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ALMOST CLOSED 1-FORMS

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Abstract. We construct an algebraic almost closed 1-form with zero scheme not expressible (even locally) as the critical locus of a holomorphic function on a non-singular variety. The result answers a question of Behrend–Fantechi. We correct here an error in our paper (D. Maulik, R Pandharipande and R. P. Thomas, Curves on K3 surfaces and modular forms, *J. Topol.* **3** (2010) 937–996. arXiv:1001.2719v3), where an incorrect construction with the same claimed properties was proposed.

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1. Introduction. An algebraic 1-form ω on a non-singular variety V is *almost closed* if $d\omega$ vanishes on the zero scheme $\mathcal{Z}(\omega) \subset V$ of ω. For example, for every non-constant polynomial $F \in \mathbb{C}[x, y]$, the 1-form

$$\omega = dF + F dz \tag{1}$$

is almost closed, but not closed, on $\mathbb{C}^3$ with coordinates x, y, z. By a construction of Behrend and Fantechi [**2**], the zero scheme $\mathcal{Z}(\omega)$ carries a natural symmetric obstruction theory in case ω is almost closed.

Following is the question asked by Behrend and Fantechi [**1**, **2**]: If ω is almost closed, can we find an analytic neighbourhood

$$\mathcal{Z} \subset U \subset V$$

and a holomorphic function Φ on U so that the equality

$$\mathcal{Z}(\omega) = \mathcal{Z}(d\Phi)$$

holds as subschemes? In other words, is $\mathcal{Z}(\omega)$ always the critical locus of a holomorphic function? In example (1), the function

$$\Phi = e^z F \tag{2}$$

provides an affirmative answer. Such a Φ is called a *potential* for $\mathcal{Z}$.

We will construct a non-reduced point at the origin

$$\mathcal{Z} \subset \mathbb{C}^2$$

which is the zero scheme of an algebraic almost closed 1-form on a Zariski open neighbourhood of the orgin in $\mathbb{C}^2$, but *not* the critical locus of any holomorphic function defined in a neighbourhood of the origin in $\mathbb{C}^2$. We also show that $\mathcal{Z}$ cannot be a critical locus of a holomorphic function defined locally on $\mathbb{C}^n$ for any n.

We will primarily study almost closed 1-forms via formal power series analysis at the origin. In Section 4 we show how these methods also yield *algebraic* almost closed 1-forms on Zariski open neighbourhoods of the origin as relevant to the Behrend–Fantechi question.

The context of the problem that this paper addresses is Donaldson–Thomas theory and its generalisations. One tries to define invariants of Calabi–Yau 3-fold X by virtual counting of (semi-stable) objects of the derived category of coherent sheaves on X. The obstruction theory of such objects is symmetric, so by Behrend [**1**] the moduli space is locally described as the zero locus of an almost closed 1-form. It is expected that these moduli spaces are in fact locally critical loci of holomorphic functions, and the example of this paper shows that this is a stronger condition than symmetry of the obstruction theory.

Recently, Pantev, Toën, Vaquie and Vezzosi [**6**] have introduced the notion of a (-1)-*shifted symplectic structure* on a derived scheme. This is a stronger condition than the existence of a symmetric obstruction theory on the underlying scheme $\mathcal{Z}$. They also show that the moduli space of objects in the derived category of a Calabi–Yau 3-fold admits such a structure; and Brav, Bussi and Joyce [**3**] have proved that a scheme $\mathcal{Z}$ is locally a critical locus if and only if it underlies a derived scheme with (-1)-shifted symplectic structure. In particular, our example cannot carry a (-1)-shifted symplectic structure.

2. Construction of the almost closed 1-form. Consider $\mathbb{C}^2$ with coordinates x and y. We will use capital letters to denote elements of the ring $\mathbb{C}[[x, y]]$. In case the element is polynomial and homogenous, the degree is designated in the subscript. As usual partial derivatives will also be denoted by subscripts, so

$$A_{d,xy} = \frac{\partial^2 A_d}{\partial x \partial y}$$

is a homogeneous polynomial of degree $d-2$.

We start by constructing an almost closed 1-form σ on $\mathbb{C}^2$ via formal power series. The construction depends upon the initial choice of a sufficiently large degree d. Let

$$\begin{aligned}\sigma &= Adx + Bdy \\ &= (A_d + A_{d+1} + A_{d+2} + \cdots)dx + (B_d + B_{d+1} + B_{d+2} + \cdots)dy,\end{aligned}$$

be a 1-form starting at order d satisfying

$$\begin{aligned}A_y - B_x = CA + DB = {} & (C_0 + C_1 + C_2 + \cdots)(A_d + A_{d+1} + A_{d+2} + \cdots) \\ & + (D_0 + D_1 + D_2 + \cdots)(B_d + B_{d+1} + B_{d+2} + \cdots),\end{aligned} \tag{3}$$

where $C, D \in \mathbb{C}[[x, y]]$.

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The zero scheme $\mathcal{Z}(\sigma)$ is Spec $\frac{\mathbb{C}[\![x,y]\!]}{(A,B)}$. We shall often use the trivialisation $dy \wedge dx$ of the 2-form on $\mathbb{C}^2$ to identify $d\sigma$ with the function $A_y - B_x$. Equation (3) is the almost closed condition,

$$d\sigma = A_y - B_x \in (A, B).$$

For sufficient large d, we will prove that the general σ satisfying (3) has zero scheme which is not the critical locus of a holomorphic function.

Fixing $C, D \in \mathbb{C}[\![x, y]\!]$, we construct solutions σ to the almost closed equation (3) by the following procedure:

- First, A_d and B_d are degree d homogeneous polynomials satisfying

$$A_{d,y} - B_{d,x} = 0. \tag{4}$$

  Thus,

$$A_d = P_{d+1,x}\,, \quad B_d = P_{d+1,y}\,, \tag{5}$$

  for any homogeneous degree $d + 1$ polynomial P_{d+1}. We pick a *general* solution: a general element of the vector space

$$\ker\left(\mathrm{Sym}^d(\mathbb{C}^2)^* \oplus \mathrm{Sym}^d(\mathbb{C}^2)^* \xrightarrow{\partial_y \oplus -\partial_x} \mathrm{Sym}^{d-1}(\mathbb{C}^2)^*\right).$$

- Next, A_{d+1} and B_{d+1} are homogeneous polynomials of degree $d + 1$ satisfying

$$A_{d+1,y} - B_{d+1,x} = C_0 A_d + D_0 B_d \tag{6}$$

  for the given constants C_0 and D_0. The above is an affine linear equation on the vector space of pairs (A_{d+1}, B_{d+1}) with a non-empty set of solutions (for example, set $B_{d+1} = 0$ and integrate to get A_{d+1}). We pick a *general* element of the affine space of solutions.
- At the kth order, we have defined $A_d, \ldots, A_{k-1}$ and $B_d, \ldots, B_{k-1}$, and we pick degree k homogeneous polynomials A_k, B_k satisfying the affine linear equation

$$A_{k,y} - B_{k,x} = \sum_{i=0}^{k-1-d} C_i A_{k-1-i} + \sum_{i=0}^{k-1-d} D_i B_{k-1-i}\,. \tag{7}$$

  Again, the affine space of solutions is non-empty, and we pick a *general* element.

Thus, we obtain a general formal power series almost-closed 1-form σ starting at order d with given $C, D \in \mathbb{C}[\![x, y]\!]$. Later, we will show σ can be modified to be algebraic.

3. Convergence. Although not necessary for our construction, the power series solution σ can be taken to be convergent when conditions[1] are placed on the series $C, D \in \mathbb{C}[\![x, y]\!]$.

[1]Stronger results are certainly possible, but the conditions we impose are sufficiently mild that the non-existence result of Theorem 3 still holds if we replace the word 'general' with 'general within the Euclidean open set of Lemma 1'.

LEMMA 1. *Suppose $C_{\geq 2} = 0 = D_{\geq 2}$. Then there is a power series solution σ starting at order d with non-zero radius of convergence. Moreover, the set of solutions of* (3) *which converge contains an Euclidean open set in the space of all solutions.*

Proof. When $C_{\geq 2} = 0 = D_{\geq 2}$, equation (7) simplifies to

$$A_{k,y} - B_{k,x} = C_0 A_{k-1} + D_0 B_{k-1} + C_1 A_{k-2} + D_1 B_{k-2}. \tag{8}$$

Let $\|p\|$ denote the maximum of the absolute values of the coefficients of the polynomial $p(x, y)$, and make the following definitions:

$$\begin{aligned}
\lambda &= \max(|C_0|, |D_0|, 2\|C_1\|, 2\|D_1\|),\\
\mu_k &= \max(\|A_k\|, \|B_k\|, \|A_{k-1}\|, \|B_{k-1}\|),\\
E_{k-1} &= C_0 A_{k-1} + D_0 B_{k-1} + C_1 A_{k-2} + D_1 B_{k-2}.
\end{aligned}$$

We easily find

$$\begin{aligned}
\|E_{k-1}\| &\leq |C_0|\|A_{k-1}\| + |D_0|\|B_{k-1}\| + 2\|C_1\|\|A_{k-2}\| + 2\|D_1\|\|B_{k-2}\|\\
&\leq 4\lambda\mu_{k-1}.
\end{aligned}$$

A general solution of (8) is provided by integrating

$$\begin{aligned}
A_{k,y} &= F_{k-1},\\
B_{k,x} &= F_{k-1} - E_{k-1},
\end{aligned}$$

for a general choice of homogeneous degree $k-1$ polynomial F_{k-1}. We now choose F_{k-1} to satisfy

$$\|F_{k-1}\| < 4\lambda\mu_{k-1},$$

which is a Euclidean open condition.

Since integration divides all coefficients by integers, we see

$$\|A_k\| < 4\lambda\mu_{k-1}, \quad \|B_k\| < 8\lambda\mu_{k-1}, \tag{9}$$

so

$$\mu_k \leq \max(1, 8\lambda)\mu_{k-1}.$$

Therefore,

$$\mu_k \leq \alpha\beta^k$$

for some constants α, β. The 1-form σ solving (3) thus has radius of convergence at least β^{-1}. □

4. Algebraic almost closed 1-forms. We return to the formal power series solutions σ of (3) starting at degree d. For general choices of A_d and B_d, the zero locus $\mathcal{Z}(\sigma)$ will be a non-reduced scheme $\mathcal{Z}_0(\sigma)$ at the origin of $\mathbb{C}^2$ (plus possibly some other disjoint loci). For sufficiently large N,

$$\mathcal{Z}_0(\sigma) \subset \mathrm{Spec}(\mathbb{C}[\![x, y]\!]/\mathfrak{m}^N),$$

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where $\mathfrak{m} = (x, y)$ is the maximal ideal of the origin. Equivalently,

$$\mathfrak{m}^N \subset (A, B) \subset \mathbb{C}[\![x, y]\!]. \tag{10}$$

Consider the polynomial 1-form

$$\begin{aligned}\sigma_{\leq N} &= A_{\leq N}\,dx + B_{\leq N}\,dy \\ &= (A_d + A_{d+1} + \cdots + A_N)dx + (B_d + B_{d+1} + \cdots + B_N)dy.\end{aligned}$$

While $\sigma_{\leq N}$ need not be almost closed on all of $\mathbb{C}^2$ (issues may arise at the zeros of σ away from the origin), $\sigma_{\leq N}$ *is* almost closed in a Zariski open neighbourhood of the origin.

LEMMA 2. *In the localised ring* $\mathbb{C}[x, y]_{(x,y)}$, *the 1-form* $\sigma_{\leq N}$ *is almost closed,*

$$(d\sigma_{\leq N}) \subset (A_{\leq N}, B_{\leq N}) \subset \mathbb{C}[x, y]_{(x,y)}.$$

Proof. First, we claim $\sigma_{\leq N}$ is almost closed when considered in the ring $\mathbb{C}[\![x, y]\!]$ of formal power series at the origin of $\mathbb{C}^2$. In fact, $d\sigma_{\leq N}$ is

$$\begin{aligned}A_{\leq N,y} - B_{\leq N,x} &= (A_y - B_x)_{\leq N-1} \\ &= (CA + DB)_{\leq N-1} \\ &= CA_{\leq N} + DB_{\leq N} + \epsilon(N),\end{aligned} \tag{11}$$

where $\epsilon(N)$ consists only of terms of degree $\geq N$ and therefore

$$\epsilon(N) \in \mathfrak{m}^N \subset (A, B)$$

by (10). So we see $\sigma_{\leq N}$ is almost closed modulo $\mathfrak{m}^N$, and we can write

$$\begin{aligned}\epsilon(N) &= C^0A + D^0B \\ &= C^0A_{\leq N} + D^0B_{\leq N} + (C^0A_{>N} + D^0B_{>N}) \\ &= C^0A_{\leq N} + D^0B_{\leq N} + \epsilon(N+1)\end{aligned} \tag{12}$$

for series $C^0, D^0 \in \mathbb{C}[\![x, y]\!]$. Here

$$\epsilon(N+1) \in \mathfrak{m}^{N+1} = \mathfrak{m}.\mathfrak{m}^N \subset \mathfrak{m}.(A, B),$$

which can therefore be written as

$$\epsilon(N+1) \;=\; C^1A_{\leq N} + D^1B_{\leq N} + \epsilon(N+2) \tag{13}$$

just as in (12), with $C^1, D^1 \in \mathfrak{m}$. Continuing inductively, we write

$$\mathfrak{m}^k.(A, B) \supset \mathfrak{m}^{N+k} \;\ni\; \epsilon(N+k) \;=\; C^kA_{\leq N} + D^kB_{\leq N} + \epsilon(N+k+1),$$

where $C^k, D^k \in \mathfrak{m}^k$ consist only of terms of degree $\geq k$. We finally obtain

$$A_{\leq N,y} - B_{\leq N,x} \;=\; \overline{C}A_{\leq N} + \overline{D}B_{\leq N},$$

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where the series

$$\overline{C} = C + C^0 + C^1 + \cdots, \quad \overline{D} = D + D^0 + D^1 + \cdots$$

are convergent in $\mathbb{C}[\![x, y]\!]$. So $\sigma_{\leq N}$ is indeed almost closed in $\mathbb{C}[\![x, y]\!]$.

The above results can be written as the vanishing

$$[A_{\leq N,y} - B_{\leq N,x}] = 0 \quad \text{in the } \mathbb{C}[\![x, y]\!]\text{-module } \frac{\mathbb{C}[\![x, y]\!]}{(A_{\leq N}, B_{\leq N})},$$

or equivalently that

$$0 \longrightarrow \mathbb{C}[\![x, y]\!] \xrightarrow{\;1\;} \mathbb{C}[\![x, y]\!] \xrightarrow{\;A_{\leq N,y} - B_{\leq N,x}\;} \frac{\mathbb{C}[\![x,y]\!]}{(A_{\leq N}, B_{\leq N})} \tag{14}$$

is exact. We are left with showing similarly that

$$0 \longrightarrow \mathbb{C}[x, y]_{(x,y)} \xrightarrow{\;1\;} \mathbb{C}[x, y]_{(x,y)} \xrightarrow{\;A_{\leq N,y} - B_{\leq N,x}\;} \frac{\mathbb{C}[x,y]_{(x,y)}}{(A_{\leq N}, B_{\leq N})} \tag{15}$$

is also exact.

Now (15) pulls back to (14) via the inclusion

$$\mathbb{C}[x, y]_{(x,y)} \hookrightarrow \mathbb{C}[\![x, y]\!]. \tag{16}$$

Since (16) is flat [**4**, Theorem 8.8] and a local map of local rings, it is faithfully flat by [**4**, Theorem 7.2]. Thus, the exactness of (15) follows from that of (14). □

5. Non-existence of a potential. Consider again formal power series solutions σ of (3) starting at degree d. We will show for almost every choice[2] of C and D, the zero scheme $\mathcal{Z}(\sigma)$ is not the critical locus of at the origin of any formal function if A and B are chosen to be *general* solutions of (7).

THEOREM 3. *If $C_{1,x} + D_{1,y} \neq 0$, then for $d \geq 18$ and general choices of A, B satisfying* (7)*, there is no potential function $\Phi \in \mathbb{C}[\![x, y]\!]$ satisfying*

$$(\Phi_x, \Phi_y) = (A, B) \subset \mathbb{C}[\![x, y]\!].$$

Assume a potential function $\Phi \in \mathbb{C}[\![x, y]\!]$ exists satisfying

$$(\Phi_x, \Phi_y) = (A, B) \subset \mathbb{C}[\![x, y]\!].$$

Then both $\Phi_x - A$ and $\Phi_y - B$ are in the ideal (A, B), so we have

$$\begin{aligned} \Phi_x - A &= XA + YB, \\ \Phi_y - B &= ZA + WB, \end{aligned} \tag{17}$$

[2]In fact, we can set $C = x$, $D = 0$ and $B_{\geq(d+2)} = 0$ and the proof still works. But the restriction gives no significant simplification in notation.

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for series $X, Y, Z, W \in \mathbb{C}[[x, y]]$. As usual, we write

$$X = X_0 + X_1 + \cdots$$

with X_k homogeneous of degree k (and similarly for Y, Z and W). To find Φ satisfying[3] (17), integrability

$$\Phi_{xy} = \Phi_{yx},$$

is a necessary and sufficient condition. In other words, we must have

$$-A_y + B_x = (XA + YB)_y - (ZA + WB)_x. \tag{18}$$

We will analyse equation (18) for X, Y, Z, W order by order modulo higher and higher powers $\mathfrak{m}^k$ of the maximal ideal. At each order, the issue is linear. For the first few orders, equation (18) can be easily solved (with several degrees of freedom). But each further stage imposes more stringent conditions on the choices at the previous stage. In degree $d + 2$, we will see the conditions become overdetermined, with no non-trivial solutions X, Y, Z, W for general A, B satisfying (3).

Degree $\mathbf{d} - \mathbf{1}$. In homogeneous degree $d - 1$, (18) yields

$$-A_{d,y} + B_{d,x} = X_0 A_{d,y} + Y_0 B_{d,y} - Z_0 A_{d,x} - W_0 B_{d,x}.$$

From (4), we have the relation $A_{d,y} = B_{d,x}$. Otherwise the partial derivatives of A_d and B_d are completely general. Therefore, the resulting equation

$$0 = (X_0 - W_0)A_{d,y} + Y_0 B_{d,y} - Z_0 A_{d,x}$$

implies the vanishing of $(X_0 - W_0)$, Y_0, and Z_0. Thus, in homogeneous degree d, equations (17) become

$$\Phi_{d+1,x} = (1 + X_0)A_d,$$
$$\Phi_{d+1,y} = (1 + X_0)B_d,$$

with Φ having no terms of degree $\leq d$. Since we require (Φ_x, Φ_y) to generate the ideal (A, B), and since

$$A_d, B_d \neq 0$$

by generality, we find $1 + X_0 \neq 0$. Rescaling Φ, we can assume

$$1 = 1 + X_0$$

without loss of generality. So we have found the following conditions at order $d - 1$:

$$X_0 = Y_0 = Z_0 = W_0 = 0. \tag{19}$$

[3]Equations (17) imply that only $(\Phi_x, \Phi_y) \subset (A, B)$ and has the trivial solution

$$X = 1 = W \quad \text{and} \quad Y = 0 = Z$$

corresponding to constant f. We will rule out the trivial solution by requiring $(\Phi_x, \Phi_y) = (A, B)$ shortly.

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Degree d. In homogeneous degree d, (18) yields

$$-A_{d+1,y} + B_{d+1,x} = (X_1 A_d)_y + (Y_1 B_d)_y - (Z_1 A_d)_x - (W_1 B_d)_x \\ + (X_0 A_{d+1})_y + (Y_0 B_{d+1})_y - (Z_0 A_{d+1})_x - (W_0 B_{d+1})_x.$$

By our work in the previous degree (19), the second line on the right vanishes identically. After substituting (6) on the left side, we find

$$-C_0 A_d - D_0 B_d \;=\; (X_1 A_d)_y + (Y_1 B_d)_y - (Z_1 A_d)_x - (W_1 B_d)_x. \tag{20}$$

We rewrite (20) as the vanishing of

$$(X_{1,y} - Z_{1,x} + C_0)A_d + X_1 A_{d,y} - Z_1 A_{d,x} \\ + (Y_{1,y} - W_{1,x} + D_0)B_d + Y_1 B_{d,y} - W_1 B_{d,x}.$$

After expanding the linear unknowns out fully via

$$X_1 = X_{1,x}x + X_{1,y}y, \quad Y_1 = Y_{1,x}x + Y_{1,y}y, \\ Z_1 = Z_{1,x}x + Z_{1,y}y, \quad W_1 = W_{1,x}x + W_{1,y}y,$$

we obtain a relation among the degree d homogeneous polynomials

$$A_d,\ xA_{d,x},\ xA_{d,y},\ yA_{d,x},\ yA_{d,y}, \\ B_d,\ xB_{d,x},\ xB_{d,y},\ yB_{d,x},\ yB_{d,y}. \tag{21}$$

Since A_d and B_d were chosen generically subject to $A_{d,y} = B_{d,x}$, the polynomials (21) are linearly independent except for the relations

$$\begin{aligned} A_{d,y} &= B_{d,x} \quad \text{(multiplied by } x, y\text{)}, \\ xA_{d,x} + yB_{d,y} &= dA_d, \\ xA_{d,x} + yB_{d,y} &= dB_d. \end{aligned} \tag{22}$$

The first is (4). The last two are the Euler homogeneity relations. A simple check shows that $d \geq 5$ is sufficient to achieve the independence.

Using the first equation of (22) to eliminate $xB_{d,x}$ and $yB_{d,x}$ and the last two to eliminate A_d, B_d, we find the following equations:

$$\begin{aligned} \text{Coefficient of } xA_{d,x}: &\quad 0 = (X_{1,y} - Z_{1,x} + C_0)/d - Z_{1,x}, \\ yA_{d,x}: &\quad 0 = -Z_{1,y}, \\ xA_{d,y}: &\quad 0 = X_{1,x} + (Y_{1,y} - W_{1,x} + D_0)/d - W_{1,x}, \\ yA_{d,y}: &\quad 0 = (X_{1,y} - Z_{1,x} + C_0)/d + X_{1,y} - W_{1,y}, \\ xB_{d,y}: &\quad 0 = Y_{1,x}, \\ yB_{d,y}: &\quad 0 = (Y_{1,y} - W_{1,x} + D_0)/d + Y_{1,y}. \end{aligned}$$

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Therefore, we find

$$\begin{aligned} Y_{1,x} &= 0 \ = \ Z_{1,y}, \\ W_{1,y} &= X_{1,y} + Z_{1,x}, \\ X_{1,x} &= Y_{1,y} + W_{1,x}, \\ C_0 &= (d+1)Z_{1,x} - X_{1,y}, \\ D_0 &= W_{1,x} - (d+1)Y_{1,y}. \end{aligned} \tag{23}$$

Equations (23) can be solved with room to spare – there is a two-dimensional affine space of solutions. However, the constraints in next degree will impose further conditions which specify X_1, Y_1, Z_1, W_1 uniquely.

Degree $\mathbf{d+1}$. In homogeneous degree $d+1$, equation (18) yields

$$\begin{aligned} -A_{d+2,y} + B_{d+2,x} = &(X_2A_d)_y + (Y_2B_d)_y - (Z_2A_d)_x - (W_2B_d)_x \\ &+ (X_1A_{d+1})_y + (Y_1B_{d+1})_y - (Z_1A_{d+1})_x - (W_1B_{d+1})_x \\ &+ (X_0A_{d+2})_y + (Y_0B_{d+2})_y - (Z_0A_{d+2})_x - (W_0B_{d+2})_x. \end{aligned}$$

By the constraints (19), the third line vanishes identically. Substituting equation (7) for $k = d+2$ on the left side yields

$$\begin{aligned} -C_0A_{d+1} - D_0B_{d+1} - C_1A_d - D_1B_d = (X_2A_d)_y + (Y_2B_d)_y - (Z_2A_d)_x - (W_2B_d)_x \\ + (X_1A_{d+1})_y + (Y_1B_{d+1})_y - (Z_1A_{d+1})_x - (W_1B_{d+1})_x. \end{aligned} \tag{24}$$

We work first modulo those degree $d+1$ polynomials generated by A_d, B_d and their first partial derivatives. More preciely, let

$$V \subset \mathrm{Sym}^{d+1}(\mathbb{C}^2)^*$$

denote the subspace spanned by x and y multiplied by A_d, B_d, and x^2, xy, y^2 multiplied by $A_{d,x}$, $A_{d,y}$, $B_{d,x}$, $B_{d,y}$. The subspace V has dimension 9 due to the relations (22). In the quotient space $\mathrm{Sym}^{d+1}(\mathbb{C}^2)^*/V$, equation (24) is

$$-C_0A_{d+1} - D_0B_{d+1} = (X_1A_{d+1})_y + (Y_1B_{d+1})_y - (Z_1A_{d+1})_x - (W_1B_{d+1})_x, \tag{25}$$

where all terms are taken mod V.

Equation (25) has form identical to equation (20) analysed in the previous degree with A_d, B_d replaced by A_{d+1}, B_{d+1}. The analysis of the previous sections applies again here: The first relation of (22) holds mod V by (6), and the two Euler relations hold with d replaced by $d+1$. By generality, for $d \geq 13$, the polynomials A_{d+1}, B_{d+1} and their partial derivatives (multiplied by x, y) are independent of A_d, B_d (multiplied by x, y) and their partial derivatives (multiplied by x^2, xy, y^2) except for relation (6) and the Euler relations. We conclude the equations corresponding to (23) hold (with d

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replaced by $d+1$):

$$\begin{aligned} Y_{1,x} = 0 \ &= \ Z_{1,y}, \\ W_{1,y} &= X_{1,y} + Z_{1,x}, \\ X_{1,x} &= Y_{1,y} + W_{1,x}, \\ C_0 &= (d+2)Z_{1,x} - X_{1,y}, \\ D_0 &= W_{1,x} - (d+2)Y_{1,y}. \end{aligned}$$

After combining with the original equations (23), we find

$$\begin{aligned} Y_{1,x} = 0 \ &= \ Y_{1,y}, \\ Z_{1,x} = 0 \ &= \ Z_{1,y}, \\ X_{1,y} = W_{1,y} \ &= \ -C_0, \\ X_{1,x} = W_{1,x} \ &= \ D_0. \end{aligned} \tag{26}$$

We have uniquely solved for X_1, Y_1, Z_1, W_1.

We next consider X_2, Y_2, Z_2, W_2. There will be no obstruction to solving for X_2, Y_2, Z_2, W_2 here. However, in the next degree, we will find further constraints: the resulting overdetermined system for X_2, Y_2, Z_2, W_2 will have solutions only if $C_{1,x} + D_{1,y} = 0$.

After substituting constraints (26) into (24), we obtained a simpler equation:

$$-C_1 A_d - D_1 B_d = (X_2 A_d)_y + (Y_2 B_d)_y - (Z_2 A_d)_x - (W_2 B_d)_x + X_1(A_{d+1,y} - B_{d+1,x}).$$

By (6) and (26), the last term $X_1(A_{d+1,y} - B_{d+1,x})$ is

$$X_1(C_0 A_d + D_0 B_d) = (D_0 x - C_0 y)(C_0 A_d + D_0 B_d).$$

We define new terms

$$\widetilde{C}_1 = C_1 + C_0(D_0 x - C_0 y), \quad \widetilde{D}_1 = D_1 + D_0(D_0 x - C_0 y).$$

Then we can write equation (24) as

$$-\widetilde{C}_1 A_d - \widetilde{D}_1 B_d = (X_2 A_d)_y + (Y_2 B_d)_y - (Z_2 A_d)_x - (W_2 B_d)_x. \tag{27}$$

Note the similarity to (20).

We solve (27) following our approach to (20). After expanding the unknowns,

$$\begin{aligned} X_2 &= X_{2,xx}\frac{x^2}{2} + X_{2,xy}xy + X_{2,yy}\frac{y^2}{2}, \\ Y_2 &= Y_{2,xx}\frac{x^2}{2} + Y_{2,xy}xy + Y_{2,yy}\frac{y^2}{2}, \\ Z_2 &= Z_{2,xx}\frac{x^2}{2} + Z_{2,xy}xy + Z_{2,yy}\frac{y^2}{2}, \\ W_2 &= W_{2,xx}\frac{x^2}{2} + W_{2,xy}xy + W_{2,yy}\frac{y^2}{2}, \end{aligned}$$

we obtain

$$
\begin{aligned}
&-(\tilde{C}_{1,x}x+\tilde{C}_{1,y}y)A_d-(\tilde{D}_{1,x}x+\tilde{D}_{1,y}y)B_d \\
&\quad=\Big(X_{2,xx}\frac{x^2}{2}+X_{2,xy}xy+X_{2,yy}\frac{y^2}{2}\Big)A_{d,y}+(X_{2,xy}x+X_{2,yy}y)A_d \\
&\qquad+\Big(Y_{2,xx}\frac{x^2}{2}+Y_{2,xy}xy+Y_{2,yy}\frac{y^2}{2}\Big)B_{d,y}+(Y_{2,xy}x+Y_{2,yy}y)B_d \\
&\qquad-\Big(Z_{2,xx}\frac{x^2}{2}+Z_{2,xy}xy+Z_{2,yy}\frac{y^2}{2}\Big)A_{d,x}-(Z_{2,xx}x+Z_{2,xy}y)A_d \\
&\qquad-\Big(W_{2,xx}\frac{x^2}{2}+W_{2,xy}xy+W_{2,yy}\frac{y^2}{2}\Big)B_{d,x}-(W_{2,xx}x+W_{2,xy}y)B_d. \qquad (28)
\end{aligned}
$$

We consider the above to be a relation among

$$
\begin{aligned}
&xA_d,\, yA_d,\, x^2A_{d,x},\, xyA_{d,x},\, y^2A_{d,x},\, x^2A_{d,y},\, xyA_{d,y},\, y^2A_{d,y}, \\
&xB_d,\, yB_d,\, x^2B_{d,x},\, xyB_{d,x},\, y^2B_{d,x},\, x^2B_{d,y},\, xyB_{d,y},\, y^2B_{d,y}.
\end{aligned}
$$

By generality, for $d \geq 7$, these are linearly independent degree $d+1$ homogeneous polynomials modulo (4) and the Euler relations (22):

$$
\begin{aligned}
A_{d,y} &= B_{d,x} && \text{(multiplied by } x^2, xy, y^2), \\
xA_{d,x}+yA_{d,y} &= dA_d && \text{(multiplied by } x, y), \qquad (29)\\
xB_{d,x}+yB_{d,y} &= dB_d && \text{(multiplied by } x, y).
\end{aligned}
$$

We use (29) to eliminate the terms

$$x^2B_{d,x},\, xyB_{d,x},\, y^2B_{d,x},\, xA_d,\, yA_d,\, xB_d,\, yB_d$$

on the right-hand side of (28). Then independence yields the following equations:

$$
\begin{aligned}
x^2A_{d,x}: &\quad 0 = \widetilde{C}_{1,x}/d - Z_{2,xx}/2 + X_{2,xy}/d - Z_{2,xx}/d, \\
xyA_{d,x}: &\quad 0 = \widetilde{C}_{1,y}/d - Z_{2,xy} + X_{2,yy}/d - Z_{2,xy}/d, \\
y^2A_{d,x}: &\quad 0 = -Z_{2,yy}/2, \\
x^2A_{d,y}: &\quad 0 = \widetilde{D}_{1,x}/d + X_{2,xx}/2 - W_{2,xx}/2 + Y_{2,xy}/d - W_{2,xx}/d, \\
xyA_{d,y}: &\quad 0 = \widetilde{C}_{1,x}/d + \widetilde{D}_{1,y}/d + X_{2,xy} - W_{2,xy} + X_{2,xy}/d \\
&\qquad\quad - Z_{2,xx}/d + Y_{2,yy}/d - W_{2,xy}/d, \\
y^2A_{d,y}: &\quad 0 = \widetilde{C}_{1,y}/d + X_{2,yy}/2 - W_{2,yy}/2 + X_{2,yy}/d - Z_{2,xy}/d, \\
x^2B_{d,y}: &\quad 0 = Y_{2,xx}/2, \\
xyB_{d,y}: &\quad 0 = \widetilde{D}_{1,x}/d + Y_{2,xy} + Y_{2,xy}/d - W_{2,xx}/d, \\
y^2B_{d,y}: &\quad 0 = \widetilde{D}_{1,y}/d + Y_{2,yy}/2 + Y_{2,yy}/d - W_{2,xy}/d.
\end{aligned}
$$

Use the second equation to eliminate $X_{2,yy}$ from the sixth equation. Use the eighth equation to eliminate $W_{2,xx}$ from the fourth equation. Finally, use the first and last equations to remove $X_{2,xy}$ and $W_{2,xy}$ from the fifth one. Tidying up, we find $Z_{2,yy} =$

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$0 = Y_{2,xx}$ and

$$\begin{aligned}(d+2)Z_{2,xx} - 2X_{2,xy} &= 2\widetilde{C}_{1,x},\\ (d+1)Z_{2,xy} - X_{2,yy} &= \widetilde{C}_{1,y},\\ X_{2,xx} - (d+3)Y_{2,xy} &= \widetilde{D}_{1,x},\\ (d+3)(Z_{2,xx} - Y_{2,yy}) &= 2(\widetilde{C}_{1,x} + \tilde{D}_{1,y}),\\ -W_{2,yy} + (d+3)Z_{2,xy} &= \widetilde{C}_{1,y},\\ (d+1)Y_{2,xy} - W_{2,xx} &= -\widetilde{D}_{1,x},\\ (d+2)Y_{2,yy} - 2W_{2,xy} &= -2\widetilde{D}_{1,y}.\end{aligned} \tag{30}$$

Therefore, we can specify that $Z_{2,xx}, Z_{2,xy}$ and Y_{2xy} arbitrarily and uniquely solve for $X_2,\ Y_2, Z_2,\ W_2$ after that. However, the conditions at the next degree will place further constraints.

Degree $\mathbf{d}+\mathbf{2}$. In homogeneous degree $d+2$, equation (18) yields

$$\begin{aligned}-A_{d+3,y} + B_{d+3,x} = (X_3A_d)_y + (Y_3B_d)_y - (Z_3A_d)_x - (W_3B_d)_x\\ +(X_2A_{d+1})_y + (Y_2B_{d+1})_y - (Z_2A_{d+1})_x - (W_2B_{d+1})_x\\ +(X_1A_{d+2})_y + (Y_1B_{d+2})_y - (Z_1A_{d+2})_x - (W_1B_{d+2})_x\\ +(X_0A_{d+3})_y + (Y_0B_{d+3})_y - (Z_0A_{d+3})_x - (W_0B_{d+3})_x.\end{aligned}$$

By constraints (19), the fourth line vanishes identically. Substituting (7) for $k = d+3$ on the left-hand side yields

$$-C_0A_{d+2} - D_0B_{d+2} - C_1A_{d+1} - D_1B_{d+1} - C_2A_d - D_2B_d.$$

Work modulo the (12-dimensional space of) degree $d+2$ homogeneous polynomials generated by A_d, B_d (multiplied by x^2, xy or y^2) and their first partial derivatives (multiplied by x^3, x^2y, xy^2, y^3). We find

$$\begin{aligned}-C_0A_{d+2} - D_0B_{d+2} - C_1A_{d+1} - D_1B_{d+1} = (X_2A_{d+1})_y + (Y_2B_{d+1})_y\\ -(Z_2A_{d+1})_x - (W_2B_{d+1})_x + (X_1A_{d+2})_y + (Y_1B_{d+2})_y - (Z_1A_{d+2})_x - (W_1B_{d+2})_x.\end{aligned}$$

Substituting (26) for $X_1,\ Y_1, Z_1,\ W_1$ into the third line and moving it to the first gives

$$\begin{aligned}-C_1A_{d+1} - D_1B_{d+1} - (D_0x - C_0y)(A_{d+2,y} - B_{d+2,x})\\ = (X_2A_{d+1})_y + (Y_2B_{d+1})_y - (Z_2A_{d+1})_x - (W_2B_{d+1})_x.\end{aligned}$$

By (7) for $k = d+2$, we know the term

$$A_{d+2,y} - B_{d+2,x} = C_0A_{d+1} + D_0B_{d+1} + C_1A_d + D_1B_d.$$

Since we are working modulo A_d, B_d, we obtain

$$-\widetilde{C}_1A_{d+1} - \widetilde{D}_1B_{d+1} = (X_2A_{d+1})_y + (Y_2B_{d+1})_y - (Z_2A_{d+1})_x - (W_2B_{d+1})_x, \tag{31}$$

where

$$\widetilde{C}_1 = C_1 + C_0(D_0x - C_0y), \quad \widetilde{D}_1 = D_1 + D_0(D_0x - C_0y)$$

as before.

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Equation (31) is precisely the same as equation (27) studied at the previous degree with d replaced by $d+1$. Also, relations (29) still hold (after replacing d by $d+1$) by (6), since we are working mod A_d, B_d, and these are the only relations which hold for general A_{d+1}, B_{d+1} once $d \geq 18$.

Hence, the analysis in the previous degree applies here verbatim. We derive the same equations (30) with d replaced by $d+1$. The first is $Z_{2,yy} = 0 = Y_{2,xx}$. And the rest are

$$\begin{aligned}
(d+3)Z_{2,xx} - 2X_{2,xy} &= 2\widetilde{C}_{1,x},\\
(d+2)Z_{2,xy} - X_{2,yy} &= \widetilde{C}_{1,y},\\
X_{2,xx} - (d+4)Y_{2,xy} &= \widetilde{D}_{1,x},\\
(d+4)(Z_{2,xx} - Y_{2,yy}) &= 2(\widetilde{C}_{1,x} + \widetilde{D}_{1,y}),\\
-W_{2,yy} + (d+4)Z_{2,xy} &= \widetilde{C}_{1,y},\\
(d+2)Y_{2,xy} - W_{2,xx} &= -\widetilde{D}_{1,x},\\
(d+3)Y_{2,yy} - 2W_{2,xy} &= -2\widetilde{D}_{1,y}.
\end{aligned}$$

From the central equation and the counterpart in (30), we obtain a necessary condition for there to exist any solutions:

$$\widetilde{C}_{1,x} + \widetilde{D}_{1,y} = 0,$$

or equivalently,

$$C_{1,x} + D_{1,y} = 0. \tag{32}$$

We have completed the proof of Theorem 3. □

Since C_1 and D_1 can be chosen arbitrarily at the beginning, choosing them to violate (32) will imply the non-existence of a potential Φ. More precisely, when A_k, B_k are general solutions of equations (7), the resulting almost closed 1-form generates an ideal at the origin which is not the critical locus of any formal power series $\Phi \in \mathbb{C}[[x, y]]$.

By Lemma 2, we can find an algebraic almost closed 1-form on a Zariski open set of the origin in $\mathbb{C}^2$ which is not the critical locus of any holomorphic (or even formal) function Φ defined near the origin of $\mathbb{C}^2$.

6. Embedding in higher dimensions. Let σ be an algebraic almost closed 1-form on a Zariski open set of the origin in $\mathbb{C}^2$ (as constructed above) whose zero locus $\mathcal{Z}(\sigma)$ is both 0-dimensional and *not* the critical locus of any holomorphic potential function Φ near the origin in $\mathbb{C}^2$.

PROPOSITION 4. *The scheme $\mathcal{Z}(\sigma) \subset \mathbb{C}^2$ cannot be written as the critical locus $\mathcal{Z}(d\Phi)$ of any holomorphic function Φ on a non-singular variety.*

Proof. For contradiction, suppose

$$\mathcal{Z}(\sigma) = \mathcal{Z}(d\Phi)$$

for some holomorphic function Φ on a non-singular analytic variety A. Let $\mathcal{Z}$ denote $\mathcal{Z}(\sigma) = \mathcal{Z}(d\Phi)$.

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We show A can be cut down to two dimensions. Since $\mathcal{Z}$ has a two-dimensional Zariski tangent space, A has a product structure (perhaps after shrinking to an Euclidean open neighbourhood),

$$A = B \times C \tag{33}$$

with B non-singular of dimension 2 and

$$\mathcal{Z} \subset B \times \{c\} \subset B \times C = A\,.$$

Let $b \in B$ be the point at which $\mathcal{Z}$ is supported, and

$$p = (b, c) \in A\,.$$

Now $D(d\Phi)|_p$ is injective on T_cC because T_cC is a complement to the kernel T_bB. Since $D(d\Phi)|_p$ is symmetric, $\operatorname{Im} D(d\Phi)|_p$ lies in the annihilator of $\ker D(d\Phi)|_p$, which is $\Omega_C|_c$. Thus, the composition

$$T_C \hookrightarrow T_A \xrightarrow{D(d\Phi)} \Omega_A \twoheadrightarrow \Omega_C$$

on $\mathcal{Z}$ is injective at p. The composition is therefore an isomorphism at p, and hence, by openness, an isomorphism in a neighbourhood of $\mathcal{Z}$.

Thus, writing $(d_B\Phi, d_C\Phi)$ for $d\Phi = (d_{B\times C/C}\Phi, d_{B\times C/B}\Phi)$ in the product structure (33), we find that the zero locus

$$B' = \mathcal{Z}(d_C\Phi)$$

of $d_C\Phi$ is tangent to B at p and smooth and two-dimensional in a neighbourhood. Shrinking if necessary, we can assume that B' is everywhere non-singular and never tangent to the C fibres of $A = B \times C$.

Now $\mathcal{Z}$ is the zero locus of $d_B\Phi$ on B'. By the tangency condition, $\mathcal{Z}$ is the same as the zero locus of $(d\Phi)|_{B'} = d(\Phi|_{B'})$. Thus, $\mathcal{Z}$ is the critical locus of $\Phi|_{B'}$, with B' non-singular and two-dimensional, contradicting Theorem 3. □

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