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On some mean value theorems of the differential calculus

J.B. Diaz and R. Výborný

A general mean value theorem, for real valued functions, is proved. This mean value theorem contains, as a special case, the result that for any, suitably restricted, function f defined on $[a, b]$, there always exists a number c in (a, b) such that $f(c) - f(a) = f'(c)(c-a)$. A partial converse of the general mean value theorem is given. A similar generalized mean value theorem, for vector valued functions, is also established.

1. Introduction

Flett's mean value theorem [6], which has attracted some attention (see, for example the book by Boas [2]), was generalized by Lakshminarasimhan [7], Trahan [9] and Reich [8]. Flett's Theorem reads: If $f(x)$ is a differentiable real valued function on $[a, b]$, and $f'(a) = f'(b)$, then there exists a number c in (a, b) such that $f(c) - f(a) = f'(c)(c-a)$. In this note, Flett's Theorem is generalized further; this generalization brings out more clearly the geometrical fact behind Flett's Theorem. Expressed in intuitive geometrical language, Lagrange's mean value theorem says that, given a smooth plane curve $\overset{\frown}{AB}$ joining two points A and B, there always is a point C, interior to the curve $\overset{\frown}{AB}$, such that the tangent to the curve at C is parallel to the chord $\overline{AB}$; whereas the present generalization of Flett's Theorem states that, if the curve intersects the chord $\overline{AB}$, then there is a point D, interior to the curve $\overset{\frown}{AB}$, such that the straight line AD is

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tangent to the curve at D . A partial converse of this theorem is also given. Besides this, a theorem for vector valued functions is also proved.

2. Real valued functions

Let f and g be real valued functions, defined on a finite closed interval $[a, b]$, where $a < b$. The set of all points $(g(x), f(x))$, for $x \in [a, b]$, will be called the graph of the couple (g, f) ; and, in the special case when $g(x) = x$, it will be simply called the graph of the function f .

For convenience, the following terminology will be adhered to: *The graph of the couple* (g, f) *is said to intersect its chord* (internally) provided that there exists a number $\bar{x} \in (a, b)$ such that

$$[f(\bar{x})-f(a)][g(b)-g(a)] = [g(\bar{x})-g(a)][f(b)-f(a)] . \tag{1}$$

The graph of the couple (g, f) is said to intersect its chord in the extended sense, if *either* there is a number $\bar{x} \in (a, b)$ such that (1) holds, *or* else $g(b) \neq g(a)$, $\lim_{x \to a+} \frac{f(x)-f(a)}{g(x)-g(a)}$ exists, and

$$\lim_{x \to a+} \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(b)-f(a)}{g(b)-g(a)} . \tag{2}$$

THEOREM 1. *Let the functions* f *and* g *satisfy the following conditions:*

(i) f *and* g *are continuous on* $[a, b]$,

(ii) $g(x) \neq g(a)$ *for* $a < x \leq b$,

(iii) the graph of the couple (g, f) *intersects its chord in the extended sense.*

Then there exists a number $c \in (a, b)$, *and two positive numbers,* δ_1, δ_2 , *such that: either both inequalities*

$$[f(c)-f(a)][g(c)-g(c-h)] \leq [g(c)-g(a)][f(c)-f(c-h)] , \tag{3}$$

and

$$[f(c+k)-f(c)][g(c)-g(a)] \leq [g(c+k)-g(c)][f(c)-f(a)] \tag{4}$$

hold for $0 < h \leq \delta_1$, *and* $0 < k \leq \delta_2$; *or both inequalities* (3), (4) *are*

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valid, with the inequality signs reversed, for $0 < h \le \delta_1$ *and* $0 < k \le \delta_2$.

Proof. Define the auxiliary function $Q(x) = \dfrac{f(x)-f(a)}{g(x)-g(a)}$ for $x > a$, and, if (2) holds, define Q also at $x = a$ by the equation $Q(x) = \dfrac{f(b)-f(a)}{g(b)-g(a)}$. No matter whether (1) or (2) holds, there is a number $\bar{x}$, with $a \le \bar{x} < b$, such that $Q(\bar{x}) = Q(b)$, and the function Q is continuous on $[\bar{x}, b]$. Consequently, the function Q attains either its maximum or its minimum, over $[\bar{x}, b]$, at a number c , with $\bar{x} < c < b$. Since the conclusion of the theorem is not affected if the function f is replaced by the function $-f$, it can be supposed that Q attains a maximum at c . Then, one has

$$(5) \qquad Q(c-h) \le Q(c) ,$$

for $0 < h \le c-\bar{x} = \delta_1$, and

$$(6) \qquad Q(c+k) \le Q(c) ,$$

for $0 < k \le b-c = \delta_2$. The inequality (5) means that

$$\frac{f(c-h)-f(a)}{g(c-h)-g(a)} \le \frac{f(c)-f(a)}{g(c)-g(a)} .$$

Since $g(x) \ne g(a)$, for $a < x \le b$, and g is continuous, one has that either $g(x) > g(a)$ for $a < x \le b$, or $g(x) < g(a)$ for $a < x \le b$. Therefore, the product $[g(c)-g(a)][g(c-h)-g(a)] > 0$, and hence

$$[f(c-h)-f(a)][g(c)-g(a)] \le [f(c)-f(a)][g(c-h)-g(a)] .$$

Adding $[f(a)-f(c)][g(c)-g(a)]$ to both sides of the last inequality, one obtains (3). Using the inequality (6) one arrives, in a similar way, at the inequality (4).

REMARK 1. If alternative (2) holds, that is $\lim\limits_{x \to a+} \dfrac{f(x)-f(a)}{g(x)-g(a)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$, then the numbers δ_1 and δ_2 can be taken to be $c - a$ and $b - c$, respectively.

REMARK 2. Assuming, further, that f is differentiable in (a, b) , and choosing $g(x) = x$, it follows from (3) and (4), by passing to the limit as $h \to 0+$ and $k \to 0+$, respectively, that

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$$f(c) - f(a) \le f'(c)(c-a) ,$$
$$f'(c)(c-a) \le f(c) - f(a) ,$$

which implies

$$f(c) - f(a) = f'(c)(c-a) .$$

This is precisely the conclusion of Flett's Theorem, but obtained here under a weaker hypothesis.

REMARK 3. Without assuming that f is differentiable, but still choosing $g(x) = x$, one obtains from the conclusion of Theorem 1 that, either

$$(7) \qquad \frac{f(c+k)-f(c)}{k} \le \frac{f(c)-f(a)}{c-a} \le \frac{f(c)-f(c-h)}{h} ,$$

or the reverse inequalities hold, for $0 < h \le \delta_1$ and $0 < k \le \delta_2$. Passing to the limit as $k \to 0+$, $h \to 0+$, and employing the usual notation for Dini derivates, one obtains that, either

$$(8^+) \qquad D^+f(c) \le \frac{f(c)-f(a)}{c-a} \le D_-f(c) ,$$

or

$$(8^-) \qquad D^-f(c) \le \frac{f(c)-f(a)}{c-a} \le D_+f(c) .$$

This is the conclusion of Theorem 1 in [8], except that, in [8], c could conceivably be b , which is excluded here. It should also be mentioned that the hypothesis

$$\left[f'(b) - \frac{f(b)-f(a)}{b-a}\right]\left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] \ge 0$$

in Theorem 1 of [8] can be shown to imply hypothesis *(iii)*, for $g(x) = x$, in the present Theorem 1, except in the trivial case when $f'(b) = \frac{f(b)-f(a)}{b-a}$.

The usual Lagrange's mean value theorem reads: if f is a real valued function, continuous on $[a, b]$, and differentiable in (a, b) , then there exists a number $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b-a)$. A generalization of Lagrange's mean value theorem, concerning the Dini derivates, appears in the work of W.H. and

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Grace Chisholm Young [*10*, p. 10], which states that, if f is a real valued function continuous on $[a, b]$, then there exists a number c, with $a < c < b$, such that either

$$(9^+) \qquad D^+f(c) \le \frac{f(b)-f(a)}{b-a} \le D_-f(c) \ ,$$

or

$$(9^-) \qquad D^-f(c) \le \frac{f(b)-f(a)}{b-a} \le D_+f(c) \ .$$

A further generalization can be given as follows [3, p. 115]: If f is a real valued function continuous on $[a, b]$, then there exists a number c such that either

$$(10) \qquad \frac{f(c+k)-f(c)}{k} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(c)-f(c-h)}{h}$$

hold for *all* positive h and k such that $c+k \in (a, b)$, $c-h \in (a, b)$, or the reverse inequalities hold with the same restrictions on h and k. The inequalities (9) and (10) bear the same relation to the Lagrange Theorem as the inequalities (8) and (7) bear to Flett's Theorem.

REMARK 4. Assuming that both f and g are differentiable, one obtains, from (3) and (4), by dividing by h and k, respectively, and then passing to the limit as $h \to 0+$ and $k \to 0+$, that

$$f'(c)[g(c)-g(a)] \le g'(c)[f(c)-f(a)] \ ,$$

$$g'(c)[f(c)-f(a)] \le f'(c)[g(c)-g(a)] \ ,$$

which implies

$$(11) \qquad g'(c)[f(c)-f(a)] = f'(c)[g(c)-g(a)] \ .$$

(One, of course, arrives at this conclusion, too, if inequalities reverse to (3) and (4) hold.) This is precisely the conclusion of Theorem 2 in [9], except that there c could conceivably be b, which is excluded here. It should also be mentioned that the hypothesis

$$(12) \qquad \left[\frac{f'(a)}{g'(a)} - \frac{f(b)-f(a)}{g(b)-g(a)}\right]\{[g(b)-g(a)]f'(b) - [f(b)-f(a)]g'(b)\} \ge 0$$

in Theorem 2 of [9] can be shown to imply hypothesis *(iii)* in the present Theorem 1, except in the trivial case, when $[g(b)-g(a)]f'(b) = [f(b)-f(a)]g'(b)$. If g' never vanishes, then (11) can be written in the form

$$\frac{f(c)-f(a)}{g(c)-g(a)} = \frac{f'(c)}{g'(c)} , \tag{13}$$

and this explains why a theorem of this sort is called a "fractional mean value theorem". As soon as a fractional mean value theorem is established, one can prove Taylor like theorems with various forms of the remainder (for example, Lagrange's, Cauchy's or Schlömilch's form). See [9], [4], [5]. This order of ideas will not be pursued further here.

REMARK 5. Assuming that only g is differentiable, one obtains from Theorem 1 (similarly as in Remark 2), that either

$$D^+f(c) \le \frac{f(c)-f(a)}{g(c)-g(a)}\, g'(c) \le D_-f(c) , \tag{14}$$

or

$$D^-f(c) \le \frac{f(c)-f(a)}{g(c)-g(a)}\, g'(c) \le D_+f(c) . \tag{15}$$

This is the conclusion of Theorem 2 in [8], except that in [8], c could be conceivably b, which is excluded here. Hypothesis (12) appears also in Theorem 2 of [8], and hence, as pointed out in Remark 4, the hypothesis of the present Theorem 1 is actually weaker, except in the trivial case, when $[g(b)-g(a)]f'(b) = [f(b)-f(a)]g'(b)$.

REMARK 6. The usual Cauchy fractional mean value theorem reads: If f and g are real valued functions continuous on $[a, b]$, differentiable on (a, b), then there exists a number c such that $[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$. A generalization of the Cauchy Theorem, concerning Dini derivates, appears in the work of W.H. and Grace Chisholm Young [10, pp. 19-24]; roughly speaking, this generalization is related to Cauchy's Theorem in a similar way as inequalities (14) and (15) are related to equation (13). A further generalization of Cauchy's Theorem can be given as follows [4, Remark 4]: If f and g are real valued continuous functions on $[a, b]$, then there exists a number c such that, *either*

$$\begin{aligned} &[f(b)-f(a)][g(c)-g(c-h)] \le [g(b)-g(a)][f(c)-f(c-h)] , \\ &[g(b)-g(a)][f(c+k)-f(c)] \le [f(b)-f(a)][g(c+k)-g(c)] \end{aligned} \tag{16}$$

hold for *all* positive h and k such that $c-h \in [a, b]$, $c+k \in [a, b]$,

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or both inequalities (16) are valid with inequality sign reversed, with the same restriction on h and k. The conclusion of this generalized mean value theorem bears the same relation to the conclusion of the Cauchy mean value theorem as the conclusion of the Theorem 1 of this paper bears to the conclusion of Theorem 2 of [9] [roughly speaking, to equation (11)], except that, in the generalization to Cauchy's theorem, the numbers h and k are only restricted by inequalities $0 < h \le c-a$, $0 < k \le b-c$.

3. Partial converse

In considering the possibility of a converse of Theorem 1, only the case when $g(x) \equiv x$ will be taken into account. A natural converse of Theorem 1 would state that, if f is continuous, and there exists a number c, with $a < c < b$, and positive numbers δ_1, δ_2 such that *either*

$$\frac{f(c+k)-f(c)}{k} \le \frac{f(c)-f(a)}{c-a} \le \frac{f(c)-f(c-h)}{h} \tag{17}$$

hold for $0 < h \le \delta_1$, and $0 < k \le \delta_2$, *or* the reverse inequalities are valid with the same restrictions on h and k, then the graph of f intersects its chord. However, this proposition is not true, as examples in Figure 1 and Figure 2 show. Nevertheless, the following theorem holds.

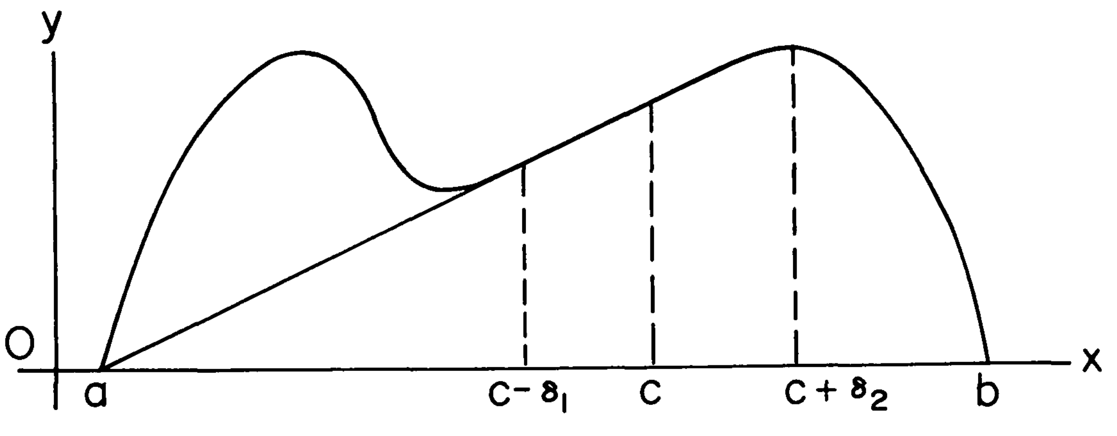

Figure 1

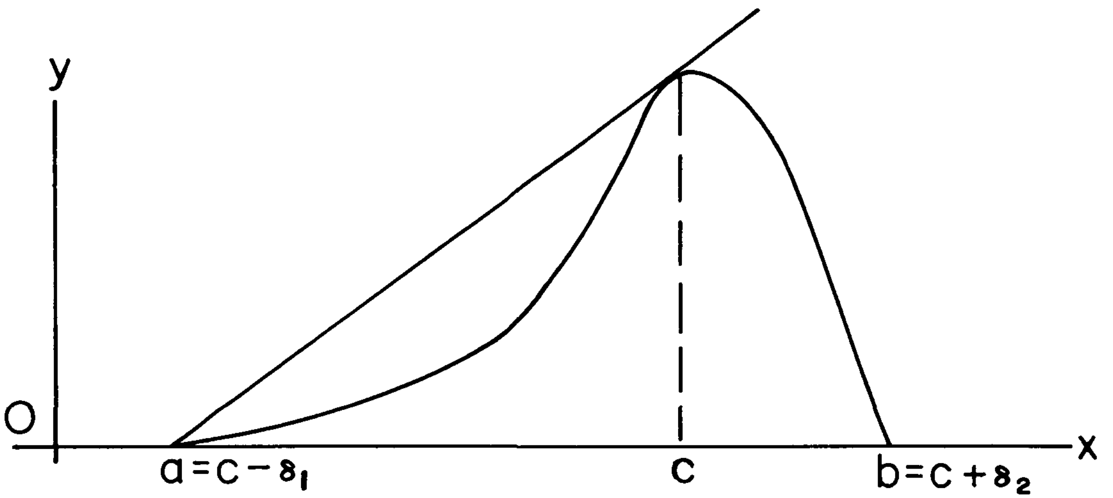

Figure 2

THEOREM 2. *Let* f *be a real valued continuous function on* $[a, b]$. *If there is a number* $c \in (a, b)$ *such that either the inequalities* (17), *or the reverse inequalities, hold for all positive* h *and* k, *with* $a \le c-h$, $c+k \le b$, *then, either* f *is linear on* $[a, c]$, *or there is a number* $d \in (c, b]$ *and a number* $\bar{x}$, *with* $a < \bar{x} < d$, *such that*

$$\frac{f(\bar{x})-f(a)}{\bar{x}-a} = \frac{f(d)-f(a)}{d-a} . \tag{18}$$

(*Thus, if* (18) *holds, then the graph of* f *intersects its chord (internally) between the points* $(a, f(a))$ *and* $(d, f(d))$.)

Proof. Suppose, first, that (17) holds. Then

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(c)-f(a)}{c-a} \tag{19}$$

holds, for $a < x \le b$. Two cases arise. In the first case, the equality sign holds for all x, with $a < x < c$. In this case, f is linear on $[a, c]$. In the second case, there is a number x_0, with $a < x_0 < c$, such that strict inequality holds, in (19), for $x = x_0$. There are two subcases; *either*,

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(20) $$\frac{f(b)-f(a)}{b-a} \geq \frac{f(x_0)-f(a)}{x_0-a} ,$$

or

(21) $$\frac{f(b)-f(a)}{b-a} < \frac{f(x_0)-f(a)}{x_0-a} .$$

If (20) holds, then the continuous function H, defined by

$$H(x) = \frac{f(x)-f(a)}{x-a}$$

has a value less than or equal to $\frac{f(b)-f(a)}{b-a}$ at $x = x_0$, by (20), and has, in view of (19), with $x = b$, a value greater than or equal to $\frac{f(b)-f(a)}{b-a}$ at $x = c$. Therefore, there exists a number $\bar{x} \in [x_0, c]$ such that

$$\frac{f(\bar{x})-f(a)}{\bar{x}-a} = \frac{f(b)-f(a)}{b-a} ,$$

that is, the equation (18) is satisfied for $d = b$. If (21) holds, then the continuous function H has a value less than $\frac{f(x_0)-f(a)}{x_0-a}$ at $x = b$, by (21), and has a value greater than $\frac{f(x_0)-f(a)}{x_0-a}$ at $x = c$, in view of (19) with $x = x_0$. Therefore, there exists a number d, with $c < d < b$, such that

$$\frac{f(d)-f(a)}{d-a} = \frac{f(x_0)-f(a)}{x_0-a} ,$$

that is, equation (18) is satisfied, with $\bar{x} = x_0$. If the inequalities reverse to (17) hold, then (17) holds with f replaced by $-f$, and, therefore, the desired conclusion follows in this case also.

4. Vector valued functions

THEOREM 3. *Let the functions F and g satisfy the following conditions:*

(i) the vector valued function F is continuous on $[a, b]$, and its values are in a linear normed space B (with the norm

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denoted by $\| \ \|$); *the real valued function* g *is continuous on* $[a, b]$,

(ii) $g(x) > g(a)$, *for* $a < x \leq b$,

(iii) *either there exists a number* $\bar{x}$ *such that*

$$\left\| \frac{F(\bar{x})-F(a)}{g(\bar{x})-g(a)} \right\| = \left\| \frac{F(b)-F(a)}{g(b)-g(a)} \right\| ,$$

or, the limit $\lim_{x \to a+} \left\| \frac{F(x)-F(a)}{g(x)-g(a)} \right\|$ *exists, and*

$$\lim_{x \to a+} \left\| \frac{F(x)-F(a)}{g(x)-g(a)} \right\| = \left\| \frac{F(b)-F(a)}{g(b)-g(a)} \right\| .$$

Then, there exists a number $c \in (a, b)$ *and a positive number* δ , *such that, either*

(22) $$\|F(c)-F(a)\|[g(c)-g(c-h)] \leq [g(c)-g(a)]\|F(c)-F(c-h)\| ,$$

for $0 < h < \delta$, *or*

(23) $$\|F(c)-F(a)\|[g(c+h)-g(c)] \leq [g(c)-g(a)]\|F(c+h)-F(c)\| ,$$

for $0 < h < \delta$.

Proof. Using Theorem 1 for f , where $f(x) = \|F(x)-F(a)\|$, one obtains: If inequalities (3) and (4) hold, then one obtains, from (3), that

(24) $$\|F(c)-F(a)\|[g(c)-g(c-h)] \leq [g(c)-g(a)]\big[\|F(c)-F(a)\|-\|F(c-h)-F(a)\|\big] ,$$

for $0 < h \leq \delta = \min(\delta_1, \delta_2)$, and, by the triangle inequality, that

(25) $$\|F(c)-F(a)\| - \|F(c-h)-F(a)\| \leq \|F(c)-F(c-h)\| .$$

Inequality (22) now follows, using *(ii)* with $x = c$, from (24) and (25). If, on the other hand, the inequalities reverse to (3) and (4) hold, then one obtains, from the inequality reverse to (4), that

(26) $$\big[\|F(c+h)-F(a)\|-\|F(c)-F(a)\|\big][g(c)-g(a)] \geq [g(c+h)-g(c)]\|F(c)-F(a)\| ,$$

for $0 < h \leq \delta = \min(\delta_1, \delta_2)$, and, by the triangle inequality, that

(27) $$\|F(c+h)-F(a)\| - \|F(c)-F(a)\| \leq \|F(c+h)-F(c)\| .$$

Inequality (23) now follows, using *(ii)* with $x = c$, from (26) and (27).

THEOREM 4. *Let the functions* F *and* g *satisfy conditions (i) and*

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(iii) of Theorem 3, and let g *be strictly monotonic. Then, there exists a number* $c \in (a, b)$ *and a positive number* δ *, such that either*

$$\left\|\frac{F(c)-F(a)}{g(c)-g(a)}\right\| \leq \left\|\frac{F(c+h)-F(c)}{g(c+h)-g(c)}\right\| , \tag{28}$$

for $0 < h \leq \delta$ *, or*

$$\left\|\frac{F(c)-F(a)}{g(c)-g(a)}\right\| \leq \left\|\frac{F(c)-F(c-h)}{g(c)-g(c-h)}\right\| , \tag{29}$$

for $0 < h \leq \delta$.

If, further, F *is strongly differentiable on* (a, b) *, and* g *is differentiable on* (a, b) *, then*

$$\left\|\frac{F(c)-F(a)}{g(c)-g(a)}\right\| \leq \left\|\frac{F'(c)}{g'(c)}\right\| . \tag{30}$$

Proof. If g is strictly increasing, then hypothesis *(ii)* of Theorem 3 holds, and (29) and (28) follow directly from (22) and (23), respectively. If g is strictly decreasing, one considers $-g$, instead of g . If F and g are differentiable, then, passing to the limit, as $h \to 0+$, in either (28) or (29), one obtains (30).

REMARK 7. Theorem 4 is a sort of a "fractional Flett-Trahan mean value theorem for vector valued functions". Using the Hahn-Banach extension theorem, it is not difficult to extend Theorem 4 to the case when F is only weakly differentiable (see, for example, [1]).

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Rensselaer Polytechnic Institute,
Troy,
New York, USA;

University of Queensland,
St Lucia, Queensland,

and

Rensselaer Polytechnic Institute,
Troy,
New York, USA.