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AVERAGING OF THE HAMILTON-JACOBI EQUATION IN INFINITE DIMENSIONS AND AN APPLICATION

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Abstract

We study the averaging of the Hamilton-Jacobi equation with fast variables in the viscosity solution sense in infinite dimensions. We prove that the viscosity solution of the original equation converges to the viscosity solution of the averaged equation and apply this result to the limit problem of the value function for an optimal control problem with fast variables.

1. Introduction

In this paper we are concerned with the averaging of the Hamilton-Jacobi-Bellman (HJB) equation with fast variables in n sense in Hilbert space X (X^* is the dual of X and $X = X^*$):

$$\begin{cases} V_t(t,x) + H(t, t/\varepsilon, x, V_x(t,x)) = 0, & (t,x) \in [0,T) \times X; \\ V(T,x) = g(x), & x \in X, \end{cases} \tag{1.1}$$

where $\varepsilon \in R^+ \equiv (0, +\infty)$; H and g are given functions and satisfy conditions given in Section 2. For the definition of the viscosity solution of the HJB equation, refer to [6, 7]. From [6, 7], we know that equation (1.1) has a unique viscosity solution $V_\varepsilon(t,x)$, which satisfies

$$|V_\varepsilon(t,x) - V_\varepsilon(\bar{t},\bar{x})| \le L_\varepsilon(|t-\bar{t}| + \|x-\bar{x}\|), \forall t, \bar{t} \in [0,T]; \|x\|, \|\bar{x}\| \le R \tag{1.2}$$

with R being a given constant and L_ε a constant which is dependent on ε and R probably.

Our purpose in this paper is to study the limiting behavior of $V_\varepsilon(t,x)$ as $\varepsilon \to 0^+$. This problem has been studied by Chaplais [3] and Barron [1] in finite dimensions

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(that is, X is R^n). The main difficulty in the passage from finite to infinite dimensions is to prove that the limit of $V_\varepsilon(t,x)$ exists. In [1, 3], the Arzela-Ascoli theorem has been used to deal with this, as *a priori* estimates of $V_\varepsilon(t,x)$ hold. However, in infinite dimensions, such a method does not apply again for there is no "appropriate" Arzela-Ascoli theorem available. In this paper, we overcome this difficulty by using the properties of the viscosity solution itself and the perturbed test function method. We directly prove that if L_ε is independent of ε, then $V_\varepsilon(t,x)$ converges to $V(t,x)$, which is a unique viscosity solution of the HJB equation

$$\begin{cases} V_t(t,x) + \overline{H}(t,x,V_x(t,x)) = 0, & (t,x) \in [0,T) \times X, \\ V(T,x) = g(x), & x \in X, \end{cases} \tag{1.3}$$

where $\overline{H} : [0,T] \times X \times X^* \to R^1$ is defined by

$$\overline{H}(t,x,p) = \int_0^1 H(t,s,x,p)\,ds, \quad \forall (t,x,p) \in [0,T] \times X \times X^*. \tag{1.4}$$

Next, we apply this result to study the limit problem of the value function of an optimal control problem with fast variables. The modeling of systems that have at least one component that oscillates rapidly is an important problem in optimal control theory. The motivation for this problem comes from the fact that the value function is used in the construction of feedback controls (see [9]).

2. Averaging of the Hamilton-Jacobi equation

First, let us make the following assumptions.

H1. $H : [0,T] \times R^+ \times X \times X^* \to R^1$ and $g : X \to R^1$ are continuous; $H(t,\cdot,x,p)$ is periodic with period 1. For any $t, \bar{t} \in [0,T]$; $s, \bar{s} \in R^+$; $x, \bar{x} \in X$; $p, q \in X^*$, there exist constants A_1, A_2, B_1, B_2, $\bar{C}$ such that

$$|H(t,s,x,p) - H(t,s,x,q)| \le (A_1\|x\| + B_1)\|p - q\|, \tag{2.1}$$

$$\begin{aligned} &|H(t,s,x,p) - H(\bar{t},\bar{s},\bar{x},p)| \\ &\qquad \le (A_2 + B_2\|p\|)(|t-\bar{t}| + |s-\bar{s}| + \|x-\bar{x}\|), \end{aligned} \tag{2.2}$$

$$|g(x) - g(\bar{x})| \le \bar{C}\|x - \bar{x}\|.$$

H2. Let $\overline{H} : [0,T] \times X \times X^* \to R^1$ be a continuous function, which satisfies

$$\overline{H}(t,x,p) = \int_0^1 H(t,s,x,p)\,ds, \quad \forall (t,x,p) \in [0,T] \times X \times X^*. \tag{2.3}$$

By assumption H1, for any $t, \bar{t} \in [0, T]$; $x, \bar{x} \in X$; $p, q \in X^*$, we have

$$|\overline{H}(t, x, p) - \overline{H}(t, x, q)| \le (A_1\|x\| + B_1)\|p - q\|, \tag{2.4}$$

$$|\overline{H}(t, x, p) - \overline{H}(\bar{t}, \bar{x}, p)| \le (A_2 + B_2\|p\|)(|t - \bar{t}| + \|x - \bar{x}\|). \tag{2.5}$$

From [6, 7], we know equation (1.1) has a unique viscosity solution which satisfies (1.2), and equation (1.3) has a unique viscosity solution $V(t, x)$ satisfying

$$|V(t, x) - V(\bar{t}, \bar{x})| \le \bar{L}(|t - \bar{t}| + \|x - \bar{x}\|), \forall t, \bar{t} \in [0, T]; \|x\|, \|\bar{x}\| \le R \tag{2.6}$$

with R being a given constant and $\bar{L}$ a constant which is only dependent on R.

LEMMA 2.1. *Let* H1, H2 *hold,* $Z(t, x, y) = V_\varepsilon(t, x) - V(t, y), \forall(t, x, y) \in [0, T] \times X \times X$; *then* $Z(\cdot, \cdot, \cdot)$ *is a viscosity subsolution of the* HJB *equation*

$$\begin{cases} Z_t(t, x, y) + H(t, t/\varepsilon, x, Z_x(t, x, y)) - \overline{H}(t, y, -Z_y(t, x, y)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall(t, x, y) \in [0, T) \times X \times X, \\ Z(T, x, y) = g(x) - g(y), \qquad (x, y) \in X \times X. \end{cases}$$

The proof can be seen in [6] or [7].

THEOREM 2.2. *Let* H1, H2 *hold and, in* (1.2), $L_\varepsilon = L$ *be independent of* ε; *then* $\lim_{\varepsilon \to 0^+} V_\varepsilon(t, x) = V(t, x)$ *uniformly on any bounded subset of* $[0, T] \times X$.

PROOF. We give a proof by contradiction. Suppose the assertion is not true. Then

$$\left.\begin{array}{l} \text{there is a constant } \sigma' > 0, \text{ a bounded subset } \mathcal{O} : [0, T] \times B_R(0), \\ (B_R(0) = \{x \in X \mid \|x\| < R\}) \text{ and a subsequence } \{\varepsilon_k\} \subset \{\varepsilon\}, \text{ such} \\ \text{that } \varepsilon_k < 1/k \text{ and } \sup\{V_{\varepsilon_k}(t, x) - V(t, x) : (t, x) \in \mathcal{O}\} > \sigma' > 0. \end{array}\right\} \tag{2.7}$$

Take $\delta_0 = \min\{\sigma'/(6L + 6\bar{L}), 2R/(2A_1R + B_1)\}$. If $|t_1 - t_2| + \|x_1 - x_2\| \le \delta_0$, then we have

$$|V_\varepsilon(t_1, x_1) - V_\varepsilon(t_2, x_2)| \le \sigma'/6, \tag{2.8}$$

$$|V(t_1, x_1) - V(t_2, x_2)| \le \sigma'/6. \tag{2.9}$$

Combining (2.7)–(2.9) we get

$$\sup\{V_{\varepsilon_k}(t, x) - V(t, x) : (t, x) \in [\delta_0/2, T] \times B_R(0)\} > \sigma'/2 > 0. \tag{2.10}$$

Let $m = [2T/\delta_0]$, $S'_i = [i\delta_0/2, (i + 1)\delta_0/2] \times B_R(0)$, $T'_i = (i + 1)T, 1 \le i \le m$ and $S'_m = [T - \delta_0/2, T) \times B_R(0)$; then $[\delta_0/2, T) \times B_R(0) = \cup_{i=1}^m S'_i$. Because $V_{\varepsilon_k}(T, x) - V(T, x) = 0$ and (2.10) holds, there exists some $1 \le i_0 \le m$ such that

$$\overline{\lim_{k \to \infty}} \sup\{V_{\varepsilon_k}(t, x) - V(t, x) : (t, x) \in S'_{i_0}\} > 0 \tag{2.11}$$

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and

$$\overline{\lim_{k\to\infty}} \sup\{V_{\varepsilon_k}(T'_{i_0}, x) - V(T'_{i_0}, x)\} \le 0. \tag{2.12}$$

Assume $i_0 = m$; we can deal with other cases similarly. Let

$$T_0 = \delta_0, \ L_0 = 2R/\delta_0, \tag{2.13}$$

$$\sigma = \frac{1}{2}\overline{\lim_{k\to\infty}} \sup\{V_{\varepsilon_k}(t, x) - V(t, x) : (t, x) \in S'_m\} > 0.$$

We replace $\{k\}$ by a subsequence of $\{k\}$, if necessary, such that

$$\sup\{V_{\varepsilon_k}(t, x) - V(t, x) : (t, x) \in S'_m\} > \sigma > 0, \ \forall k.$$

Because

$$\begin{aligned} S'_m &\subset \{(t, x) \in [T - \delta_0/2, T) \times X : \|x\| < L_0(t - T + T_0)\} \\ &\subset S_1 = \{(t, x) \in (T - T_0, T) \times X : \|x\| < L_0(t - T + T_0)\}, \end{aligned}$$

then

$$\sup\{V_{\varepsilon_k}(t, x) - V(t, x) : (t, x) \in S_1\} > \sigma > 0. \tag{2.14}$$

By (2.4) and (2.13), $\forall (t, x) \in S_1$; $p, q \in X^*$ we have

$$\begin{aligned} |\overline{H}(t, x, p) - \overline{H}(t, x, q)| &\le (A_1\|x\| + B_1)\|p - q\| \\ &\le [A_1L_0(t - T + T_0) + B_1]\|p - q\| \\ &\le (A_1L_0T_0 + B_1)\|p - q\| \le L_0\|p - q\|. \end{aligned} \tag{2.15}$$

Similarly,

$$|H(t, s, x, p) - H(t, s, x, q)| \le L_0\|p - q\|. \tag{2.16}$$

We split the following proof into several steps.

Step 1. Definition of auxiliary functions and sets.

Define

$$S = \{(t, x, y) \in (T - T_0, T) \times X \times X : \|x\|, \|y\| < L_0(t - T + T_0)\}. \tag{2.17}$$

Take $\beta, \delta > 0$ with $\beta + \delta < L_0T_0$. Let

$$d(x, y) = \|x - y\|, \tag{2.18}$$

$$\nu(x) = (\beta + \|x\|^2)^{1/2}. \tag{2.19}$$

By (1.2) and (2.6) (theorem assumptions), $V_\varepsilon(\cdot,\cdot)$ is locally bounded. Therefore, we can choose $K > 0$ such that

$$K > \sup\{V_{\varepsilon_k}(t,x) - V(t,y) : (t,x,y) \in S\}. \tag{2.20}$$

Let $G(\cdot) \in C^\infty(R^1)$, which satisfies

$$G'(r) \geq 0, \quad G(r) = \begin{cases} 0, & r \leq -\delta, \\ 2K, & r \geq 0. \end{cases} \tag{2.21}$$

For $0 < \alpha < 1$, $\lambda > 0$, we set

$$\begin{aligned} \phi(t,x,y) = {} & V_{\varepsilon_k}(t,x) - V(t,y) \\ & - \{d(x,y)^2/\alpha + G(\nu(x) - L_0(t - T + T_0)) - \lambda(t-T)\}. \end{aligned} \tag{2.22}$$

Step 2. Properties of $\phi(t,x,y)$.
By (2.17) and (2.22), we see that $\overline{S}$ (the closure of S) is bounded, convex and $\phi(\cdot,\cdot,\cdot)$ is bounded, continuous on $\overline{S}$. According to Stegall [12],

$$\left.\begin{array}{l} \text{there are elements } a_k \in R^1;\ p_k, q_k \in X^* \text{ which satisfy } |a_k| + \|p_k\| + \\ \|q_k\| < \alpha, \text{ such that } (t,x,y) \to \phi(t,x,y) + a_k t + \langle p_k, x\rangle + \langle q_k, y\rangle \\ \text{attains its maximum over } \overline{S} \text{ at some point } (t_k, x_k, y_k). \end{array}\right\} \tag{2.23}$$

Let

$$\varphi(t,x,y) = \phi(t,x,y) + a_k t + \langle p_k, x\rangle + \langle q_k, y\rangle \tag{2.24}$$

and so

$$\varphi(t_k, x_k, y_k) \geq \varphi(t_k, x_k, x_k). \tag{2.25}$$

Using (2.22), (2.23) and (2.24) we have

$$d(x_k, y_k)^2/\alpha \leq V(t_k, x_k) - V(t_k, y_k) + 4(L_0 + 1)T\alpha. \tag{2.26}$$

By (2.6), we can get

$$d(x_k, y_k) \leq L_1 \alpha^{1/2}, \tag{2.27}$$

$$d(x_k, y_k)^2/\alpha \leq L_2 \alpha^{1/2}, \tag{2.28}$$

where L_1, L_2 are constants which are only dependent on L_0, T_0, L and $\bar{L}$. Let

$$M_{\varepsilon_k} = \sup\{\phi(t,x,y) : (t,x,y) \in S\}. \tag{2.29}$$

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Set

$$S_{\beta,\delta} = \{(t,x) \in S_1 : (\beta + \|x\|^2)^{1/2} \le L_0(t - T + T_0) - \delta\}. \tag{2.30}$$

We can choose β and δ so small that $S_{\beta,\delta} \neq \emptyset$. Since

$$G(\nu(x) - L_0(t - T + T_0)) = 0, \quad \forall (t,x) \in S_{\beta,\delta}.$$

Using (1.2), (2.6), (2.14) and (2.22), we choose β, δ and λ so small that

$$\begin{aligned}
&\sup\{\phi(t,x,x) : (t,x) \in S_{\beta,\delta}\} \\
&\quad = \sup\{V_{\varepsilon_k}(t,x) - V(t,x) + \lambda(t - T) : (t,x) \in S_{\beta,\delta}\} > \sigma/2.
\end{aligned}$$

By (2.29), for any k, we have

$$M_{\varepsilon_k} > \sigma/2. \tag{2.31}$$

Consider the following three possible cases

(I) $t_k \to T$ or

(II) $L_0(t_k - T + T_0) - \max\{\|x_k\|, \|y_k\|\} \to 0$ or

(III) for some $\eta > 0$, $N_1 > 0$, if $k \ge N_1$, then $\eta < t_k < T - \eta$; $\|x_k\|, \|y_k\| < L_0(t_k - T + T_0) - \eta$.

By passing to a subsequence of $\{(t_k, x_k, y_k)\}$ if necessary, we can always reduce to a case in which one of (I)–(III) holds.

Step 3. We claim that we can choose α so small that case (I) is impossible. In fact, noticing that $V_{\varepsilon_k}(T,x) = V(T,x) = g(x)$, $\forall x \in X$, by (2.31), (2.29), (2.23), (2.22), (1.2) and (2.6), we have

$$\begin{aligned}
\sigma/2 < M_{\varepsilon_k} &\le \phi(t_k, x_k, y_k) + L_1\alpha \\
&\le V_{\varepsilon_k}(t_k, x_k) - V(t_k, y_k) + L_1\alpha \\
&\le (V_{\varepsilon_k}(t_k, x_k) - V_{\varepsilon_k}(T, x_k)) + (V_{\varepsilon_k}(T, x_k) - V(t_k, y_k)) \\
&\quad - (V(T, x_k) - V(T, y_k)) + L_1\alpha \\
&\le (L + \bar{L})|T - t_k| + L_1\bar{C}\alpha^{1/2} + L_1\alpha.
\end{aligned}$$

From the above, we see that case (I) is not possible if we choose α small enough. Thus, our claim holds.

Step 4. We claim that case (II) is not possible provided that α is small enough. Using (2.31), (2.29) and (2.22), we have

$$\begin{aligned}
\sigma/2 < M_{\varepsilon_k} &\le \phi(t_k, x_k, y_k) + L_1\alpha \\
&\le V_{\varepsilon_k}(t_k, x_k) - V(t_k, y_k) - G(\nu(x_k) - L_0(t_k - T + T_0)) + L_1\alpha.
\end{aligned}$$

If $L_0(t_k - T + T_0) - \|x_k\| \to 0$, then

$$G(\nu(x_k) - L_0(t_k - T + T_0)) = -2K.$$

By (2.20), we have

$$\sigma/2 \le -K + L_1\alpha.$$

If α is small enough, we get a contradiction. Similarly, if $L_0(t_k - T + T_0) - \|y_k\| \to 0$, we also get a contradiction. Hence, our claim holds.

Step 5. Finally we consider case (III).

Let

$$\begin{aligned}\Delta = \{(t, x, y) \in (T - T_0, T) \times X \times X : \eta < t < T - \eta;\\ \|x\|, \|y\| < L_0(t - T + T_0) - \eta\}.\end{aligned}$$

Let case (III) hold, then there exists $\bar{r} > 0$ which is independent of k, such that

$$S_k = \{(t, x, y) \in S : |t - t_k|^2 + \|x - x_k\|^2 + \|y - y_k\|^2 \le \bar{r}^2\} \subset S, \tag{2.32}$$

and $S_k \ne \emptyset$. Let

$$\overline{\phi}(t, x, y) = d(x, y)^2/\alpha + G(\nu(x) - L_0(t - T + T_0)) \tag{2.33}$$

and so

$$\overline{\phi}_t(t, x, y) = -L_0 G'(\nu(x) - L_0(t - T + T_0)), \tag{2.34}$$

$$\begin{aligned}\overline{\phi}_x(t, x, y) &= 2d(x, y)d_x(x, y)/\alpha + G'(\nu(x) - L_0(t - T + T_0))D\nu(x)\\ &= 2(x - y)/\alpha + G'(\nu(x) - L_0(t - T + T_0))x/\nu(x),\end{aligned} \tag{2.35}$$

$$\overline{\phi}_y(t, x, y) = 2d(x, y)d_y(x, y)/\alpha = -2(x - y)/\alpha. \tag{2.36}$$

Let

$$\frac{dr_1(s)}{ds} = \overline{H}(t_k, x_k, \overline{\phi}_x(t_k, x_k, y_k)) - H(t_k, s, x_k, \overline{\phi}_x(t_k, x_k, y_k)). \tag{2.37}$$

By assumptions H1 and H2, we know that $r_1(\cdot)$ is periodic with period 1 and $r_1(\cdot) \in C^1(R^1)$. Since $G \in C^\infty(R^1)$, there exists a constant $\bar{C}$ such that

$$\sup\{|G'(r)|, r \le L_0 T\} < \bar{C}.$$

By (2.35), we have

$$\begin{aligned}&\overline{H}(t_k, x_k, \overline{\phi}_x(t_k, x_k, y_k)) - H(t_k, s, x_k, \overline{\phi}_x(t_k, x_k, y_k))\\ &\le [\|\phi_x(t_k, x_k, y_k)\| + 1]L(1 + \|x_k\|)\\ &\le (2L_0T_0/\alpha + \bar{C} + 1)(1 + L_0T_0)L.\end{aligned}$$

Hence $r_1(\cdot)$ is uniformly bounded in k. Assume for any k, $\sup|r_1(\cdot)| < \bar{R}$ ($\bar{R}$ may be dependent on α). Consider

$$\begin{aligned}\psi(t,x,y) = \varphi(t,x,y) - \varepsilon_k r_1(t/\varepsilon_k)\\ - 3[\bar{R}/(k\bar{r}^2)](d(x,x_k)^2 + d(y,y_k)^2 + |t-t_k|^2)\end{aligned} \tag{2.38}$$

and so

$$\psi(t_k,x_k,y_k) \geq \varphi(t_k,x_k,y_k) - \bar{R}/k \geq \varphi(t,x,y) - \bar{R}/k. \tag{2.39}$$

As $(t,x,y) \in \partial S_k$, we have

$$\begin{aligned}\psi(t,x,y) &= \varphi(t,x,y) - \varepsilon_k r_1(t/\varepsilon_k) - 3\bar{R}/k\\ &\leq \varphi(t,x,y) - 2\bar{R}/k\\ &\leq \psi(t_k,x_k,y_k) - \bar{R}/k.\end{aligned} \tag{2.40}$$

According to Stegall [12], for any $\theta > 0$,

there are elements $a_2 \in R^1$ and $p_2, q_2 \in X^*$ which satisfy $|a_2| + \|p_2\| + \|q_2\| \leq \min\{\theta,\alpha\}$ such that $(t,x,y) \to \psi(t,x,y) + a_2 t + \langle p_2, x\rangle + \langle q_2, y\rangle$ attains its maximum over S_k at some point $(\bar{t},\bar{x},\bar{y})$. (2.41)

Take $\theta < \bar{R}/[k(4L_0T_0 + T)]$; by (2.38), the maximum point must be an interior point of S_k. Let

$$\begin{gathered} Z(t,x,y) = V_{\varepsilon_k}(t,x) - V(t,y),\\ a = a_k + a_2,\ p = p_k + p_2,\ q = q_k + q_2.\end{gathered}$$

Therefore $(t,x,y) \to Z(t,x,y) - \{\overline{\phi}(t,x,y) - \lambda(t-T) + \varepsilon_k r_1(t/\varepsilon_k) + 3[\bar{R}/(k\bar{r}^2)] \times (d(x,x_k)^2 + d(y,y_k)^2 + |t-t_k|^2) - at - \langle p,x\rangle - \langle q,y\rangle\}$ attains its maximum over S_k at $(\bar{t},\bar{x},\bar{y})$. By Lemma 2.1 and the definition of the viscosity solution, we have

$$\begin{aligned}\lambda \leq \overline{\phi}_t(\bar{t},\bar{x},\bar{y}) + \frac{dr_1(\bar{t}/\varepsilon_k)}{ds} + 6[\bar{R}/(k\bar{r}^2)]|\bar{t}-t_k| - a\\ + H(\bar{t},\bar{t}/\varepsilon_k,\bar{x},\overline{\phi}_x(\bar{t},\bar{x},\bar{y}) + 6[\bar{R}/(k\bar{r}^2)](\bar{x}-x_k) - p)\\ - \overline{H}(\bar{t},\bar{y},-\overline{\phi}_y(\bar{t},\bar{x},\bar{y}) - 6[\bar{R}/(k\bar{r}^2)](\bar{y}-y_k) + q).\end{aligned} \tag{2.42}$$

Thus

$$\begin{aligned}\lambda \leq \overline{\phi}_t(\bar{t},\bar{x},\bar{y}) + 6[\bar{R}/(k\bar{r}^2)]|t-t_k| - a + \{\overline{H}(t_k,x_k,\overline{\phi}_x(t_k,x_k,y_k))\\ - \overline{H}(t_k,y_k,-\overline{\phi}_y(t_k,x_k,y_k) - 6[\bar{R}/(k\bar{r}^2)](\bar{y}-y_k) + q)\}\\ + \{H(\bar{t},\bar{t}/\varepsilon_k,\bar{x},\overline{\phi}_x(\bar{t},\bar{x},\bar{y}) + 6[\bar{R}/(k\bar{r}^2)](\bar{x}-x_k) - p)\\ - H(t_k,\bar{t}/\varepsilon_k,x_k,\overline{\phi}_x(t_k,x_k,y_k))\}.\end{aligned} \tag{2.43}$$

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Noticing that $G \in C^\infty(R^1)$, $\overline{\phi}_x$, $\overline{\phi}_y$ are Lipschitz continuous in all arguments and combining (2.34)–(2.36) with the assumptions, we have

$$\lambda \le A_3\bar{r} + A_4\bar{r}/\alpha + 16\bar{R}/(k\bar{r}) + 3\alpha, \tag{2.44}$$

where A_3, A_4 are constants which are independent of α, k and $\bar{r}$. In the above, first let $k \to \infty$, then let $\bar{r} \to 0$, finally let $\alpha \to 0$; then we get

$$\lambda \le 0.$$

Thus, we obtain a contradiction. So Theorem 2.2 is proved.

3. Application: the limit problem of the value function for an optimal control problem

Consider the following state equation in Hilbert space X:

$$\begin{cases} \dfrac{dx(r)}{dr} = Ax(r) + f(r, r/\varepsilon, x(r), u(r)), & r \in (t, T], \\ x(t) = x, & x \in X, \end{cases} \tag{3.1}$$

where $A : \mathscr{D}(A) \subset X \to X$ is the generator of some C_0 semigroup e^{At} and f : $[0, T] \times R^1 \times X \times U \to X$ is a given map with U being a metric space in which the control $u(\cdot)$ takes values. Let

$$\mathscr{U}[0, T] = \{u(\cdot) : [0, T] \to U | u(\cdot) \text{ measurable}\}.$$

The cost function is given by

$$J^\varepsilon_{t,x}(u(\cdot)) = \int_t^T f^0\left(r, \frac{r}{\varepsilon}, x(r), u(r)\right) dr + g(x(T)); \tag{3.2}$$

the value function is defined by

$$V_\varepsilon(t, x) = \inf_{u(\cdot) \in \mathscr{U}[t,T]} J^\varepsilon_{t,x}(u(\cdot)). \tag{3.3}$$

Assume $f(t, \cdot, x, u)$, $f^0(t, \cdot, x, u)$ are periodic with period 1. Our aim is to see whether the limit of V_ε exists as $\varepsilon \to 0^+$ and, if the limit exists, how to determine it.

In this section, we need the following assumptions.

H3. Assume $f : [0, T] \times R^+ \times X \times U \to X, f^0 : [0, T] \times R^+ \times X \times U \to R^1 (R^+ = [0, \infty))$ and $g : X \to R^1$ are continuous; $f(t, \cdot, x, u)$ and $f^0(t, \cdot, x, u)$ are periodic with period 1. There exists a constant $L > 0$ and a continuous function

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$\omega : R^+ \times R^+ \to R^+$ which is increasing in all arguments with $\omega(r, 0) = 0, \forall\, r \geq 0$. For any $x, \bar{x} \in X$; $t, \bar{t} \in [0, T]$; $s, \bar{s} \in R^+$; $u \in U$, the following hold:

$$\begin{cases} \|f(t, s, x, u) - f(\bar{t}, \bar{s}, \bar{x}, u)\| \leq L(|t - \bar{t}| + \|x - \bar{x}\|) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \omega(\max\{\|x\|, \|\bar{x}\|\}, |s - \bar{s}|), \\ \|f(t, s, 0, u)\| \leq L, \end{cases}$$

$$\begin{cases} |f^0(t, s, x, u) - f^0(\bar{t}, s, \bar{x}, u)| \leq L(|t - \bar{t}| + \|x - \bar{x}\|), \\ |f^0(t, s, 0, u)| \leq L, \end{cases}$$

$$\begin{cases} |g(x) - g(\bar{x})| \leq L\|x - \bar{x}\|, \\ |g(0)| \leq L. \end{cases}$$

H4. Assume that there exists a sequence of linear bounded operators $\{A_\mu, \mu \subset (0, +\infty)\}$, $\sup \mu = +\infty$, for any $z(\cdot) \in C([0, T]; X)$,

$$\lim_{\mu \to +\infty} \sup_{0 \leq s \leq t \leq T} \left\| \int_s^t \left[e^{A(t-r)} - e^{A_\mu(t-r)}\right] f\left(r, \frac{r}{\varepsilon}, z(r), u(r)\right) dr \right\| = 0$$

uniformly in $u(\cdot) \in \mathscr{U}$, $\varepsilon \in (0, +\infty)$. For any $x \in X$, $\lim_{\mu \to +\infty} \left\|e^{A_\mu t}x - e^{At}x\right\| = 0$ uniformly in $t \in [0, T]$, and $\|e^{A_\mu t}\| \leq Le^{\omega' t}$, $t \in [0, T]$, $\mu > 0$, where ω' is a constant.

Consider the following optimal control problem:

$$\begin{cases} \dot{x}^\mu(r) = A_\mu x^\mu(r) + f(r, r/\varepsilon, x^\mu(r), u(r)), & r \in (t, T], \\ x^\mu(t) = x, & x \in X. \end{cases} \tag{3.4}$$

The cost function is given by

$$J_{t,x}^{\mu,\varepsilon}(u(\cdot)) = g(x^\mu(T)) + \int_t^T f^0\left(r, \frac{r}{\varepsilon}, x^\mu(r), u(r)\right) dr; \tag{3.5}$$

the value function is defined by

$$V_\varepsilon^\mu(t, x) = \inf_{u(\cdot) \in \mathscr{U}} J_{t,x}^{\mu,\varepsilon}(u(\cdot)). \tag{3.6}$$

THEOREM 3.1. *Let* H3, H4 *hold.* $V_\varepsilon^\mu(\cdot, \cdot)$ *is a unique viscosity solution of the* HJB *equation*

$$\begin{cases} V_t(t, x) + H_\mu(t, t/\varepsilon, x, V_x(t, x)) = 0, & \forall (t, x) \in [0, T) \times X, \\ V(T, x) = g(x), & x \in X, \end{cases} \tag{3.7}$$

where $H_\mu(t, s, x, p) = \langle p, A_\mu x\rangle + \inf_{u \in U}\{p \cdot f(t, s, x, u) + f^0(t, s, x, u)\}$.

The proof of Theorem 3.1 can be seen in [1, 7, 8].

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THEOREM 3.2. *Let* H3, H4 *hold. Then for any* $R > 0$ *there exist constants* M *and* C_1, *which are dependent on* R, *such that*

$$\begin{aligned} &\text{(i)} \quad |V_\varepsilon^\mu(t,x)| \le M, \\ &\text{(ii)} \quad |V_\varepsilon^\mu(t,x) - V_\varepsilon^\mu(\bar{t},\bar{x})| \le C_1(|t-\bar{t}| + \|x-\bar{x}\|), \end{aligned} \tag{3.8}$$

$\forall \varepsilon \in R^+$; $t, \bar{t} \in [0,T]$; $\|x\|, \|\bar{x}\| \le R$.

The proof of the theorem is similar to the proof of [1, Lemma 1.2].

Define

$$\overline{f}(t,x,u(\cdot)) = \int_0^1 f(t,s,x,u(s))\,ds,$$
$$\overline{f^0}(t,x.u(\cdot)) = \int_0^1 f^0(t,s,x,u(s))\,ds,$$

where $u(\cdot) \in \mathscr{A} = \{u : [0,1] \to U \mid u \text{ measurable }\}$.

Define the averaging of the Hamiltonian

$$\overline{H}_\mu(t,x,p) = \inf_{u(\cdot)\in\mathscr{A}}\{p\cdot\overline{f}(t,x,u(\cdot)) + \overline{f^0}(t,x,u(\cdot))\} + \langle p, A_\mu x\rangle,$$
$$\forall (t,x,p) \in [0,T]\times X \times X^*. \tag{3.9}$$

LEMMA 3.3. *Let* H3, H4 *hold. Then* $\forall (t,x,p) \in [0,T]\times X\times X^*$,

$$\int_0^1 H_\mu(t,s,x,p)ds = \overline{H}_\mu(t,x,p). \tag{3.10}$$

PROOF. Define the functional

$$J(y) = \int_y^1 \{p\cdot f(t,r,x,u(r)) + f^0(t,r,x,u(r))\}\,dr.$$

Defined the value function

$$U(y) = \inf_{u(\cdot)\in\mathscr{A}} J(y).$$

According to [6], we know that $U(\cdot)$ is a unique viscosity solution of the HJB equation

$$\begin{cases} \dfrac{dU(y)}{dy} + \inf\limits_{u\in U}\{p\cdot f(t,y,x,u) + f^0(t,y,x,u)\} = 0, \quad y \in [0,1), \\ U(1) = 0. \end{cases} \tag{3.11}$$

Obviously, $U(y) = \int_y^1 \inf_{u\in U}\{p\cdot f(t,r,x,u) + f^0(t,r,x,u)\}\,dr \in C^1[0,1]$. Thus, the Lemma holds.

By Lemma 3.3 and H3, H4 we have the following theorem.

THEOREM 3.4. *Let* H3, H4 *hold. Then* $\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x) = \overline{V^\mu}(t,x)$, $\forall (t,x) \in [0,T]\times X$ *and* $\overline{V^\mu}$ *is a unique viscosity solution of the* HJB *equation*

$$\begin{cases} V_t(t,x) + \inf_{u(\cdot)\in\mathscr{A}}\{D_x V(t,x)\overline{f}(t,x,u(\cdot)) + \overline{f^0}(t,x,u(\cdot))\} \\ \qquad +\langle D_x V(t,x), A_\mu x\rangle = 0, \qquad \forall (t,x)\in[0,T)\times X, \\ V_t(T,x) = g(x), \qquad x\in X. \end{cases} \tag{3.12}$$

Finally we give a method to determine the limit of the original value function.

THEOREM 3.5. *Let* H3, H4 *hold. Then* $\lim\limits_{\mu\to\infty}\lim\limits_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x) = \lim\limits_{\varepsilon\to 0^+} V^\varepsilon(t,x)$, $\forall (t,x) \in [0,T]\times X$.

Before proving the above theorem we give the following lemma.

LEMMA 3.6. *Let* H3, H4 *hold and* $x_{t,x}(\cdot, u(\cdot))$, $x_{t,x}^\mu(\cdot,u(\cdot))$ *stand for the solution of systems* (3.1) *and* (3.4) *respectively with the initial value being* $(t,x)\in[0,T]\times X$ *and the control being* $u(\cdot)\in\mathscr{U}$. *Then*

(i) $\lim_{\mu\to\infty}\|x_{t,x}(r,u(\cdot)) - x_{t,x}^\mu(r,u(\cdot))\| = 0$ *uniformly in* $u(\cdot)\in\mathscr{U}$, $\varepsilon\in R^+$, $r\in[0,T]$.

(ii) $\overline{\lim\limits_{\varepsilon\to 0^+}} \sup_{u(\cdot)\in\mathscr{U}} \sup_{t\le r\le T} \|x_{t,x}^\mu(r,u(\cdot)) - x_{t,x}(r,u(\cdot))\| = 0$,
$\underline{\lim\limits_{\varepsilon\to 0^+}} \sup_{u(\cdot)\in\mathscr{U}} \sup_{t\le r\le T} \|x_{t,x}^\mu(r,u(\cdot)) - x_{t,x}(r,u(\cdot))\| = 0$.

PROOF. By H3, we have

$$\begin{aligned}\left\|x_{t,x}(r) - x_{t,x}^\mu(r)\right\| \le{}& \left\|\left(e^{A(r-t)} - e^{A_\mu(r-t)}\right)x\right\| \\ &+ \left\|\int_t^r \left[e^{A(r-s)} - e^{A_\mu(r-s)}\right] f(s, s/\varepsilon, x(s), u(s))ds\right\| \\ &+ Le^{\omega T}\int_t^r \left\|x_{t,x}(s) - x_{t,x}^\mu(s)\right\| ds.\end{aligned}$$

Using Gronwall's inequality we obtain our conclusion. Part (ii) is a corollary of (i).

PROOF OF THEOREM 3.5. For any $u(\cdot)\in\mathscr{U}$, by H3, we have $|J_{t,x}^{\mu,\varepsilon}(u(\cdot)) - J_{t,x}^\varepsilon(u(\cdot))| \le L\|x_{t,x}^\mu(T) - x_{t,x}(T)\| + \int_t^T \|x_{t,x}^\mu(r) - x_{t,x}(r)\| dr$ and so

$$|J_{t,x}^{\mu,\varepsilon}(u(\cdot)) - J_{t,x}^\varepsilon(u(\cdot))| \le \bar{C}\sup_{t\le r\le T}\|x_{t,x}^\mu(r) - x_{t,x}(r)\|.$$

In the above, taking the infimum in $u(\cdot) \in \mathscr{U}$ on both sides respectively, we have

$$|V_\varepsilon^\mu(t,x) - V_\varepsilon(t,x)| \le \bar{C} \sup_{u(\cdot)\in\mathscr{U}} \sup_{t\le r\le T} \|x_{t,x}^\mu(r) - x_{t,x}(r)\|.$$

In the above, taking the superior limit and the inferior limit in $\varepsilon \to 0^+$ on both sides respectively, we have

$$\begin{aligned} &\left|\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x) - \overline{\lim_{\varepsilon\to 0^+}} V_\varepsilon(t,x)\right| \\ &\qquad \le \bar{C}\,\overline{\lim_{\varepsilon\to 0^+}} \sup_{u(\cdot)\in\mathscr{U}} \sup_{t\le r\le T} \left\|x_{t,x}^\mu(r) - x_{t,x}(r)\right\|, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\left|\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x) - \underline{\lim}_{\varepsilon\to 0^+} V_\varepsilon(t,x)\right| \\ &\qquad \le \bar{C}\,\overline{\lim_{\varepsilon\to 0^+}} \sup_{u(\cdot)\in\mathscr{U}} \sup_{t\le r\le T} \left\|x_{t,x}^\mu(r) - x_{t,x}(r)\right\|. \end{aligned} \tag{3.14}$$

In (3.13) and (3.14), taking the superior limit and the inferior limit in $\mu \to \infty$ on both sides respectively, we get

$$\begin{aligned} &\left|\underline{\lim}_{\mu\to\infty}\left(\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x)\right) - \overline{\lim_{\varepsilon\to 0^+}} V_\varepsilon(t,x)\right| \\ &\qquad \le \bar{C} \lim_{\mu\to\infty} \overline{\lim_{\varepsilon\to 0^+}} \sup_{u\in\mathscr{U}} \sup_{t\le r\le T} \left\|x_{t,x}^\mu(r) - x_{t,x}(r)\right\|, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\left|\overline{\lim_{\mu\to\infty}}\left(\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x)\right) - \underline{\lim}_{\varepsilon\to 0^+} V_\varepsilon(t,x)\right| \\ &\qquad \le \bar{C} \lim_{\mu\to\infty} \overline{\lim_{\varepsilon\to 0^+}} \sup_{\mu\in\mathscr{U}} \sup_{t\le r\le T} \left\|x_{t,x}^\mu(r) - x_{t,x}(r)\right\|. \end{aligned} \tag{3.16}$$

By Lemma 3.6, we know that the right side of (3.15), (3.16) equals zero, so

$$\underline{\lim}_{\mu\to\infty}\left(\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x)\right) = \overline{\lim_{\varepsilon\to 0^+}} V_\varepsilon(t,x) \ge \underline{\lim}_{\varepsilon\to 0^+} V_\varepsilon(t,x) = \overline{\lim_{\mu\to\infty}}\left(\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x)\right).$$

Thus $\lim_{\varepsilon\to 0^+} V_\varepsilon(t,x) = \lim_{\mu\to\infty}\lim_{\varepsilon\to 0^+} V_\varepsilon^\mu(t,x)$. Hence Theorem 3.5 holds.

Combining Theorems 3.4 and 3.5, we can determine the limit of $V_\varepsilon(\cdot,\cdot)$ as $\varepsilon \to 0^+$.

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References

[1] E. N. Barron, "Averaging in Lagrange and minimax problems of optimal control", *SIAM J. Control and Optim.* **31** (1993) 1630–1652.

[2] A. Bensoussan, J. L. Lions and G. Papanicolaon, *Asymptotic Analysis for Periodic Structures* (North Holland, New York, 1987).

[3] F. Chaplais, "Averaging and deterministic optimal control", *SIAM J. Control and Optim.* **25** (1987) 767–780.

[4] M. G. Crandall, L. C. Evans and P. L. Lions, "Some properties of viscosity solutions of Hamilton-Jacobi equations", *Trans. Amer. Math. Soc.* **282** (1984) 487–502.

[5] M. G. Crandall and P. L. Lions, "Viscosity solutions of Hamilton-Jacobi equations", *Trans. Amer. Math. Soc.* **277** (1983) 1–42.

[6] M. G. Crandall and P. L. Lions, "Hamilton-Jacobi equations in infinite dimension. I Uniqueness of viscosity solutions", *J. Funct. Anal.* **62** (1985) 379–396.

[7] M. G. Crandall and P. L. Lions, "Hamilton-Jacobi equations in infinite dimension. II Existence of viscosity solutions", *J. Funct. Anal.* **65** (1986) 368–405.

[8] M. G. Crandall and P. L. Lions, "Hamilton-Jacobi equations in infinite dimension. IV Hamilton with unbounded linear term", *J. Funct. Anal.* **90** (1990) 237–283.

[9] X. J. Li and J. M. Yong, *Optimal Control Theory for Infinite Dimensional Systems* (Birkhäuser, Boston, 1995).

[10] P. L. Lions, *Generalized Solutions of Hamilton-Jacobi Equations* (Pitman, London, 1982).

[11] A. Pazy, *Semigroup of Linear Operators and Application to Partial Differential Equations* (Springer, New York, Berlin, 1983).

[12] C. Stegall, "Optimization of functions on certain subsets of Banach space", *Math. Ann.* **236** (1978) 171–176.

[13] J. M. Yong, *Dynamic Programming Method and HJB Equation* (Shanghai Science Technology Press, Shanghai, 1992).