}(m) \times \mathrm{GL}(n)$ on $\mathfrak{g}_1$ is given by $(A, B) \cdot x = AxB^{-1}$ for $A \in \mathrm{GL}(m)$, $B \in \mathrm{GL}(n)$ and $x \in \mathfrak{g}_1$.

The G_0-orbit structure of $\mathfrak{g}_1$ is given as follows. The orbits are

$$(\mathfrak{g}_1)_r = \{x \in \mathfrak{g}_1 \mid \mathrm{rank}(x) = r\}$$

for $0 \leqslant r \leqslant \min(m, n)$ and, in particular, we have

$$(\mathfrak{g}_1)_r = G_0 \cdot x_r$$

where x_r is any fixed matrix of rank r.

The closure of $(\mathfrak{g}_1)_r$ is

$$\overline{(\mathfrak{g}_1)_r} = \{x \in \mathfrak{g}_1 \mid \mathrm{rank}(x) \leqslant r\};$$

thus $(\mathfrak{g}_1)_r \subset \overline{(\mathfrak{g}_1)_s}$ if and only if $r \leqslant s$. Hence, the graph (Hasse diagram) which describes the partial ordering given by inclusion of orbit closures is a simple chain.

For $M \in \mathcal{F}$, let $\mathcal{V}_{\mathfrak{g}_1}(M)$ denote the support variety of M as defined in § 3.2. In particular, $\mathcal{V}_{\mathfrak{g}_1}(\mathbb{C}) = \mathfrak{g}_1$. It follows, since $\mathcal{V}_{\mathfrak{g}_1}(M)$ is a closed G_0-invariant subvariety of $\mathcal{V}_{\mathfrak{g}_1}(\mathbb{C})$, that $\mathcal{V}_{\mathfrak{g}_1}(M) = \overline{(\mathfrak{g}_1)_r}$ for some r. So $\mathcal{V}_{\mathfrak{g}_1}(M)$ is always irreducible and can be computed by applying the rank variety description to a representative from each of the $\min(m, n)$ orbits. Note that a similar description of G_0-orbits on $\mathfrak{g}_{-1}$ also holds.

2.7 Example

Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$. The simple modules in the principal block $\mathcal{B}_0$ are one-dimensional and indexed by $L(\lambda\varepsilon_1 - \lambda\varepsilon_2)$ where $\lambda \in \mathbb{Z}$. In [BKN11], we proved that $c_{\mathcal{F}}(L(\lambda\varepsilon_1 - \lambda\varepsilon_2)) = 2$ for all $\lambda \in \mathbb{Z}$ by constructing an explicit minimal projective resolution. However, the relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ has Krull dimension one and so is not large enough to use for constructing a support variety theory which measures the complexity.

In this case, one can consider the subalgebra $\mathfrak{f} \cong \mathfrak{sl}(1|1)$ in $\mathfrak{g}$ as defined in § 9.1. The Krull dimension of $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})$ is two. In § 9.3, we will show that one can define a support variety theory for $\mathfrak{f}$ which measures the complexity of modules for $\mathfrak{gl}(1|1)$. This construction does not easily generalize to $\mathfrak{gl}(m|n)$ when $m, n > 1$. It remains an open question as to whether there exists a theory of varieties for modules for these classical Lie superalgebras which can be used to compute the rate of growth of projective resolutions.

3. Support varieties

3.1 In [BKN10] we showed that for a classical Lie superalgebra $\mathfrak{g}$, the relative cohomology ring for the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ is finitely generated. We then used this ring to construct a support variety theory for objects of $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. These varieties provide an important geometric interpretation for the atypicality of simple $\mathfrak{g}$-modules. In this section, for type I Lie superalgebras, we will prove properties of support varieties for $\mathfrak{g}_{\pm 1}$ (respectively, the pair $(\mathfrak{p}^{\pm}, \mathfrak{g}_0)$) which will be used to measure the complexity for $\mathfrak{g}_{\pm 1}$-modules (respectively, modules in $\mathcal{F}_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}$). The results in this section will later be used to compute the complexity of Kac modules in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$.

Let $\mathfrak{g}$ be a classical Lie superalgebra, let $R := \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$, and let M_1 and M_2 be in $\mathcal{F} := \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. According to [BKN10, Theorem 2.5.3], $\mathrm{Ext}^\bullet_{\mathcal{F}}(M_1, M_2)$ is a finitely generated R-module. Set $J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, M_2) := \mathrm{Ann}_R(\mathrm{Ext}^\bullet_{\mathcal{F}}(M_1, M_2))$ (i.e. the annihilator ideal of this module). The *relative support variety of the pair* (M_1, M_2) is

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, M_2) := \mathrm{MaxSpec}(R/J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, M_2)). \tag{3.1.1}$$

In the case where $M = M_1 = M_2$, set $J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, M)$ and

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) := \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, M).$$

The variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ is called the *support variety* of M. In this situation, $J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \mathrm{Ann}_R(\mathrm{Id})$ where Id is the identity morphism in $\mathrm{Hom}_{\mathcal{F}}(M, M)$.

3.2 The case where $\mathfrak{g} = \mathfrak{g}_{\pm 1}$

Observe that for type I Lie superalgebras, both $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are abelian Lie superalgebras and, consequently,

$$R_{\pm} := \mathrm{H}^\bullet(\mathfrak{g}_{\pm 1}, \mathbb{C}) = \mathrm{H}^\bullet(\mathfrak{g}_{\pm 1}, \{0\}; \mathbb{C}) \cong S(\mathfrak{g}^*_{\pm 1})$$

as graded algebras. Let $\mathcal{F}(\mathfrak{g}_{\pm 1})$ be the category of finite-dimensional $\mathfrak{g}_{\pm 1}$-modules. If M is an object in $\mathcal{F}(\mathfrak{g}_{\pm 1})$, then one can define the $\mathfrak{g}_{\pm 1}$ *support variety* of M,

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \mathcal{V}_{(\mathfrak{g}_{\pm 1},0)}(M).$$

Since $\mathfrak{g}_{\pm 1}$ is abelian, the arguments given in [BKN10, § 5] for detecting subalgebras apply here as well, and one has that $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ is canonically isomorphic to the following rank variety:

$$\mathcal{V}^{\mathrm{rank}}_{\mathfrak{g}_{\pm 1}}(M) := \{x \in \mathfrak{g}_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\},$$

where $U(\langle x \rangle)$ denotes the enveloping algebra of the Lie subsuperalgebra generated by $x \in \mathfrak{g}_{\pm 1}$. We will identify $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ and $\mathcal{V}^{\mathrm{rank}}_{\mathfrak{g}_{\pm 1}}(M)$ via this canonical isomorphism. As a consequence of this alternate description, $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ satisfies the various properties of a rank variety (e.g. it satisfies the tensor product rule and detects $\mathfrak{g}_{\pm 1}$ projectivity; cf. [BKN10, §§ 5 and 6]).

3.3 The case where $\mathfrak{g} = \mathfrak{p}^{\pm}$

For a type I classical Lie superalgebra, we show for modules in $\mathcal{F} = \mathcal{F}_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}$ that the theory of support varieties for $\mathfrak{g}_{\pm 1}$, as presented in the previous section, does measure complexity. Recall that if $L_0(\lambda)$ is a finite-dimensional simple $\mathfrak{g}_0$-module, then it is canonically a simple $\mathfrak{p}^{\pm}$-module via inflation. Furthermore, as λ ranges over X_0^+, this provides a complete, irredundant set of simple $\mathfrak{p}^{\pm}$-modules in $\mathcal{F}_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}$.

Theorem 3.3.1. *Let $\mathfrak{g}$ be a type I classical Lie superalgebra and let M be a module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}$. Then*

$$c_{\mathcal{F}}(M) = \dim \mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \dim \mathcal{V}^{\mathrm{rank}}_{\mathfrak{g}_{\pm 1}}(M).$$

Proof. Let M be in $\mathcal{F} = \mathcal{F}_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}$. According to Proposition 2.2.1, it follows that

$$c_{\mathcal{F}}(M) = r\left(\mathrm{Ext}^\bullet_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}\left(M, \bigoplus_{\lambda \in X_0^+} L_0(\lambda)^{\dim P(\lambda)}\right)\right).$$

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In this instance, the projective cover is $P(\lambda) \cong U(\mathfrak{p}^{\pm}) \otimes_{U(\mathfrak{g}_0)} L_0(\lambda)$ in $\mathcal{F}$. Set

$$L = \bigoplus_{\lambda \in X_0^+} L_0(\lambda)^{\dim P(\lambda)}.$$

Observe that, as a G_0-module,

$$\begin{aligned} L &\cong \Big[\bigoplus_{\lambda \in X_0^+} L_0(\lambda) \otimes L_0(\lambda)^* \Big] \otimes \Lambda^\bullet(\mathfrak{g}_{\pm 1}) \\ &\cong k[G_0] \otimes \Lambda^\bullet(\mathfrak{g}_{\pm 1}), \end{aligned} \tag{3.3.1}$$

where in the first isomorphism the action of G_0 on $\Lambda^\bullet(\mathfrak{g}_{\pm 1})$ and on each $L_0(\lambda)^*$ is trivial, and in the second isomorphism the action on $k[G_0]$ is by left translation while the action on $\Lambda^\bullet(\mathfrak{g}_{\pm 1})$ is trivial. The second isomorphism is a well-known fact from the representation theory of reductive algebraic groups (cf. [Jan03, I.3.7]).

Next, observe that $\mathfrak{g}_{\pm 1}$ is an ideal of $\mathfrak{p}^{\pm}$. There exists a Lyndon–Hochschild–Serre spectral sequence for the pair $(\mathfrak{g}_{\pm 1}, \{0\})$ in $(\mathfrak{p}^{\pm}, \mathfrak{g}_0)$ (cf. [BW80, Theorem 6.5]):

$$E_2^{i,j} = \operatorname{Ext}^i_{(\mathfrak{g}_0,\mathfrak{g}_0)}(\mathbb{C}, \operatorname{Ext}^j_{(\mathfrak{g}_{\pm 1},\{0\})}(M, L)) \Rightarrow \operatorname{Ext}^{i+j}_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}(M, L).$$

The higher extension groups $\operatorname{Ext}^i_{(\mathfrak{g}_0,\mathfrak{g}_0)}(-,-)$ vanish because we are considering extensions in the category of finitely semisimple $\mathfrak{g}_0$-modules; thus the spectral sequence collapses and yields the first of the following isomorphisms:

$$\begin{aligned} \operatorname{Ext}^d_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}(M, L) &\cong \operatorname{Hom}_{\mathfrak{g}_0}(\mathbb{C}, \operatorname{Ext}^d_{\mathfrak{g}_{\pm 1}}(M, L)), \\ &\cong \operatorname{Hom}_{G_0}(\mathbb{C}, \operatorname{Ext}^d_{\mathfrak{g}_{\pm 1}}(M, \mathbb{C}) \otimes L), \\ &= \operatorname{Hom}_{G_0}(\mathbb{C}, \operatorname{Ext}^d_{\mathfrak{g}_{\pm 1}}(M, \mathbb{C}) \otimes k[G_0] \otimes \Lambda^\bullet(\mathfrak{g}_{\pm 1})), \\ &\cong \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, \operatorname{Ext}^d_{\mathfrak{g}_{\pm 1}}(M, \mathbb{C}) \otimes \Lambda^\bullet(\mathfrak{g}_{\pm 1})). \end{aligned}$$

The second isomorphism follows from the fact that $\mathfrak{g}_{\pm 1}$ acts trivially on L, and the equality follows from (3.3.1). The third isomorphism follows from the tensor identity, the fact that $k[G_0] \cong \operatorname{ind}_1^{G_0} \mathbb{C}$, and Frobenius reciprocity.

Hence, for all $d \geqslant 0$, we see that

$$\dim \operatorname{Ext}^d_{(\mathfrak{p}^{\pm},\mathfrak{g}_0)}(M, L) = \dim \operatorname{Ext}^d_{\mathfrak{g}_{\pm 1}}(M, \mathbb{C}) \otimes \Lambda^\bullet(\mathfrak{g}_{\pm 1}),$$

and, since $\Lambda^\bullet(\mathfrak{g}_{\pm 1})$ is finite-dimensional, we have the first of the following equalities:

$$c_{\mathcal{F}}(M) = r(\operatorname{Ext}^\bullet_{\mathfrak{g}_{\pm 1}}(M, \mathbb{C})) = c_{\mathcal{F}(\mathfrak{g}_{\pm 1})}(M) = \dim \mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \dim \mathcal{V}^{\mathrm{rank}}_{\mathfrak{g}_{\pm 1}}(M).$$

To obtain the subsequent equalities, we use [BKN11, Theorem 2.9.1] along with the fact that the abelian superalgebra $\mathfrak{g}_{\pm 1}$ has only a single simple module, namely the trivial module. □

4. Kazhdan–Lusztig polynomials

4.1 Given $\lambda, \mu \in X_0^+$, we define the 'naive' Kazhdan–Lusztig polynomial $p_{\lambda,\mu}(q)$ by

$$p_{\lambda,\mu}(q) = q^{l(\mu)-l(\lambda)} \sum_{n \geqslant 0} \dim \operatorname{Ext}^n_{\mathcal{F}}(K(\lambda), L(\mu)) q^{-n}.$$

In the notation of [Bru03, Theorem 4.51], we have

$$p_{\lambda,\mu}(q) = q^{l(\mu)-l(\lambda)} l_{\lambda,\mu}(-q). \tag{4.1.1}$$

By the same theorem, $p_{\lambda,\mu}(q)$ has constant term 1.

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4.2 For the purposes of our computation we will need to use the following fact that the set of Kazhdan–Lusztig polynomials is finite.

Theorem 4.2.1. *For a fixed $\mathfrak{gl}(m|n)$, the set $\{p_{\lambda,\mu}(q) \mid \lambda, \mu \in X_0^+\}$ is finite.*

Proof. If $K(\lambda)$ and $L(\mu)$ lie in different blocks, then $p_{\lambda,\mu} = 0$. So we may assume that λ and μ lie in the same block. Now fix a block $\mathcal{B}$ of atypicality k. In [SZ07, § 3.9], Su and Zhang combinatorially define a bijection on highest weights between any block of atypicality k and the principal block of $\mathfrak{gl}(k|k)$. Let $\phi : \mathcal{B} \to \mathcal{B}_{0,k|k}$ denote the Su–Zhang bijection from $\mathcal{B}$ to the principal block of $\mathfrak{gl}(k|k)$. We discuss the bijection in greater detail in § 7.2. For the moment, however, we only need the following fact. By [SZ07, Theorem 3.29] and the fact that the Su–Zhang bijection satisfies $l(\gamma) = l(\phi(\gamma))$ for all $\gamma \in \mathcal{B}$, we have $p_{\lambda,\mu}(q) = p_{\phi(\lambda),\phi(\mu)}(q)$ for all $\lambda, \mu \in \mathcal{B}$. As a consequence,

$$\{p_{\lambda,\mu}(q) \mid \lambda, \mu \in X_0^+\} = \{p_{\lambda,\mu}(q) \mid \lambda, \mu \in \mathcal{B}_{0,k|k},\ k = 1, \ldots, \min(m, n)\} \cup \{0\}.$$

Therefore we may assume without loss of generality that $\mathcal{B} = \mathcal{B}_0$ is the principal block of $\mathfrak{gl}(k|k)$.

Since by definition the coefficients of the polynomials $p_{\lambda,\mu}(q)$ are nonnegative integers, it suffices to show that there is an absolute bound on their degree and on the sum of their coefficients. In order to bound the degree, observe that

$$\begin{aligned} \dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)) &= \dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), L(\mu)) \\ &= \dim \operatorname{Ext}^d_{(\mathfrak{p}^+,\mathfrak{g}_0)}(L_0(\lambda), L(\mu)) \\ &= \dim \operatorname{Hom}_{\mathfrak{g}_0}(L_0(\lambda), \mathrm{H}^d(\mathfrak{g}_1, 0; L(\mu))) \\ &\leqslant \dim \operatorname{Hom}_{\mathfrak{g}_0}(L_0(\lambda), S^d(\mathfrak{g}_1^*) \otimes L(\mu)) \\ &\leqslant \dim \operatorname{Hom}_{\mathfrak{g}_0}(L_0(\lambda), S^d(\mathfrak{g}_1^*) \otimes \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes L_0(\mu)). \end{aligned} \tag{4.2.1}$$

The first line is (2.2.1), the second is Frobenius reciprocity, the third comes from application of a Lyndon–Hochschild–Serre spectral sequence, the fourth is because in this case relative cohomology is a subquotient of the complex $S^\bullet(\mathfrak{g}_1^*) \otimes L(\mu)$ (cf. [BKN10, § 2.5]), and the last is because $L(\mu)$ is a quotient of the Kac module $K(\mu)$ so that any $\mathfrak{g}_0$ composition factor of $S^n(\mathfrak{g}_1^*) \otimes L(\mu)$ is a composition factor of $S^d(\mathfrak{g}_1^*) \otimes K(\mu) \cong S^n(\mathfrak{g}_1^*) \otimes \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes L_0(\mu)$ (this isomorphism is as $\mathfrak{g}_0$-modules).

Now consider the element $c := \sum_{k=1}^m E_{k,k} \in \mathfrak{g}_0$. Then c is central in the enveloping algebra of $\mathfrak{g}_0$. Furthermore, if $L_0(\gamma)$ is a simple $\mathfrak{g}_0$-module of highest weight $\gamma \in \mathfrak{h}^*$, then c acts on $L_0(\gamma)$ by the scalar $|\gamma|$. From this fact and the description of $\mathfrak{g}_{\pm 1}$ as $\mathfrak{g}_0$-modules, we see that $L_0(\lambda)$ is a composition factor of $S^d(\mathfrak{g}_1^*) \otimes \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes L_0(\mu)$ only if

$$|\lambda| = -d - b + |\mu|$$

for some $b \in \{0, 1, \ldots, \dim \mathfrak{g}_{-1}\}$. By (4.2.1), it follows that

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)) \neq 0 \implies -d = |\lambda| - |\mu| + b \tag{4.2.2}$$

where $b \in \{0, 1, \ldots, \dim \mathfrak{g}_{-1}\}$. This statement, along with the fact that $|\lambda| = l(\lambda)$ for all λ in the principal block of $\mathfrak{gl}(k|k)$, shows by the definition of $p_{\lambda,\mu}(q)$ that the degree of this polynomial is bounded by $\dim \mathfrak{g}_{-1}$.

Next, we prove that the sum of the coefficients of $p_{\lambda,\mu}(q)$ is absolutely bounded. Observe, using (4.1.1), that this sum is given by $p_{\lambda,\mu}(1) = l_{\lambda,\mu}(-1)$. There is a convenient alternate

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description of $l_{\lambda,\mu}(-q^{-1})$ given in [SZ07, Theorem 3.24], which defines it as a sum of monomials indexed by a subset of the symmetric group on k letters. From this it is evident that $p_{\lambda,\mu}(1) \leqslant k!$. □

5. Projective modules

5.1 The formula in Proposition 2.2.1 indicates that in order to compute the complexity of a module in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$, one requires effective bounds on the dimension of $P(\mu)$ (the projective cover of the simple $\mathfrak{gl}(m|n)$-module $L(\mu)$). In order to accomplish this, we relate the dimension of $P(\mu)$ to the dimension of $L_0(\mu)$.

First observe that as a $\mathfrak{g}_0$-module, $L(\mu)$ contains $L_0(\mu)$ as a composition factor. Consequently, $\dim P(\mu) \geqslant \dim L_0(\mu)$. On the other hand, by the PBW theorem for Lie superalgebras, we see that $U(\mathfrak{g})$ is a free $U(\mathfrak{g}_0)$-module and thus $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\mu)$ is a projective $\mathfrak{g}$-module. Furthermore, by applying Frobenius reciprocity, we see that $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\mu)$ surjects onto $L(\mu)$. Thus, $P(\mu)$ is a direct summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\mu)$. However, by the PBW theorem, $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} L_0(\mu) \cong \Lambda^\bullet(\mathfrak{g}_{\bar{1}}) \otimes L_0(\mu)$ as vector spaces. Therefore we have $2^{\dim \mathfrak{g}_{\bar{1}}} \dim L_0(\mu) \geqslant \dim P(\mu)$. In summary, we have

$$2^{\dim \mathfrak{g}_{\bar{1}}} \dim L_0(\mu) \geqslant \dim P(\mu) \geqslant \dim L_0(\mu). \tag{5.1.1}$$

5.2 We now obtain an upper bound on the dimension of projective indecomposables appearing in a minimal projective resolution of an object of $\mathcal{F}$.

Theorem 5.2.1. *Let M be a $\mathfrak{gl}(m|n)$-module which lies in a block of atypicality k. Let $P_\bullet \to M$ be a minimal projective resolution for M. Then there is a positive constant C depending only on m, n and M such that if $P(\mu)$ appears as a direct summand of P_d, then*

$$\dim P(\mu) \leqslant C d^{(m+n-k-1)k}.$$

Proof. We first consider the case where $M = L(\lambda)$ is a simple module. Let $P(\mu)$ be a direct summand of P_d in the minimal projective resolution of $L(\lambda)$. By tensoring by sufficiently many copies of the one-dimensional supertrace representation (cf. § 2.4), we may assume without loss of generality that $\lambda_i, \mu_i > 0$ for $i = 1, \ldots, m$ and $\lambda_i, \mu_i < 0$ for $i = m+1, \ldots, m+n$. Furthermore, by (5.1.1), it suffices to show that $\dim L_0(\mu)$ is bounded above by $C d^{(m+n-k-1)k}$.

Since $\mathfrak{g}_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, it follows by Weyl's dimension formula (cf. [GW98]) that

$$\dim L_0(\mu) = \prod_{\alpha \in \Phi_m^+} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\alpha \in \Phi_n^+} \frac{(\mu + \rho_n, \alpha)}{(\rho_n, \alpha)}. \tag{5.2.1}$$

We can decompose the first factor as follows:

$$\prod_{\alpha \in \Phi_m^+} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} = \prod_{\alpha \in A_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\alpha \in B_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\alpha \in C_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)}, \tag{5.2.2}$$

where $A_m = A_m(\mu)$, $B_m = B_m(\mu)$ and $C_m = C_m(\mu)$ are as in § 2.5.

Consider the factor involving A_m. Since $P(\mu)$ lies in the same block as $L(\lambda)$, we have by Proposition 2.5.1 that the core of μ equals the core of λ. In particular, we have an equality of multisets

$$\{(\mu + \rho, \alpha) \mid \alpha \in A_m(\mu)\} = \{(\lambda + \rho, \alpha) \mid \alpha \in A_m(\lambda)\},$$

and hence we have an equality of multisets

$$\{(\mu + \rho_m, \alpha) \mid \alpha \in A_m(\mu)\} = \{(\lambda + \rho_m, \alpha) \mid \alpha \in A_m(\lambda)\}.$$

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From this and the fact that $(\rho_m, \alpha) \geqslant 1$ for all $\alpha \in \Phi_m^+$, we deduce that

$$\prod_{\alpha \in A_m(\mu)} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \leqslant \prod_{\alpha \in A_m(\mu)} (\mu + \rho_m, \alpha) = \prod_{\alpha \in A_m(\lambda)} (\lambda + \rho_m, \alpha) =: C_1. \tag{5.2.3}$$

In particular, C_1 is a constant which depends only on m, n and λ.

Next, consider the B_m factor in (5.2.2). Since $P(\mu)$ appears in P_d, we have $\operatorname{Hom}_{\mathcal{F}}(P_d, L(\mu)) \neq 0$. But as $P_\bullet$ is a minimal projective resolution, we have

$$\operatorname{Ext}^d_{\mathcal{F}}(L(\lambda), L(\mu)) = \operatorname{Hom}_{\mathcal{F}}(P_d, L(\mu)) \neq 0.$$

By [Bru03, Corollary 4.52] we have

$$0 \neq \dim \operatorname{Ext}^d_{\mathcal{F}}(L(\lambda), L(\mu)) = \sum_{i+j=d} \sum_{\sigma \in \mathcal{B}} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\lambda)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)),$$

where $\mathcal{B}$ is the block containing $L(\lambda)$ and $L(\mu)$. Therefore, there must be a $\sigma \in \mathcal{B}$ and i, j with $i + j = d$ such that

$$\dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\lambda)) \neq 0 \quad \text{and} \quad \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \neq 0.$$

Now this implies by (4.2.2) that

$$i = |\lambda| - |\sigma| - b_1,$$
$$j = |\mu| - |\sigma| - b_2,$$

where $b_1, b_2 \in \{0, 1, \ldots, \dim \mathfrak{g}_{-1}\}$. Upon taking the difference we obtain

$$-d \leqslant i - j = |\lambda| - |\mu| - b_1 + b_2,$$

and hence we find that

$$|\mu| \leqslant d + D$$

where D is a constant depending only on m, n and λ.

Since $\mu_1, \ldots, \mu_m \geqslant 0$, we deduce that

$$\mu_t \leqslant d + D$$

for $t = 1, \ldots, m$. Let $\alpha = \varepsilon_r - \varepsilon_s \in B_m(\mu)$. Then

$$(\mu, \alpha) \leqslant \mu_r + \mu_s \leqslant 2(d + D).$$

Let $z = \min\{(\rho_m, \alpha) \mid \alpha \in \Phi_m^+\} > 0$ and $Z = \max\{(\rho_m, \alpha) \mid \alpha \in \Phi_m^+\}$. We then have

$$\prod_{\alpha \in B_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} = \prod_{\alpha \in B_m} \frac{(\mu, \alpha) + (\rho_m, \alpha)}{(\rho_m, \alpha)} \leqslant \prod_{\alpha \in B_m} \frac{2(d + D) + Z}{z} \leqslant C_2 d^{(m-k)k}, \tag{5.2.4}$$

where C_2 is a constant depending only on D, Z and z and so, ultimately, only on m, n and λ. The last inequality also uses the observation that there are precisely $(m - k)k$ elements in the set B_m.

We now consider the C_m factor. An identical calculation, using the fact that there are precisely $(k^2 - k)/2$ elements in the set C_m, yields

$$\prod_{\alpha \in C_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \leqslant C_3 d^{(k^2-k)/2} \tag{5.2.5}$$

where, again, C_3 depends only on m, n and λ.

Finally, we put (5.2.3)–(5.2.5) into (5.2.2) and obtain

$$\prod_{\alpha\in\Phi_m^+}\frac{(\mu+\rho_m,\alpha)}{(\rho_m,\alpha)}\leqslant C_1C_2C_3d^{(m-k)k+(k^2-k)/2}. \tag{5.2.6}$$

We now turn to the second factor in (5.2.1). An identical analysis (with m replaced by n) yields

$$\prod_{\alpha\in\Phi_n^+}\frac{(\mu+\rho_n,\alpha)}{(\rho_n,\alpha)}\leqslant C_1'C_2'C_3'd^{(n-k)k+(k^2-k)/2}. \tag{5.2.7}$$

Inserting (5.2.6) and (5.2.7) into (5.2.1), we obtain

$$\dim L_0(\mu)\leqslant Cd^{(m+n-k-1)k},$$

where C is some constant depending only on m,n and λ. As we explained at the beginning of the proof, this suffices to prove the desired result for $L(\lambda)$.

To prove the general case, we use the horseshoe lemma to argue by induction on the length of a composition series for M. □

6. Complexity for Kac modules

6.1 We begin by establishing general bounds using the geometry of support varieties for the complexity of a Kac module for a type I Lie superalgebra $\mathfrak{g}$. Let $K(\lambda)$ be a Kac module and let

$$\cdots\to P_2\to P_1\to P_0\to L_0(\lambda)\to 0$$

be a minimal projective resolution of $L_0(\lambda)$ in $\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}$. We can apply the exact functor $U(\mathfrak{g})\otimes_{U(\mathfrak{p}^+)}-$ to this resolution to get a projective resolution with the same rate of growth for $K(\lambda)$ in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$:

$$\cdots\to U(\mathfrak{g})\otimes_{U(\mathfrak{p}^+)}P_2\to U(\mathfrak{g})\otimes_{U(\mathfrak{p}^+)}P_1\to U(\mathfrak{g})\otimes_{U(\mathfrak{p}^+)}P_0\to K(\lambda)\to 0.$$

This shows that

$$c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda))\leqslant c_{\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}}(L_0(\lambda))\leqslant\dim\mathfrak{g}_1, \tag{6.1.1}$$

where the last inequality is by Theorem 3.3.1.

Next, observe that any projective resolution in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ of a module M (such as $K(\lambda)$) will restrict to a projective resolution of M in $\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}$. Therefore,

$$c_{\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}}(M)\leqslant c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(M).$$

Combining this statement with Theorem 3.3.1, we have

$$\dim\mathcal{V}_{\mathfrak{g}_1}^{\mathrm{rank}}(M)\leqslant c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(M). \tag{6.1.2}$$

6.2 Complexity of Kac modules in the principal block for $\mathfrak{gl}(k|k)$

As a step towards solving the general problem for $\mathfrak{gl}(m|n)$, we now compute the complexity of the Kac and dual Kac modules in the principal block $\mathcal{B}_0$ of $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ for $\mathfrak{gl}(k|k)$.

Theorem 6.2.1. *Let $K(\lambda)$ (respectively, $K^-(\lambda)$) be a Kac (respectively, dual Kac) module in the principal block $\mathcal{B}_0$ of $\mathcal{F}=\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ for $\mathfrak{g}=\mathfrak{gl}(k|k)$. Then*

$$c_{\mathcal{F}}(K(\lambda))=c_{\mathcal{F}}(K^-(\lambda))=\mathrm{atyp}(\lambda)^2.$$

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Proof. Let us first consider the case of Kac modules. If $K(\lambda)$ lies in the principal block $\mathcal{B}_0$, then $\operatorname{atyp}(\lambda) = k$. According to (6.1.1) and (6.1.2),

$$\dim \mathcal{V}_{\mathfrak{g}_1}^{\mathrm{rank}}(K(\lambda)) \leqslant c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda)) \leqslant \dim \mathfrak{g}_1 = k^2.$$

Therefore, it suffices to prove that $\mathcal{V}_{\mathfrak{g}_1}^{\mathrm{rank}}(K(\lambda)) = \mathfrak{g}_1$.

Let

$$I_k = E_{1,k+1} + E_{2,k+2} + \cdots + E_{k,2k} \in \mathfrak{g}_1.$$

The element I_k has rank k, so by § 2.6 we have $\mathfrak{g}_1 = \overline{G_0 \cdot I_k}$. The variety $\mathcal{V}_{\mathfrak{g}_1}^{\mathrm{rank}}(K(\lambda))$ is closed and stable under G_0 (because $K(\lambda)$ is a $\mathfrak{g}$-module), so we need only demonstrate that $K(\lambda)$ is not free as a $U(\langle I_k \rangle)$-module.

Since $K(\lambda)$ is in $\mathcal{B}_0$, $L_0(\lambda) \cong S \boxtimes S^*$ where S is a simple $\mathrm{GL}(k)$-module. Let ΔG_0 (respectively, $\Delta\mathfrak{g}_0$) be the image of the diagonal embedding of $\mathrm{GL}(k) \hookrightarrow \mathrm{GL}(k) \times \mathrm{GL}(k)$ (respectively, $\mathfrak{gl}(k) \hookrightarrow \mathfrak{gl}(k) \times \mathfrak{gl}(k)$). As a ΔG_0-module, $L_0(\lambda) \cong S \otimes S^* \cong \mathbb{C} \oplus N$ for some module N.

One can verify directly that

$$[\mathfrak{g}_{-1}, I_k] \subseteq \Delta\mathfrak{g}_0.$$

By the fact that $\mathfrak{g}$ is $\mathbb{Z}$-graded,

$$[\Delta\mathfrak{g}_0, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_{-1}.$$

The inclusions above imply that as a $U(\langle I_k \rangle)$-module, $K(\lambda)$ decomposes as

$$K(\lambda)|_{U(\langle I_k \rangle)} \cong (U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} \mathbb{C}) \oplus (U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} N). \tag{6.2.1}$$

Now consider $K(0) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} \mathbb{C}$ and observe that this is the first component of the decomposition in (6.2.1) as a $U(\langle I_k \rangle)$-module. Using the $\mathbb{Z}$-grading on $\mathfrak{g}$, one has

$$[\mathfrak{p}^+, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

and

$$[\mathfrak{g}_{-1}, \mathfrak{g}_0] \subseteq \mathfrak{g}_{-1}.$$

These two relations imply that, as $U(\mathfrak{p}^+)$-modules, we have

$$K(0)|_{U(\mathfrak{p}^+)} \cong (1 \otimes \mathbb{C}) \oplus (U(\mathfrak{g}_{-1})\mathfrak{g}_{-1} \otimes \mathbb{C}) \cong \mathbb{C} \oplus (U(\mathfrak{g}_{-1})\mathfrak{g}_{-1} \otimes \mathbb{C}).$$

Since $I_k \in \mathfrak{p}^+$, it follows that $K(\lambda)$ as a $U(\langle I_k \rangle)$-module has $\mathbb{C}$ as a direct summand, which proves that $K(\lambda)$ is not free as a $U(\langle I_k \rangle)$-module.

The proof for dual Kac modules follows the same line of reasoning but with $\mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_{\mp 1}$ interchanged. □

6.3 General case

We can now compute the complexity of an arbitrary Kac module for $\mathfrak{gl}(m|n)$. This requires the use of the Gruson–Serganova block equivalences. Specifically, Gruson and Serganova proved that a block of $\mathcal{F} = \mathcal{F}(\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)_0)$ of atypicality k is equivalent to the principal block of $\mathcal{F}(\mathfrak{gl}(k|k), \mathfrak{gl}(k|k)_0)$. This was also proven by entirely different means by Brundan and Stroppel [BS12]. However, for our purposes we require the explicit equivalence constructed by Gruson and Serganova. We only sketch what we need; the full details can be found in [GS10].

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Let $\mathcal{B}$ be a fixed block of $\mathcal{F}$ with atypicality k. On a given module in $\mathcal{B}$, the functor defining the Gruson–Serganova equivalence is given by

$$\mathrm{Res}_\mu \circ T_t \circ \cdots \circ T_1, \tag{6.3.1}$$

where t is some natural number determined by the module, and the functors T_i are certain translation functors which provide equivalences between blocks of $\mathcal{F}$. Let $\mathfrak{g}'$ denote the subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{gl}(k|k)$ spanned by the matrix units $E_{i,j}$ $(i, j = m-k+1, \ldots, m+k)$. The functor Res_μ refines the restriction from $\mathfrak{g}$ to $\mathfrak{g}'$.

We examine how complexity is affected by the translation functors. Let T be a translation functor defined by tensoring by a finite-dimensional $\mathfrak{g}$-module E and projecting onto a block. Let $P_\bullet \to M$ be a minimal projective resolution of M. As translation functors are exact, if we apply T to this resolution we obtain a (not necessarily minimal) projective resolution $T(P_\bullet) \to T(M)$. Furthermore, we have $\dim(T(P_d)) \leqslant \dim(P_d) \cdot \dim(E)$ for all $d \geqslant 0$. Therefore we have

$$c_{\mathcal{F}}(T(M)) \leqslant c_{\mathcal{F}}(M).$$

However, in the case of the Gruson–Serganova construction, each T_i is an equivalence of categories between two blocks with the inverse functor also given by a translation functor. From this we immediately obtain

$$c_{\mathcal{F}}(T_i(M)) = c_{\mathcal{F}}(M)$$

for all i in (6.3.1).

In what follows, we use the notation of § 2.6. For the computation of the dimension in the following theorem we use the well-known formula for the dimension of the variety $\overline{(\mathfrak{g}_1)_k}$ (see, e.g., [Har92, Proposition 12.2]).

THEOREM 6.3.1. *Let $K(\lambda)$ (respectively, $K^-(\lambda)$) be a Kac (respectively, dual Kac) module for $\mathfrak{gl}(m|n)$ with $\mathrm{atyp}(\lambda) = k$. Then:*

(a) $c_{\mathcal{F}}(K(\lambda)) = \dim \overline{(\mathfrak{g}_1)_k} = (m+n)k - k^2$;

(b) $c_{\mathcal{F}}(K^-(\lambda)) = \dim \overline{(\mathfrak{g}_{-1})_k} = (m+n)k - k^2$.

Proof. We will prove (a); assertion (b) follows by a similar reasoning. Our first step entails showing that $c_{\mathcal{F}}(K(\lambda)) \geqslant \dim \overline{(\mathfrak{g}_1)_k}$. As $\mathrm{atyp}(\lambda) = k$, $K(\lambda)$ lies in a block of atypicality k. By the work of Gruson and Serganova [GS10] discussed above, this block is equivalent to the principal block of $\mathfrak{gl}(k|k)$. On a fixed module, the functor defining this equivalence is of the form given by (6.3.1). As discussed above, the complexity of a module is unaffected by the functors $T_1, \ldots, T_t$. Thus the module $M = T_t \circ \cdots \circ T_1(K(\lambda))$ has the same complexity as $K(\lambda)$. By the definition of the functor Res_μ we have, as $\mathfrak{g}'$-modules,

$$M = \mathrm{Res}_\mu(M) \oplus G_\mu(M) \tag{6.3.2}$$

for a $\mathfrak{g}'$-module $G_\mu(M)$ (cf. [BKN09, (4.7.4)]). The functor (6.3.1) takes Kac modules to Kac modules and, in particular, $\mathrm{Res}_\mu(M)$ is a Kac module in the principal block of $\mathfrak{gl}(k|k)$. Now, it was shown in the proof of Theorem 6.2.1 that the element $I_k \in \mathfrak{g}'_1 \subseteq \mathfrak{g}_1$ lies in $\mathcal{V}_{\mathfrak{g}'_1}(\mathrm{Res}_\mu(M))$. By the rank variety description, it immediately follows from (6.3.2) that $I_k \in \mathcal{V}_{\mathfrak{g}_1}(M)$. But since I_k is a rank-k matrix and $\mathcal{V}_{\mathfrak{g}_1}(M)$ is a G_0-stable closed subvariety of $\mathfrak{g}_1$, we have

$$\overline{G_0 \cdot I_k} = \overline{(\mathfrak{g}_1)_k} \subseteq \mathcal{V}_{\mathfrak{g}_1}(M). \tag{6.3.3}$$

Combining this with (6.1.2) gives the desired inequality

$$c_{\mathcal{F}}(K(\lambda)) = c_{\mathcal{F}}(M) \geqslant \dim \mathcal{V}_{\mathfrak{g}_1}(M) \geqslant \dim \overline{(\mathfrak{g}_1)_k}.$$

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We now consider the reverse inequality. By Proposition 2.2.1 we have

$$c_{\mathcal{F}}(K(\lambda)) = r\Big(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus L(\mu)^{\dim P(\mu)}\Big)\Big), \tag{6.3.4}$$

where the direct sum is over all simple modules in the block containing $K(\lambda)$. However, for fixed d,

$$\dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus L(\mu)^{\dim P(\mu)}\Big) = \sum \dim P(\mu)\cdot \dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), L(\mu)). \tag{6.3.5}$$

If $P_\bullet \to K(\lambda)$ is a minimal projective resolution, then $\mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), L(\mu)) = \mathrm{Hom}_{\mathcal{F}}(P_d, L(\mu))$ being nonzero implies that $P(\mu)$ is a summand of P_d. By Theorem 5.2.1, this implies that $\dim P(\mu) \leqslant Cd^{(m+n-k-1)k}$ for some constant C which depends only on m, n and λ. Thus,

$$\dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus L(\mu)^{\dim P(\mu)}\Big) \leqslant Cd^{(m+n-k-1)k} \dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus L(\mu)\Big). \tag{6.3.6}$$

Therefore, it suffices to prove that

$$\dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus L(\mu)\Big) \leqslant Kd^{k-1} \tag{6.3.7}$$

for some constant K, where the direct sum is over all simple modules in the block containing $K(\lambda)$.

Because the left-hand side of (6.3.7) is invariant under the Gruson–Serganova categorical equivalence between blocks, we may assume without loss of generality that $m = n = k$ and that $K(\lambda)$ is a Kac module in the principal block for $\mathfrak{gl}(k|k)$. We now consider the case where

$$\dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), L(\sigma)) \neq 0. \tag{6.3.8}$$

Put $\lambda = \sum_{i=1}^{2k} \lambda_i \varepsilon_i$ and $\sigma = \sum_{i=1}^{2k} \sigma_i\varepsilon_i$. As (6.3.8) is nonzero for only finitely many σ, by tensoring with sufficiently many copies of the one-dimensional supertrace representation we may assume without loss of generality that $(\lambda_1, \dots, \lambda_k)$ and $(\sigma_1, \dots, \sigma_k)$ are always partitions. By (4.2.2), the nonvanishing of (6.3.8) implies that $|\sigma| = |\lambda| + d + b$ where $b \in \{0, \dots, \dim \mathfrak{g}_{-1} = k^2\}$. Therefore,

$$|\lambda| + d \leqslant |\sigma| \leqslant |\lambda| + d + k^2.$$

That is, in order for (6.3.8) not to vanish, $(\sigma_1, \dots, \sigma_k)$ must be a partition of an integer between $|\lambda| + d$ and $|\lambda| + d + k^2$.

Now, the number of partitions of i into no more than k parts is bounded by $C_1 i^{k-1}$, where C_1 is a constant depending only on k (see [Nat00, Corollary 15.1]). Furthermore, the dimension of $\mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), L(\sigma))$ is the coefficient of a Kazhdan–Lusztig polynomial of the type considered in § 4. By Theorem 4.2.1, the coefficients of these polynomials are uniformly bounded by some constant C_2. Taken together, these observations imply that

$$\begin{aligned}\dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus L(\mu)\Big) &= \sum \dim \mathrm{Ext}^{d}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), L(\mu))\\ &\leqslant C_1 \cdot C_2 \cdot \sum_{i=|\lambda|+d}^{|\lambda|+d+k^2} i^{k-1}\\ &\leqslant Kd^{k-1},\end{aligned}$$

where K is some constant depending only on m, n, k and λ. This verifies (6.3.7) and hence completes the proof. □

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6.4 Let

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} \mid [x, x] = 0\}.$$

If M is in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$, then Duflo and Serganova [DS05] define an associated variety[3] by

$$\mathcal{X}_M = \{x \in \mathcal{X} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}$$

and show that their varieties capture a number of interesting features of $\mathcal{F}$.

In the next result, we compute the Duflo–Serganova varieties for the Kac and dual Kac modules. We also indicate how the varieties $\mathcal{X}_{K^{\pm}(\lambda)}$ and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K^{\pm}(\lambda))$ measure the complexity of $K^{\pm}(\lambda)$ in $\mathcal{F}$.

Theorem 6.4.1. *Let $K(\lambda)$ (respectively, $K^{-}(\lambda)$) be a Kac (respectively, dual Kac) module for $\mathfrak{gl}(m|n)$ with* $\operatorname{atyp}(\lambda) = k$. *Then:*

(a) $\mathcal{X}_{K^{\pm}(\lambda)} = \mathcal{V}_{\mathfrak{g}_{\pm 1}}(K^{\pm}(\lambda)) = \overline{(\mathfrak{g}_{\pm 1})_k}$;

(b) $c_{\mathcal{F}}(K^{\pm}(\lambda)) = \dim \mathcal{X}_{K^{\pm}(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K^{\pm}(\lambda))$.

Proof. We will restrict our attention to Kac modules in the proof; the arguments for dual Kac modules are similar.

(a) We first prove the second equality. Since the $\mathfrak{g}_1$-support varieties have the tensor product property, it follows by an argument similar to the one given for complexity in § 6.3 that

$$\mathcal{V}_{\mathfrak{g}_1}(K(\lambda)) = \mathcal{V}_{\mathfrak{g}_1}(M),$$

where $M = T_t \circ \cdots \circ T_1(K(\lambda))$ is the $\mathfrak{gl}(m|n)$ module given in the proof of Theorem 6.3.1. This, along with (6.3.3), implies that

$$\overline{(\mathfrak{g}_1)_k} \subseteq \mathcal{V}_{\mathfrak{g}_1}(K(\lambda)).$$

On the other hand, by (6.1.2) and Theorem 6.3.1 we have that

$$\dim \mathcal{V}_{\mathfrak{g}_1}(K(\lambda)) \leqslant c_{\mathcal{F}(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda)) = \dim \overline{(\mathfrak{g}_1)_k}.$$

Since $\mathcal{V}_{\mathfrak{g}_1}(K(\lambda))$ is a closed G_0-stable subvariety of $\mathfrak{g}_1$, the description of the G_0-orbits given in § 2.6 implies that the two varieties are equal.

We now consider the first equality. By definition we have

$$\mathcal{V}_{\mathfrak{g}_1}(K(\lambda)) = \mathcal{X}_{K(\lambda)} \cap \mathfrak{g}_1 \subseteq \mathcal{X}_{K(\lambda)}.$$

To prove that the inclusion is in fact an equality, we argue by contradiction. Suppose $y \in \mathcal{X}_{K(\lambda)}$ but $y \notin \mathfrak{g}_1$. As $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, we can write $y = y_{-1} + y_1$ with $y_k \in \mathfrak{g}_k$ for $k = -1, 1$ and, by assumption, $y_{-1} \neq 0$.

Fix $a \in \mathbb{R}$ with $0 < a < 1$. Since $\mathfrak{g}$ is $\mathbb{Z}$-graded, we have that $U(\mathfrak{g})$ is $\mathbb{Z}$-graded and, in turn, $K(\lambda)$ inherits a $\mathbb{Z}$-grading. We can then define an action of $\mathbb{Z}$ (written multiplicatively with fixed generator t) on $\mathfrak{g}$ (respectively, $K(\lambda)$) by $t \cdot x = a^l x$ for $x \in \mathfrak{g}_l$ (respectively, $t \cdot m = a^l m$ for $m \in K(\lambda)_l$), where $l \in \mathbb{Z}$. We note that $t \cdot (xm) = (t \cdot x)(t \cdot m)$ for all $x \in \mathfrak{g}$ and $m \in K(\lambda)$. Now, from the definition of $\mathcal{X}_{K(\lambda)}$ and [BKN10, Proposition 5.2.1] it follows that when $K(\lambda)$ is considered as a $\langle y \rangle$-module, a trivial module appears as a direct summand; say it is spanned by $m \in K(\lambda)$. We then check that $t \cdot m$ spans a trivial direct summand of $K(\lambda)$ as a $\langle t \cdot y \rangle$-module. Hence $t \cdot y \in \mathcal{X}_{K(\lambda)}$. Thus $\mathcal{X}_{K(\lambda)}$ is stable under the action of $\mathbb{Z}$ on $\mathfrak{g}_{\bar{1}}$. Since $\mathcal{X}_{K(\lambda)}$ is stable under the action of t, it follows that $t^l y = a^{-l} y_{-1} + a^l y_1 \in \mathcal{X}_{K(\lambda)}$ for all $l > 0$. Since $\mathcal{X}_{K(\lambda)}$

[3] The definition given here is different from but equivalent to the one originally given by Duflo and Serganova (cf. [BKN09, § 3.6]).

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is also conical, we can scale by a^l and see that $y_{-1} + a^{2l} y_1 \in \mathcal{X}_{K(\lambda)}$ for all $l > 1$. However, as $\mathcal{X}_{K(\lambda)}$ is closed, it follows by letting l go to infinity that $y_{-1} \in \mathcal{X}_{K(\lambda)}$. In other words, by definition we have

$$0 \neq y_{-1} \in \mathcal{V}_{\mathfrak{g}_{-1}}(K(\lambda)).$$

But this contradicts the fact that $\mathcal{V}_{\mathfrak{g}_{-1}}(K(\lambda)) = \{0\}$ by [BKN11, Theorem 3.3.1].

For part (b), observe that we have proved that $c_{\mathcal{F}}(K^{\pm}(\lambda)) = \dim \mathcal{X}_{K^{\pm}(\lambda)}$ by part (a) and Theorem 6.3.1. The statement now follows by using the fact that $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K^{\pm}(\lambda)) = \{0\}$ from [BKN09, Corollary 3.3.1]. □

7. An alternative to support varieties

7.1 A polytope calculation

When computing the complexity of Kac modules, a key ingredient is the lower bound provided by the dimension of support varieties. However, for simple modules the known geometric tools are insufficient (see § 2.7). Thus we need to find a suitable replacement. To do so, we use Ehrhart's theorem on counting lattice points in polytopes to obtain 'enough' highest weights to provide an effective lower bound.

Lemma 7.1.1. *Fix an integer $k > 1$. For any integer $d > 0$, let $\tilde{S}(d)$ denote the set of all points $(b, a) = (b_1, \ldots, b_k, a_1, \ldots, a_k) \in \mathbb{Z}^{2k}$ which satisfy the following conditions.*

First, we require the equality

$$(b_1 + \cdots + b_k) - 2(a_1 + \cdots + a_k) = d. \tag{7.1.1}$$

In addition, we require the following inequalities to be satisfied:

$$\begin{aligned} b_u - b_{u+1} &\geqslant d/2k^2, \\ b_1 &\leqslant -d/2k^2, \\ a_u - a_{u+1} &\geqslant 0, \\ a_1 &\leqslant 0, \\ 0 \leqslant (b_1 + \cdots + b_k) - (a_1 + \cdots + a_k) &\leqslant d, \\ a_v &\leqslant b_v, \end{aligned} \tag{7.1.2}$$

where $u = 1, \ldots, k-1$ and $v = 1, \ldots, k$.

As a function of d, the number of elements of $\tilde{S}(d)$ is bounded below by a polynomial, $Q(d)$, of degree $2k - 1$ with positive leading coefficient.

Proof. Let $\mathcal{H}$ be the affine hyperplane in $\mathbb{R}^{2k}$ defined by (7.1.1) when $d = 1$. Let $\mathcal{P}$ be the region in $\mathcal{H}$ defined by points in $\mathcal{H}$ which are simultaneous solutions to the inequalities (7.1.2) when $d = 1$. Then $\mathcal{P}$ is a polytope within $\mathcal{H}$ and hence is of dimension not more than $2k - 1$. To see that $\mathcal{P}$ is of dimension $2k - 1$, it suffices to exhibit a point which simultaneously satisfies (7.1.1) when $d = 1$ and *strictly* satisfies the inequalities (7.1.2) when $d = 1$; in other words, show that there exists a point in the interior of $\mathcal{P}$. To see that such a point exists, let $\delta > 1/2$ and $0 < \delta' < k$ be real numbers. Let

$$\begin{aligned} b &= (-(1+\delta)/k^2, -(1+2\delta)/k^2, \ldots, -(1+k\delta)/k^2), \\ a &= (-(1+\delta+\delta')/k^2, -(1+2\delta+\delta')/k^2, \ldots, -(1+k\delta+\delta')/k^2). \end{aligned}$$

Then, for $u = 1, \ldots, k-1$ and $v = 1, \ldots, k$,

$$\begin{gathered} b_u - b_{u+1} = \delta/k^2 > 1/2k^2, \\ b_1 = -1/k^2 - \delta/k^2 < -1/2k^2, \\ a_u - a_{u+1} = \delta/k^2 > 0, \\ a_1 = -(1+\delta+\delta')/k^2 < 0, \\ 0 < (b_1 + \cdots + b_k) - (a_1 + \cdots + a_k) = \delta'/k < 1, \\ a_v = b_v - \delta'/k^2 < b_v. \end{gathered}$$

Thus all the inequalities are strictly satisfied for any such δ and δ'. Furthermore,

$$\begin{aligned} (b_1 + \cdots + b_k) - 2(a_1 + \cdots a_k) &= [(1+\delta+2\delta') + (1+2\delta+2\delta') + \cdots + (1+k\delta+2\delta')]/k^2 \\ &= \frac{1}{k}\left(1 + \frac{k+1}{2}\delta + 2\delta'\right). \end{aligned}$$

Since $k \geqslant 2$, one can choose (for example) $\delta = 3/5$ and $\delta' = (7k-13)/20$ so that this expression equals 1. Hence, there is an interior point in the polytope $\mathcal{P}$.

Now, for any integer $d > 0$, let $d\mathcal{P}$ be the dilated polytope

$$d\mathcal{P} = \{(db_1, \ldots, db_k, da_1, \ldots, da_k) \mid (b_1, \ldots, b_k, a_1, \ldots, a_k) \in \mathcal{P}\}.$$

We observe that the integer lattice points of $d\mathcal{P}$ are precisely those which satisfy the conditions of the lemma. Let $L_{\mathcal{P}}(d)$ denote the number of such points within $d\mathcal{P}$, that is, the cardinality of $\tilde{S}(d)$.

As the coefficients of the hyperplanes defining $\mathcal{P}$ are rational, $\mathcal{P}$ is a rational polytope of dimension $2k-1$. By Ehrhart's theorem for rational polytopes (e.g. [BR07, Theorem 3.23]), $L_{\mathcal{P}}(d)$ is given by a quasipolynomial of degree $2k-1$; that is, there exist a fixed positive integer M independent of d and a sequence of polynomials $Q_1(d), \ldots, Q_M(d)$ each of degree $2k-1$ such that when $d \equiv i \pmod{M}$, $L_{\mathcal{P}}(d) = Q_i(d)$. Furthermore, the leading coefficient of each polynomial $Q_1(d), \ldots, Q_M(d)$ is the volume of $\mathcal{P}$.

From this we can construct a single polynomial $Q(d)$ of degree $2k-1$ such that $Q_i(d) \geqslant Q(d)$ for all $d \in \mathbb{Z}_{>0}$; specifically, for $j = 0, \ldots, 2k-1$ we can take the coefficient of d^j in $Q(d)$ to be the minimum among the coefficients of d^j among the polynomials $Q_i(d)$. In particular, note that the leading coefficient of $Q(d)$ will again be the volume of $\mathcal{P}$ and, hence, positive.

Therefore, there exists a polynomial of degree $2k-1$ with positive leading coefficient such that

$$|\tilde{S}(d)| = L_{\mathcal{P}}(d) \geqslant Q(d)$$

for all $d \in \mathbb{Z}_{>0}$. □

Remark 7.1.2. We note that when $k = 1$ the polytope $\mathcal{P}$ degenerates to a single point, $(-1, -1)$. As a consequence, we treat $k = 1$ as a separate case in the following arguments.

7.2 The Su–Zhang bijection

Su and Zhang defined a bijection on highest weights between a block of $\mathcal{F}_{(\mathfrak{gl}(m|n),\mathfrak{gl}(m|n)_0)}$ of atypicality k and the principal block of $\mathcal{F}_{(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}$; see [SZ07]. We now use that bijection to define a set of pairs of highest weights which will provide an effective lower bound for the complexity of a simple module.

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7.2.1 *The $k>1$ case.* If $k>1$, then we can use the set $\tilde{S}(d)$ to define a set of pairs of highest weights in a certain block of $\mathfrak{gl}(m|n)$ as follows. Using Proposition 2.5.1, let us write $\mathcal{B}$ for the block of $\mathfrak{gl}(m|n)$ of atypicality k and with core

$$(\{2m-k, 2m-k-2, \dots, k+2\}, \{2m-k+2, 2m-k+4, \dots, 2m+2n-3k\}).$$

If $k=m$ (respectively, $k=n$), then we intend for the left-hand (respectively, right-hand) set in the core to be empty. In particular, in the case of $m=n=k$ we take $\mathcal{B}$ to be the principal block.

Set $p:=m-k$ and $q:=2m-2k$. Define an injective map $\zeta:\mathbb{R}^k \to \bigoplus_{i=1}^{m+n}\mathbb{R}\varepsilon_i$ by

$$\begin{aligned}\zeta(x_1,\dots,x_k) = p\varepsilon_1 + (p-1)\varepsilon_2 + \cdots + \varepsilon_{m-k}\\ + x_1\varepsilon_{m-k+1} + \cdots + x_k\varepsilon_m\\ - x_k\varepsilon_{m+1} - \cdots - x_1\varepsilon_{m+k}\\ - (q+1)\varepsilon_{m+k+1} - (q+2)\varepsilon_{m+k+2} - \cdots - (q+n-k)\varepsilon_{m+n}.\end{aligned}$$

For example, in what follows we will frequently refer to the special weight

$$\nu := \zeta(0,\dots,0) = p\varepsilon_1 + \cdots + \varepsilon_{m-k} - (q+1)\varepsilon_{m+k+1} - \cdots - (q+n-k)\varepsilon_{m+n}. \tag{7.2.1}$$

Then $\nu \in X_0^+$ and $L(\nu)$ is a simple $\mathfrak{gl}(m|n)$-module in $\mathcal{B}$. More generally, observe that if $\zeta(x_1,\dots,x_k)\in X_0^+$, then $L(\zeta(x_1,\dots,x_k))$ lies in the block $\mathcal{B}$.

Define

$$\omega:\mathbb{R}^{2k} \to \bigoplus_{i=1}^{m+n}\mathbb{R}\varepsilon_i \times \bigoplus_{i=1}^{m+n}\mathbb{R}\varepsilon_i$$

by

$$\omega(x_1,\dots,x_k,y_1,\dots,y_k) = (\zeta(x_1,\dots,x_k), \zeta(y_1,\dots,y_k)).$$

Then ω is clearly injective.

We define

$$S(d) = \omega(\tilde{S}(d)),$$

where $\tilde{S}(d)$ is the subset of $\mathbb{Z}^{2k}$ defined in Lemma 7.1.1. Then $S(d)\subset \mathcal{B}\times\mathcal{B}$ and, since ω is injective, the cardinality of $S(d)$ equals the cardinality of $\tilde{S}(d)$.

We now introduce the bijection on highest weights defined in [SZ07, Theorem 3.29],

$$\phi:\mathcal{B}\to\mathcal{B}_{0,k|k}, \tag{7.2.2}$$

where $\mathcal{B}_{0,k|k}$ denotes the principal block of $\mathfrak{gl}(k|k)$. The interested reader can find the full definition in [SZ07]. However, we only require the value of this map on elements of $\mathcal{B}$ which appear in an element of $S(d)$. For our purposes, it suffices to note that for any $\zeta(x_1,\dots,x_k)\in\mathcal{B}$ we have

$$\phi(\zeta(x_1,\dots,x_k)) = x_1\varepsilon_1 + \cdots + x_k\varepsilon_k - x_k\varepsilon_k - \cdots - x_k\varepsilon_{2k}.$$

In particular, observe that for the weight ν defined in (7.2.1),

$$\phi(\nu)=0.$$

It is easy to see from [SZ07, (3.13)] or by using the definition of the Bruhat order in [Bru03] that for all $\zeta(x_1,\dots,x_k)$ and $\zeta(y_1,\dots,y_k)$ which lie in $\mathcal{B}$, we have

$$\zeta(x_1,\dots,x_k)\preccurlyeq\zeta(y_1,\dots,y_k) \text{ if and only if } \phi(\zeta(x_1,\dots,x_k))\preccurlyeq\phi(\zeta(y_1,\dots,y_k)) \tag{7.2.3}$$

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and

$$l(\zeta(x_1,\ldots,x_k)) = l(\phi(\zeta(x_1,\ldots,x_k))) = x_1 + \cdots + x_k. \tag{7.2.4}$$

Now, since $S(d) \subseteq \mathcal{B} \times \mathcal{B}$, we use (7.1.2), (7.2.1), (7.2.3) and (7.2.4) to see that the elements $(\mu,\sigma) \in S(d)$ satisfy

$$\begin{gathered} \sigma \preccurlyeq \mu, \\ \sigma \preccurlyeq \nu, \\ 0 \leqslant l(\mu) - l(\sigma) \leqslant d; \end{gathered} \tag{7.2.5}$$

$$-l(\sigma) = \frac{d - l(\mu)}{2}; \tag{7.2.6}$$

$$\begin{gathered} -\mu_{m-k+1} \geqslant d/2k^2, \\ \mu_i - \mu_{i+1} \geqslant d/2k^2 \end{gathered} \tag{7.2.7}$$

for $i = m-k+1,\ldots,m-1$.

7.2.2 *The $k=1$ case.* We now consider the case where $k=1$. Let $\mathcal{B}$ denote the block of $\mathfrak{gl}(m|n)$ of atypicality one and with core

$$(\{2m-2, 2m-4, \ldots, 2\}, \{2m, 2m+2, \ldots, 2m+2n-4\}).$$

Let p and q be as defined near the beginning of § 7.2.1, so that we have $p = m-1$ and $q = 2m-2$. For $d > 6(m+n)$ and $2d/3 < a \leqslant d$, let $b = a + m - n$ and set

$$\mu^{(a)} = a\varepsilon_1 + p\varepsilon_2 + \cdots + \varepsilon_m - (q+1)\varepsilon_{m+1} - (q+2)\varepsilon_{m+2} - \cdots - (q+n-1)\varepsilon_{m+n-1} - b\varepsilon_{m+n}.$$

For $d > 6(m+n)$ we then set

$$S(d) = \{(\mu^{(a)}, \mu^{(a)}) \mid a \in \mathbb{Z},\ 2d/3 < a \leqslant d\}.$$

Note that by our assumption on d and the fact that $b \geqslant a > 2d/3$, we have $S(d) \subset \mathcal{B} \times \mathcal{B}$.

We will need the value of the Su–Zhang bijection $\phi : \mathcal{B} \to \mathcal{B}_{0,1|1}$ on weights of the form $\mu^{(a)}$. From the definition of ϕ it is easy to see that

$$\phi(\mu^{(a)}) = (a-n+1)\varepsilon_1 - (a-n+1)\varepsilon_2.$$

Finally, set $\nu \in \mathcal{B}$ to be the highest weight given by

$$\nu = \phi^{-1}(0). \tag{7.2.8}$$

The interested reader who wishes to compute ν will need to refer to the definition of ϕ given in [SZ07]. However, for our purposes all we require is that it goes to 0 under the bijection.

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7.2.3 Finally, we record a crucial property of the Su–Zhang bijection.

Lemma 7.2.1. *Let $\phi : \mathcal{B} \to \mathcal{B}_{0,k|k}$ be the Su–Zhang bijection. Then for all $d \geqslant 0$ and all $\lambda, \mu \in \mathcal{B}$ we have*

$$\dim \operatorname{Ext}^d_{\mathcal{F}(\mathfrak{gl}(m|n),\mathfrak{gl}(m|n)_0)}(K(\lambda), L(\mu)) = \dim \operatorname{Ext}^d_{\mathcal{F}(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}(K(\phi(\lambda)), L(\phi(\mu))),$$
$$\dim \operatorname{Ext}^d_{\mathcal{F}(\mathfrak{gl}(m|n),\mathfrak{gl}(m|n)_0)}(L(\lambda), L(\mu)) = \dim \operatorname{Ext}^d_{\mathcal{F}(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}(L(\phi(\lambda)), L(\phi(\mu))).$$

Proof. Both results follow from [Bru03, Corollary 4.52] and [SZ07, Theorem 3.29(2)]. □

7.3 A lower bound on dimensions of projectives

Lemma 7.3.1. *Let $\mathcal{B}$ be the block given in the previous section and let $(\mu, \sigma) \in S(d) \subset \mathcal{B} \times \mathcal{B}$. Then, for d sufficiently large,*

$$\dim P(\mu) \geqslant C d^{(m+n-k-1)k}$$

where C is a positive constant which is independent of μ and σ.

Proof. By (5.1.1) it suffices to use the Weyl dimension formula to obtain a lower bound on the dimension of $L_0(\mu)$:

$$\dim L_0(\mu) = \prod_{\alpha \in \Phi_m^+} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\alpha \in \Phi_n^+} \frac{(\mu + \rho_n, \alpha)}{(\rho_n, \alpha)}. \tag{7.3.1}$$

We first consider the case where $k = 1$ (and so $p = m - 1$, $q = 2m - 2$ and $d > 6(m + n)$). From (7.3.1) and the definition of $\mu = \mu^{(a)}$ we have

$$\begin{aligned}\dim L_0(\mu) &\geqslant \prod_{\substack{\alpha=\varepsilon_1-\varepsilon_t\\ t=2,\dots,m}} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\substack{\alpha=\varepsilon_t-\varepsilon_{m+n}\\ t=m+1,\dots,m+n-1}} \frac{(\mu + \rho_n, \alpha)}{(\rho_n, \alpha)}\\ &= \prod_{\substack{\alpha=\varepsilon_1-\varepsilon_t\\ t=2,\dots,m}} \frac{a - (p - t + 2) + (\rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\substack{\alpha=\varepsilon_t-\varepsilon_{m+n}\\ t=m+1,\dots,m+n-1}} \frac{b - (q + t - m) + (\rho_n, \alpha)}{(\rho_n, \alpha)}.\end{aligned}$$

However, since $a > 2d/3$ and $d/3 \geqslant p$, we have

$$a - (p - t + 2) \geqslant a - p > 2d/3 - d/3 = d/3.$$

Similarly, since $b > 2d/3$ and $d/3 \geqslant q + n - 1$, we have

$$b - (q + t - m) \geqslant b - (q + n - 1) > 2d/3 - d/3 = d/3.$$

Substituting yields

$$\begin{aligned}\dim L_0(\mu) &\geqslant \prod_{\substack{\alpha=\varepsilon_1-\varepsilon_t\\ t=2,\dots,m}} \frac{d/3 + (\rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\substack{\alpha=\varepsilon_t-\varepsilon_{m+n}\\ t=m+1,\dots,m+n-1}} \frac{d/3 + (\rho_n, \alpha)}{(\rho_n, \alpha)}\\ &\geqslant C d^{m-1} d^{n-1} = C d^{m+n-2},\end{aligned}$$

where C is a constant independent of μ and σ. This proves the desired result when $k = 1$.

We now consider the case of $k > 1$. We begin by studying the first factor in (7.3.1). We have

$$\prod_{\alpha \in \Phi_m^+} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} = \prod_{\alpha \in A_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\alpha \in B_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \prod_{\alpha \in C_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)}, \tag{7.3.2}$$

where A_m, B_m and C_m are as defined in § 2.5.

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If we let $Z = \max\{(\rho_m, \alpha) \mid \alpha \in \Phi_m^+\}$, then we have

$$\prod_{\alpha \in A_m} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \geqslant \prod_{\alpha \in A_m} \frac{(\mu + \rho_m, \alpha)}{Z} =: C_0.$$

But by the definition of A_m, the value of C_0 depends only on the core of μ and not on μ itself. Hence it depends only on m, n and $\mathcal{B}$.

For $(\mu, \sigma) \in S(d)$ we have

$$(\mu + \rho_m, \alpha) \geqslant (\mu, \alpha) \geqslant d/2k^2$$

for all $\alpha \in B_m$ and $\alpha \in C_m$. Taken together with the fact that B_m has cardinality $(m-k)k$ and C_m has cardinality $k(k-1)/2$, we see that

$$\prod_{\alpha \in \Phi_m^+} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \geqslant C_1 d^{(m-k)k + k(k-1)/2}$$

for some constant C_1 which is independent of μ and σ. Similarly,

$$\prod_{\alpha \in \Phi_n^+} \frac{(\mu + \rho_m, \alpha)}{(\rho_m, \alpha)} \geqslant C_2 d^{(n-k)k + k(k-1)/2}.$$

Upon combining these, we see that

$$\dim P(\mu) \geqslant C_1 C_2 d^{(m-k)k + k(k-1)/2 + (n-k)k + k(k-1)/2} = C d^{(m+n-k-1)k}$$

for some constant C which is independent of μ and σ, as desired. □

8. Complexity for simple modules

8.1 We first observe that for a fixed $\mathfrak{gl}(m|n)$, any two simple modules with the same atypicality have the same complexity.

THEOREM 8.1.1. *Let $L(\lambda)$ and $L(\mu)$ be two simple modules for $\mathfrak{gl}(m|n)$ with* $\operatorname{atyp}(\lambda) = \operatorname{atyp}(\mu)$. *Then*

$$L(\lambda)^* \otimes L(\lambda) \otimes L(\mu) \cong L(\mu) \oplus U \tag{8.1.1}$$

for some $\mathfrak{gl}(m|n)$-module U. Furthermore, the complexity of $L(\lambda)$ equals the complexity of $L(\mu)$.

Proof. Let $L(\lambda)$ and $L(\mu)$ be two simple modules with the same atypicality. Let $P_\bullet \to L(\lambda)$ be a minimal projective resolution of $L(\lambda)$. Upon tensoring this resolution on the left by $L(\lambda)^*$ and on the right by $L(\mu)$ we obtain a (not necessarily minimal) projective resolution of $L(\lambda)^* \otimes L(\lambda) \otimes L(\mu)$ with rate of growth equal to the rate of growth of $P_\bullet$. Therefore, we deduce that

$$c_{\mathcal{F}}(L(\lambda)) \geqslant c_{\mathcal{F}}(L(\lambda)^* \otimes L(\lambda) \otimes L(\mu)). \tag{8.1.2}$$

By [Ser10, Corollary 6.6], every simple $\mathfrak{gl}(m|n)$-module admits an ambidextrous trace in the sense of [GKP11]. In particular, $L(\mu)$ has an ambidextrous trace and, by [GKP11, Theorem 3.3.2], this trace defines a modified dimension function, $\mathsf{d}_{L(\mu)}$, on the ideal of $\mathcal{F}$ generated by $L(\mu)$. By the generalized Kac–Wakimoto conjecture, stated for basic classical Lie superalgebras in [GKP11, Conjecture 6.3.2] and proven for $\mathfrak{gl}(m|n)$ in [Ser10, Corollary 6.7], it follows that $L(\lambda)$ is in the ideal generated by $L(\mu)$ and that $\mathsf{d}_{L(\mu)}(L(\lambda)) \neq 0$. However, by [GKP11, Corollary 4.3.3], this implies that the canonical surjection induced by the evaluation map,

$$L(\lambda)^* \otimes L(\lambda) \otimes L(\mu) \to L(\mu),$$

splits. In short, because $L(\lambda)$ and $L(\mu)$ have the same atypicality, we have

$$L(\lambda)^* \otimes L(\lambda) \otimes L(\mu) \cong L(\mu) \oplus U$$

for some $\mathfrak{gl}(m|n)$-module U (where the isomorphism preserves the $\mathbb{Z}_2$-grading). Using Proposition 2.2.1 and the additivity of Ext, we then see that

$$c_{\mathcal{F}}(L(\lambda)^* \otimes L(\lambda) \otimes L(\mu)) \geqslant c_{\mathcal{F}}(L(\mu)). \tag{8.1.3}$$

Combining (8.1.2) and (8.1.3), we obtain

$$c_{\mathcal{F}}(L(\lambda)) \geqslant c_{\mathcal{F}}(L(\mu)).$$

The argument is symmetric under switching $L(\lambda)$ and $L(\mu)$, and so we have equality. □

8.2 We now compute the complexity of a simple $\mathfrak{gl}(m|n)$-module of atypicality k. The end result will be the following theorem.

Theorem 8.2.1. *Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality k. Then*

$$c_{\mathcal{F}}(L(\lambda)) = \dim \overline{(\mathfrak{g}_1)_k} + k = (m+n)k - k^2 + k = \dim \mathcal{X}_{L(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)).$$

By Theorem 8.1.1 it suffices to compute complexity for the simple module $L(\nu)$ in the block $\mathcal{B}$ given in § 7.2; recall the definition of ν in (7.2.1) for $k>1$ and (7.2.8) for $k=1$. The second equality of Theorem 8.2.1 follows from the well-known formula for the dimension of the variety $\overline{(\mathfrak{g}_1)_k}$ (see [Har92, Proposition 12.2]). The third equality follows from [BKN09, Theorem 4.8.1] and [DS05, Theorems 4.5 and 5.4]. Thus we focus on computing the complexity of $L(\nu)$ by computing sharp upper and lower bounds for the expression given by Proposition 2.2.1. In particular, we see that combining Propositions 8.3.2 and 8.4.1 (below) proves the first equality of Theorem 8.2.1.

8.3 The upper bound

We first prove an intermediate result.

Lemma 8.3.1. *Let $L(\nu)$ and $\mathcal{B}$ be as above. Then*

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\left(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)\right) \leqslant Dd^{2k-1},$$

where D is a positive constant.

Proof. By our choice of ν, we may apply Lemma 7.2.1 and assume without loss of generality that $\mathfrak{gl}(m|n) = \mathfrak{gl}(k|k)$, $L(\nu) = \mathbb{C}$, and $\mathcal{B} = \mathcal{B}_0$ is the principal block of $\mathcal{F}$. We first analyze an individual term in the direct sum. Fix $\mu \in \mathcal{B}_0$. We have

$$\begin{aligned}\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}, L(\mu)) &= \sum_{i+j=d}\ \sum_{\sigma\in\mathcal{B}_0} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma),\mathbb{C}) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu))\\ &= \sum_{i+j=d}\ \sum_{\sigma\in\mathcal{B}_0} \dim \operatorname{Ext}^i_{(\mathfrak{p}^+,\mathfrak{g}_0)}(L_0(\sigma),\mathbb{C}) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu))\\ &= \sum_{i+j=d}\ \sum_{\sigma\in\mathcal{B}_0} \dim \operatorname{Hom}_{\mathfrak{g}_0}(L_0(\sigma), S^i(\mathfrak{g}_1^*)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)), \quad (8.3.1)\end{aligned}$$

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where the first line is by [Bru03, Theorem 4.51 and Corollary 4.52], the second line is by Frobenius reciprocity, and the third line comes from using a spectral sequence argument (cf. [BKN11, (3.3.2)]).

Suppose that there is a nonzero term in the last sum. Then, by [Sch69], the Hom-space is one-dimensional and $\sigma \preceq 0$ with $i = -l(\sigma)$ (recalling that in the principal block of $\mathfrak{gl}(k|k)$ we have $l(\sigma) = |\sigma|$ and the alternate description of the Bruhat order). Also, $\sigma \preceq \mu$ and $j = l(\mu) - l(\sigma) - b$ where $b \in \{0, 1, \ldots, \dim \mathfrak{g}_{-1} = k^2\}$ (cf. Theorem 4.2.1 and [Bru03, Theorem 4.5.1]). In particular,

$$l(\mu) \geqslant l(\sigma) = -i \geqslant -d. \tag{8.3.2}$$

Also,

$$l(\mu) + i - k^2 = l(\mu) - l(\sigma) - k^2 \leqslant j = d - i \leqslant l(\mu) - l(\sigma) = l(\mu) + i,$$

which in turn implies that

$$\frac{d - l(\mu)}{2} \leqslant i \leqslant \frac{d + k^2 - l(\mu)}{2}. \tag{8.3.3}$$

But the last inequality, together with $i \geqslant 0$, implies $l(\mu) \leqslant d + k^2$. Combining this with (8.3.2), we obtain

$$-d \leqslant l(\mu) \leqslant d + k^2. \tag{8.3.4}$$

We will also require another estimate on the entries of μ. From the conditions $\sigma \preceq 0$, $\sigma \preceq \mu$, $i = l(0) - l(\sigma)$, $j \geqslant l(\mu) - l(\sigma) - k^2$ and $i + j = d$ we deduce that

$$[l(0) - l(\sigma)] + [l(\mu) - l(\sigma)] \leqslant d + k^2. \tag{8.3.5}$$

There is a greatest element σ^0 of X_0^+ dominated by both 0 and μ in the Bruhat order, with coordinates defined by $\sigma_j^0 = \min(0, \mu_j)$, $1 \leqslant j \leqslant k$. Note that $[l(0) - l(\sigma^0)] + [l(\mu) - l(\sigma^0)] = \sum_{j=1}^k |\mu_j|$. Since $\sigma \preceq \sigma^0$, (8.3.5) implies that

$$\sum_{j=1}^{k} |\mu_j| \leqslant d + k^2. \tag{8.3.6}$$

So we may assume that μ satisfies (8.3.4) and (8.3.6). Taking into account (8.3.3) and the fact that $\dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu))$ is the coefficient of a Kazhdan–Lusztig polynomial, which by Theorem 4.2.1 is bounded by a constant C_0, we have

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}, L(\mu)) \leqslant C_0 \sum_i \#\{0 \succeq \sigma \in X_0^+ \mid l(\sigma) = -i\},$$

where the sum is over $\max(0, (d - l(\mu))/2) \leqslant i \leqslant \min(d, (d + k^2 - l(\mu))/2)$. Now, $0 \succeq \sigma \in X_0^+$ and $l(\sigma) = -i$ means that $-\sigma$ is a partition of i into at most k (positive) parts; and there is a constant C_1 (depending only on k) such that the number of such partitions is at most $C_1 i^{k-1}$ (see [Nat00, Corollary 15.1]). Thus

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}, L(\mu)) \leqslant C_0 C_1 \sum_i i^{k-1}, \tag{8.3.7}$$

with the same conditions on i as before. Assuming without loss of generality that $2d \geqslant k^2$, the last expression is maximized when $l(\mu) = -d + k^2$, giving

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}, L(\mu)) \leqslant C_0 C_1 \sum_{d - k^2/2 \leqslant i \leqslant d} i^{k-1} \leqslant C_2 d^{k-1}$$

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for some constant C_2 depending only on k. Alternatively, to bound (8.3.7) it suffices to note that for d sufficiently large, the number of terms in the sum is bounded by a constant independent of both d and μ, and that $i \leqslant (d+k^2-l(\mu))/2 \leqslant d+k^2/2$ (where (8.3.4) is used to obtain the second inequality).

Lastly, in order for (8.3.1) to be nonzero, the condition (8.3.6) certainly implies that each $|\mu_j| \leqslant d+k^2$ for $1 \leqslant j \leqslant k$, so the total number of such μ in $\mathcal{B}_0$ is bounded by $C_3 d^k$ for some constant C_3 depending only on k.

Putting the ingredients together, we have

$$\begin{aligned}\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(\mathbb{C}, \bigoplus_{\mu\in\mathcal{B}_0} L(\mu)\Big) &= \sum_{\mu\in\mathcal{B}_0} \dim \operatorname{Ext}^d_{\mathcal{F}}(\mathbb{C}, L(\mu)) \\ &\leqslant \sum_{\substack{\mu\in\mathcal{B}_0,\\ \sum|\mu_k|\leqslant d+k^2}} C_2 d^{k-1} \\ &\leqslant C_3 C_2 d^k d^{k-1} \\ &= D d^{2k-1}. \end{aligned}$$ □

We now prove an upper bound for the complexity of $L(\nu)$.

Proposition 8.3.2. *Let $L(\nu)$ and $\mathcal{B}$ be as above. Then for all d we have*

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{\dim P(\mu)}\Big) \leqslant K d^{(m+n-k+1)k-1},$$

where K is a positive constant.

Proof. Recall that if $P_\bullet \to L(\nu)$ is a minimal projective resolution, then since

$$\operatorname{Ext}^d_{\mathcal{F}}(L(\nu), L(\mu)) \cong \operatorname{Hom}_{\mathcal{F}}(P_d, L(\mu)),$$

we have that this vector space is nonzero if and only if $P(\mu)$ is a direct summand of P_d. Therefore, combining Lemma 8.3.1 with Theorem 5.2.1, we obtain

$$\begin{aligned}\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{P(\mu)}\Big) &\leqslant C d^{(m+n-k-1)k} \dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)\Big) \\ &\leqslant CD d^{(m+n-k-1)k} d^{2k-1} \\ &= K d^{(m+n-k+1)k-1}. \end{aligned}$$ □

8.4 The lower bound

We now compute a lower bound for the complexity of $L(\nu)$.

Proposition 8.4.1. *Let $L(\nu)$ and $\mathcal{B}$ be as above. Then as a function of d for all d sufficiently large,*

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{\dim P(\mu)}\Big)$$

is bounded below by a polynomial of degree $(m+n-k+1)k-1$ with positive leading coefficient.

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Proof. We first consider the case of $k=1$. Let $P_\bullet \to \mathbb{C}$ be the minimal projective resolution of $\mathbb{C}$ as a $\mathfrak{gl}(1|1)$-module given in [BKN11]. Combining the basic properties of a minimal projective resolution with Lemmas 7.2.1 and 7.3.1, we obtain

$$\begin{aligned}
&\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{\dim P(\mu)}\Big)\\
&\quad\geqslant \sum_{(\mu,\mu)\in S(d)} \dim P(\mu) \dim \operatorname{Ext}^d_{\mathcal{F}}(L(\nu), L(\mu))\\
&\quad= \sum_{(\mu,\mu)\in S(d)} \dim P(\mu) \dim \operatorname{Ext}^d_{\mathcal{F}_{(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}}(L(\phi(\nu)), L(\phi(\mu)))\\
&\quad\geqslant C d^{(m+n-k)k-k} \sum_{(\mu,\mu)\in S(d)} \dim \operatorname{Ext}^d_{\mathcal{F}_{(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}}(\mathbb{C}, L(\phi(\mu)))\\
&\quad= C d^{(m+n-k)k-k} \sum_{(\mu,\mu)\in S(d)} \dim \operatorname{Hom}_{\mathcal{F}_{(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}}(P_d, L(\phi(\mu))).
\end{aligned}$$

From the construction of P_d and $S(d)$, we can use § 7.2.2 to see that this Hom-space is nonzero for each $\mu=\mu^{(a)}$ when $a-n+1$ has the same parity as d. Therefore, the total dimension of the Hom-spaces given above is bounded below by a linear function in $|S(d)|$ which, in turn, is bounded below by a linear function in d. This implies the desired result for $k=1$.

We now consider the case where $k>1$. As in (8.3.1), we have

$$\begin{aligned}
&\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{\dim P(\mu)}\Big)\\
&\quad= \sum_{i+j=d} \sum_{\mu,\sigma\in\mathcal{B}} \dim P(\mu) \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\nu)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu))\\
&\quad\geqslant \sum_{i+j=d} \sum_{(\mu,\sigma)\in S(d)} \dim P(\mu) \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\nu)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)). \qquad (8.4.1)
\end{aligned}$$

Each $(\mu,\sigma)\in S(d)$ satisfies the conditions (7.2.5), so by [Bru03, Theorem 4.51] we have that

$$\dim \operatorname{Ext}^{l(\mu)-l(\sigma)}_{\mathcal{F}}(K(\sigma), L(\mu)) = 1.$$

By Lemma 7.2.1 and the argument used in (8.3.1), we have

$$\begin{aligned}
\dim \operatorname{Ext}^{d-l(\mu)+l(\sigma)}_{\mathcal{F}}(K(\sigma), L(\nu)) &= \dim \operatorname{Ext}^{d-l(\mu)+l(\sigma)}_{\mathcal{F}_{(\mathfrak{gl}(k|k),\mathfrak{gl}(k|k)_0)}}(K(\phi(\sigma)), \mathbb{C})\\
&= \dim \operatorname{Hom}_{\mathfrak{gl}(k|k)_0}(L_0(\phi(\sigma)), S^{d-l(\mu)+l(\sigma)}(\mathfrak{g}_1^*)). \qquad (8.4.2)
\end{aligned}$$

The element $(\mu,\sigma)\in S(d)$ also satisfies (7.2.6), so upon using (7.2.3) and (7.2.4) to translate (7.2.6) to the analogous conditions on $(\phi(\mu),\phi(\sigma))$ we may apply [Sch69] to (8.4.2) and obtain

$$\dim \operatorname{Ext}^{d-l(\mu)+l(\sigma)}_{\mathcal{F}}(K(\sigma), L(\nu)) = 1.$$

Taken together, we see that

$$\sum_{i+j=d} \sum_{(\mu,\sigma)\in S(d)} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\nu)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \geqslant |S(d)|.$$

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Applying this along with Lemma 7.3.1 to (8.4.1) yields

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(L(\nu), \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{\dim P(\mu)}\Big) \geqslant Cd^{(m+n-k)k-k}|S(d)| \geqslant Cd^{(m+n-k)k-k}Q(d).$$

The last inequality follows from the fact that the cardinality of $S(d)$ equals the cardinality of $\tilde{S}(d)$ and so is bounded below by a polynomial, $Q(d)$, of degree $2k-1$ with positive leading coefficient, by Lemma 7.1.1. This proves the desired result when $k>1$. □

9. A categorical invariant

9.1 We will first assume that $\mathfrak{g}$ is a classical Lie superalgebra and M is a module in $\mathcal{F}=\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. It is natural to consider

$$z_{\mathcal{F}}(M) = r\Big(\operatorname{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}\Big(M, \bigoplus S\Big)\Big),$$

where the direct sum runs over all simple modules of $\mathcal{F}$. Unlike complexity, $z_{\mathcal{F}}(-)$ has the advantage of being invariant under category equivalences.

Using our complexity calculations, we compute this invariant for the Kac, dual Kac and simple modules of $\mathfrak{gl}(m|n)$.

Theorem 9.1.1. *Let $\mathfrak{g}=\mathfrak{gl}(m|n)$ and let $X(\lambda)$ (respectively, $L(\lambda)$) be a Kac or dual Kac (respectively, simple) module of atypicality k in $\mathcal{F}$. Then*

$$z_{\mathcal{F}}(X(\lambda)) = k$$

and

$$z_{\mathcal{F}}(L(\lambda)) = 2k.$$

Proof. We consider $K(\lambda)$; the proof for $K^-(\lambda)$ is the same. By (6.3.7) there is a positive constant K such that for all $d\geqslant 1$,

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}\Big(K(\lambda), \bigoplus S\Big) \leqslant Kd^{k-1}.$$

On the other hand, if the left-hand side could be bounded above by $K'd^{k-2}$ for some positive constant K', then this would imply by the proof of Theorem 6.3.1 that the complexity of $K(\lambda)$ is strictly less than $(m+n)k-k^2$, contradicting the conclusion of that proof. Therefore, the power $k-1$ is sharp and $z_{\mathcal{F}}(K(\lambda))=k$.

An identical argument applies to $L(\lambda)$, using Lemma 8.3.1. □

9.2 Recalling that we assume $n\leqslant m$, we set $\mathfrak{f}_{\bar{1}}\subset\mathfrak{g}_{\bar{1}}$ to be the span of the matrix units $E_{m-t+1,m+t}$ and $E_{m+t,m-t+1}$ for $t=1,\dots,n$. Set $\mathfrak{f}_0=\mathfrak{f}_{\bar{0}}=[\mathfrak{f}_{\bar{1}},\mathfrak{f}_{\bar{1}}]$. We then define a subalgebra of $\mathfrak{g}$ by

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}}\oplus\mathfrak{f}_{\bar{1}}.$$

The Lie superalgebra $\mathfrak{f}$ is classical (and of type I) and so has a support variety theory. Furthermore, as $[\mathfrak{f}_0,\mathfrak{f}_{\bar{1}}]=0$, it follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$.

The subalgebra $\mathfrak{f}$ is a 'detecting' Lie subsuperalgebra of $\mathfrak{g}$, which first appeared in [BKN10] and can be seen to have a remarkable cohomological detection property due to work of Lehrer, Nakano and Zhang [LNZ10] (where it is called $\tilde{\mathfrak{f}}$). We now show that these detecting subalgebras naturally capture the above categorical invariant for Kac, dual Kac and simple modules.

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THEOREM 9.2.1. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and let $X(\lambda)$ denote a Kac, dual Kac, or simple module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$. Then*

$$z_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda)) = c_{\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}}(X(\lambda)).$$

Proof. The second equality is immediate from [BKN09, Proposition 5.2.2] and [BKN11, Theorem 2.9.1(c)]. We now consider the first equality.

We first obtain an upper bound on $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda))$. Let $y \in \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda))$, and choose $y_{-1} \in \mathfrak{f}_{\bar{1}} \cap \mathfrak{g}_{-1}$ and $y_1 \in \mathfrak{f}_{\bar{1}} \cap \mathfrak{g}_1$ so that $y = y_{-1} + y_1$. Since $X(\lambda)$ inherits a $\mathbb{Z}$-grading from the $\mathbb{Z}$-grading on $\mathfrak{g}$, the argument using the $\mathbb{Z}$-action on $X(\lambda)$ in the proof of Theorem 6.4.1 shows that $y_{-1}, y_1 \in \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda))$. From this we conclude that

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda)) \subseteq (\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{-1}}(X(\lambda))) \times (\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_1}(X(\lambda))). \tag{9.2.1}$$

If $X(\lambda)$ is a Kac module, we can use Theorem 6.4.1 and [BKN11, Theorem 3.3.1] to deduce that $\dim(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_1}(X(\lambda))) = k$ and $\dim(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{-1}}(X(\lambda))) = 0$ and, hence, that the dimension of the right-hand variety in (9.2.1) is k. The dual Kac module is handled similarly by using Theorem 6.4.1 and [BKN11, Theorem 3.3.2].

If $X(\lambda)$ is a simple module, then we use [BKN11, (3.8.1)] (which ultimately depends on calculations in [DS05]) to deduce that $\dim(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_i}(X(\lambda))) = k$ for $i = -1, 1$, so that the dimension of the right-hand variety in (9.2.1) is $2k$.

We next obtain a lower bound. From the rank variety description it is clear that

$$(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{-1}}(X(\lambda))) \cup (\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_1}(X(\lambda))) \subseteq \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda)). \tag{9.2.2}$$

If $X(\lambda)$ is a Kac or dual Kac module, then by the above calculations the left-hand variety is k-dimensional. Therefore, the dimension of $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(X(\lambda))$ equals k, and applying Theorem 9.1.1 proves the theorem for Kac and dual Kac modules.

Now, if $X(\lambda) = L(\lambda)$ is a simple module and $L(\gamma)$ is another simple $\mathfrak{gl}(m|n)$-module of atypicality k, then by (8.1.1) and the basic properties of rank varieties [BKN10, Proposition 6.3.1 and Theorem 6.4.2] we have

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(L(\gamma)).$$

In particular, applying this to the Gruson–Serganova block equivalences introduced in § 6.3, we see that we may assume without loss that $\mathrm{Res}_\mu L(\lambda)$ is isomorphic to the trivial module for the subalgebra $\mathfrak{g}' \cong \mathfrak{gl}(k|k)$. If we set $\mathfrak{f}' = \mathfrak{f} \cap \mathfrak{g}'$, then $\mathfrak{f}'$ is the corresponding detecting subalgebra for $\mathfrak{g}'$. By the rank variety description, it is immediate that

$$\mathfrak{f}'_{\bar{1}} = \mathcal{V}_{(\mathfrak{f}',\mathfrak{f}'_0)}(\mathrm{Res}_\mu L(\lambda)) \subseteq \mathcal{V}_{(\mathfrak{f}',\mathfrak{f}'_0)}(L(\lambda)) \subseteq \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(L(\lambda)).$$

Thus, the dimension of $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(L(\lambda))$ is at least $2k$ and, hence, equals $2k$. Combining this with Theorem 9.1.1 proves the theorem for simple modules. □

9.3 The $\mathfrak{gl}(1|1)$ case

We now show that the $\mathfrak{f}$ support varieties capture the invariant defined in the previous section and the complexity for an arbitrary $\mathfrak{gl}(1|1)$-module.

THEOREM 9.3.1. *Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $M \in \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$. Then*

$$z_{\mathcal{F}}(M) = c_{\mathcal{F}}(M) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(M).$$

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Proof. We may reduce to the case where M is in the principal block $\mathcal{B}_0$ of $\mathcal{F}$; otherwise M will be projective. The simple modules of the principal block, $L(\lambda) := (\lambda\epsilon_1 - \lambda\epsilon_2)$, are one-dimensional, where $\lambda \in \mathbb{Z}$; the projective cover, $P(\lambda)$, of $L(\lambda)$ is four-dimensional. Therefore,

$$z_{\mathcal{F}}(M) = r\left(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_0)}\left(M, \bigoplus_{\lambda\in\mathbb{Z}} L(\lambda)\right)\right) = r\left(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_0)}\left(M, \bigoplus_{\lambda\in\mathbb{Z}} L(\lambda)^{\dim P(\lambda)}\right)\right) = c_{\mathcal{F}}(M).$$

Next, observe that $\mathfrak{f} \cong \mathfrak{sl}(1|1)$, which is an ideal in $\mathfrak{gl}(1|1)$ with $\mathfrak{gl}(1|1)/\mathfrak{sl}(1|1) \cong \mathfrak{h}$ where $\mathfrak{h}$ is the one-dimensional subalgebra spanned by the 2×2 diagonal matrix diag$\{1, -1\}$. One can now consider the spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{(\mathfrak{h},\mathfrak{h})}(\mathbb{C}, \mathrm{Ext}^j_{(\mathfrak{f},\mathfrak{f}_0)}(M, \oplus_{\lambda\in\mathbb{Z}} L(\lambda)^{\oplus 4})) \Rightarrow \mathrm{Ext}^{i+j}_{(\mathfrak{g},\mathfrak{g}_0)}(M, \oplus_{\lambda\in\mathbb{Z}} L(\lambda)^{\oplus 4}).$$

This spectral sequence collapses and yields the following isomorphisms:

$$\begin{aligned}\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_0)}(M, \oplus_{\lambda\in\mathbb{Z}} L(\lambda)^{\oplus 4}) &\cong \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}, \mathrm{Ext}^{\bullet}_{(\mathfrak{f},\mathfrak{f}_0)}(M, \oplus_{\lambda\in\mathbb{Z}} L(\lambda)^{\oplus 4}))\\ &\cong \oplus_{\lambda\in\mathbb{Z}}\ \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}, \mathrm{Ext}^{\bullet}_{(\mathfrak{f},\mathfrak{f}_0)}(M, \mathbb{C}^{\oplus 4}) \otimes (2\lambda))\\ &\cong \oplus_{\lambda\in\mathbb{Z}}\ \ \mathrm{Ext}^{\bullet}_{(\mathfrak{f},\mathfrak{f}_0)}(M, \mathbb{C}^{\oplus 4})_{-2\lambda}\\ &\subseteq \mathrm{Ext}^{\bullet}_{(\mathfrak{f},\mathfrak{f}_0)}(M, \mathbb{C}^{\oplus 4}).\end{aligned}$$

The subscript -2λ on the third line indicates the -2λ weight space under the action of the aforementioned matrix in $\mathfrak{h}$. Now we use the following facts: (i) M is in the principal block for $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ and thus is in the principal block of $\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}$; (ii) the principal block of $\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}$ has one simple module (namely the trivial module); and (iii) the projective cover of the trivial module in $\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}$ is four-dimensional. These facts, in conjunction with the above calculation, show that

$$c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(M) \leqslant c_{\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}}(M).$$

In order to show equality, one can use the fact that any projective resolution in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ will restrict to a projective resolution in $\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}$. Finally, we apply [BKN11, Theorem 2.9.1] and the fact that there is only one simple module in the principal block of $\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}$ to conclude that $c_{\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_0)}}(M) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_0)}(M)$. □

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