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VARIATIONS ON THOMPSON'S CHARACTER DEGREE THEOREM

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Abstract. If P is a Sylow-p-subgroup of a finite p-solvable group G, we prove that $G' \cap \mathbf{N}_G(P) \subseteq P$ if and only if p divides the degree of every irreducible non-linear p-Brauer character of G. More generally if π is a set of primes containing p and G is π-separable, we give necessary and sufficient group theoretic conditions for the degree of every irreducible non-linear p-Brauer character to be divisible by some prime in π. This can also be applied to degrees of ordinary characters.

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A well-known Theorem of Thompson (12.2 of [**3**]) states that a finite group G has a normal p-complement if the degree of every non-linear irreducible complex character of G is divisible by p. Gow and Humphreys [**1**] proved that the same conclusion holds if p divides the degree of every non-linear irreducible q-Brauer character for a prime $q \neq p$ when G is q-solvable. They also showed that there exist p-solvable groups of arbitrary p-length, where p divides the degree of every non-linear irreducible p-Brauer character. We use known results about the McKay conjecture to give necessary and sufficient conditions for p to divide the degree of every non-linear irreducible p-Brauer character of a p-solvable group G.

Theorem A. *The degree of every non-linear p-Brauer character of a p-solvable group G is divisible by p if and only* if $G' \cap \mathbf{N}_G(P) \subseteq P$ *whenever P is a Sylow-p-subgroup of G.*

More generally, given a set of primes π with $p \in \pi$, we give in Theorem B necessary and sufficient group-theoretic conditions for the degree of each non-linear irreducible p-Brauer character of a π-separable group G to be divisible by some prime in π, at least when G satisfies some separability conditions. This can also be applied to ordinary characters of G by choosing p with $(p,|G|)=1$.

Actually Theorem A is valid whenever G satisfies McKay's conjecture for p-Brauer characters of p'-degree. Theorem A is not true for all finite groups, at least

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when $p = 2$. The simple group of order 168 has a self-normalizing Sylow-2-subgroup, but it also has two irreducible 2-Brauer characters of degree 3. The other direction also fails in general since the simple group of order 60 does not have a self-normalizing Sylow-2-subgroup, yet the degrees of its irreducible 2-Brauer characters are 1,2,2, and 4.

We conjecture that Theorem A is valid for all finite groups when p is odd, but it seems hard to find groups G satisfying either condition with $p < 2$ and G not p-solvable. An example of a group G that is not p-solvable and satisfies both conditions with $p = 3 = |G{:}G'|$ has $G' \cong L_2(27)$. (See [**4**].) For $p > 3$, a theorem of Glauberman shows that no Sylow-p-subgroup of a simple non-abelian group is self-normalizing. (See the Corollary to Theorem 5.14 of [**7**].)

For $P \in \mathrm{Syl}_p(G)$, the McKay conjecture asserts that the number of irreducible characters of G with p'-degree equals the number of such characters of $\mathbf{N}_G(P)$. For p-solvable groups, E. Dade first proved this, and a simpler proof due to Okuyama and Wajima appears in [**5**]. The following generalization is the main result of [**8**].

THEOREM 1. *If G is p-solvable and π-separable for a set of primes π and H is a Hall-π-subgroup of G, then*

$$|\{\varphi \in \mathrm{IBr}_p(G) | \varphi(1)$$
$$\textit{is a } \pi'\text{-number}|\} = |\{\eta \in \mathrm{IBr_p}\ (\mathrm{N_G(H)}) | \alpha(1) \textit{ is a } \pi'\textit{-number}|.$$

Theorem A is immediate from Theorem B by setting $\pi = \{p\}$.

THEOREM B. *Let H be a Hall-π-subgroup of a π-separable group G for a set π of primes with $p \in \pi$. Assume that G is p-solvable. Then the degree of every non-linear irreducible p-Brauer character of G is divisible by some prime in π if and only if $G' \cap \mathbf{N}_G(H)/H'$ is a p-group.*

Proof. We first prove the theorem when H is normal in G. Let $P \in Syl_p(G)$. Now $H'P \subseteq H$ and $H'P/H'$ is a normal Sylow-p-subgroup of G/H'. Since $H/H'P$ is a normal abelian Hall-π-subgroup of the p'-group $G/H'\mathrm{P}$, the degree of every $\varphi \in \mathrm{Irr}(G/H'P) = \mathrm{IBr}_p(G/H'P)$ is a π'-number. Now assume that the degree of every non-linear irreducible p-Brauer character of G is divisible by some prime in π. Then every $\varphi \in \mathrm{Irr}(G/H'P)$ is linear, from which it follows that $G' \subseteq H'P$ and that G'/H' is a p-group. For the converse, assume that G'/H' is a p-group. Then $G' \subseteq H'P$ and every $\varphi \in \mathrm{Irr}(G/H'P)$ is linear. Since $H'P/H'$ is a p-group, every irreducible p-Brauer character of G/H' is linear. Consequently if β is a non-linear irreducible p-Brauer character of G, then an irreducible constituent γ of $\beta_{H'}$ is not principal. We may choose $\boldsymbol{\alpha} \in \mathrm{IBr}_p(H)$ that is an irreducible constituent of β_H that lies over γ. By p-solvability, the degree of α divides $|\mathrm{H}|$. Also α is not linear because the irreducible constituents of H' are not principal. Hence $\alpha(1)$ is divisible by some prime in π. Furthermore $\alpha(1)$ divides $\beta(1)$ by Clifford's Theorem. (See [**6**, Corollary 8.7].) Hence $\beta(1)$ is divisible by some prime in π. This establishes the Theorem when H is normal in G. Setting $N = \mathbf{N}_G(H)$, we may now assume that $N < G$.

Now $G'H$ is a normal subgroup of G of π'-index, so that the Frattini argument implies that $G = (G'H)N = G'N$ and $G/G' \cong N/G' \cap N$. Let $J_{\pi'}(G)$ denote the set of irreducible p-Brauer characters of G of π'-degree. Since G is p-solvable and π-separable, Theorem 1 implies that $|J_{\pi'}(G)| = |J_{\pi'}(N)|$. Assume that all $\varphi \in J_{\pi'}(G)$ are linear. Then $J_{\pi'}(G) = \mathrm{IBr}_p(G/G')$. Since $G/G' \cong N/N \cap G'$ and $|J_{\pi'}(G)| = |J_{\pi'}(N)|$, it follows that $J_{\pi'}(N) = IBr_p(N/N \cap G')$. Thus every $\mu \in J_{\pi'}(N)$ is linear and also

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$N \cap G'/N'$ must be a p-group. Since H is normal in N, we have by the first paragraph, that N'/H' is a p-group. Then $N \cap G'/H'$ is a p-group, as desired.

Conversely assume that $N \cap G'/N'$ is a p-group. Then $N'H'$ is a p-group and so every $\mu \in J_{\pi'}(N)$ is linear, by the first paragraph. Now $N \cap G'/N'$ is a p-group and so $J_{\pi'}(N) = \mathrm{IBr}_p(N/N \cap G')$. Since $G/G' \cong N/N \cap G'$ and $|J_{\pi'}(G)| = |J_{\pi'}(N)|$, we have $J_{\pi'}(G) = IBr_p(G/G')$ and all $\varphi \in J_{\pi'}(G)$ are linear, as desired. □

We used p-solvability twice in the proof of Theorem B, namely to conclude that $|J_{\pi'}(G)| = |J_{\pi'}(N)|$ via Theorem 1 and to assume that $\beta(1)||H|$ for all $\beta \in (\mathrm{IBr}_p(H)$.

If G satisfies the conditions of Theorem B, it follows from the Theorem that $\mathbf{N}_G(H)$ does too. But the converse is false, as is evidenced by a semi-direct product $G = EP$, where E is an extra-special r-group for a prime $r \neq p$, P has order p, and $\mathbf{C}_E(P) = \mathbf{Z}(E)$. Then $\mathbf{N}_G(P)' = 1$ and the degree of every non-linear irreducible p-Brauer character of $\mathbf{N}_G(P)$ is divisible by p (vacuously), while G has an irreducible p-Brauer character of degree r.

Corollary 1. *Suppose that G is p-solvable and the degree of every non-linear p-Brauer character of G is divisible by p. Let* $K = \mathbf{O}^{p'}(G)$ *and* $P \in \mathrm{Syl}_p(G)$.

(i) $G/K \cong \mathbf{N}_G(P)/N_K(P)$ *is abelian.*
(ii) *The degree of every non-linear p-Brauer character of K is divisible by p*
(iii) $\mathbf{N}_K(P) = P$.

Proof. The Frattini argument shows that $G = K\mathbf{N}_G(P)$ and so $G/K \cong \mathbf{N}_G(P)/\mathbf{N}_K(P)$. The hypotheses imply that every $\chi \in \mathrm{Irr}(G/K) = \mathrm{IBr}_p(G/K)$ is linear and thus G/K is abelian. Applying Theorem A twice, we have that $K' \cap \mathbf{N}_K(P) \subseteq G' \cap \mathbf{N}_G(P) \subseteq P$ and the degree of every non-linear p-Brauer character of K is divisible by p.

Since K has no non-trivial p'-factor groups, it follows that $K'P = K$ and hence that $\mathbf{N}_K(P)/P \cong K' \cap \mathbf{N}_K(P)/K' \cap P$. But $K' \cap \mathbf{N}_K(P) \subseteq P$ by the last paragraph. Thus $\mathbf{N}_K(P)/P$ is both a p-group and a p'-group, whence $\mathbf{N}_K(P) = P$. □

That $G/K \cong \mathbf{N}_G(P)/\mathbf{N}_K(P)$ in part (i) above is solely a consequence of the Frattini argument. The conclusion above that $\mathbf{N}_K(P) = P$ thus shows that $\mathbf{N}_G(P)$ is as small as possible. Applying the classification of finite simple groups, we get the following result.

Corollary 2. *If the degree of every non-linear p-Brauer character of a p-solvable group G is divisible by p, then G is solvable.*

Proof. Let $K = \mathbf{O}^{p'}(G)$. By Corollary 1, we have that G/K is abelian and so we may choose a minimal normal subgroup M of G with $M \subseteq K$. Arguing by induction on $|G|$, we conclude that G/M is solvable and so we may assume that M is a p'-group. Now $\mathbf{C}_M(P) \subseteq \mathbf{N}_K(P) = P$ by Corollary 1. Thus M admits a coprime automorphism group P with trivial centralizer and it is a well-known consequence of the classification of simple groups that M must be solvable. Thus G is solvable. □

As sketched by the referee, there is a reasonably direct proof of Corollary 2 that bypasses Theorem A, but that still uses the classification. By choosing the prime p to be coprime to $|G|$ in Theorem B, we next get results about ordinary characters.

Corollary 3. *Let H be a Hall-π-subgroup of a π-separable group G. The degree of every non-linear $\chi \in \mathrm{Irr}(G)$ is divisible by some prime in π if and only if $G' \cap \mathbf{N}_G(H) = H'$.*

When π is a singleton $\{q\}$ in Corollary 3, the hypothesis that G is q-solvable is unnecessary, because G must be q-nilpotent by the character theoretic condition and Thompson's Theorem (12.2 of [**3**]) or by the group theoretic condition and Tates's Theorem (IV 4.7 of [**2**]). In Corollary 3, the hypothesis that G is π-separable may be replaced by the hypothesis that the number of irreducible characters of G of π'-degree and the number of such characters of $\mathbf{N}_G(H)$ are equal, as evidenced by remarks after Theorem B.

Thompson's Theorem (12.2 of [**3**]) on character degrees and Corollary 3 show that a prime q divides the degree of every non-linear ordinary character of a finite group G if and only if G has a normal q-complement M and $\mathbf{C}_{M'}(Q) = 1$ when $Q \in \mathrm{Syl}_q(G)$, an equivalence derived in [**1**]. This condition on character degrees not only implies q-solvability, but even solvability. (See the proof of Corollary 2.)

Corollary 4. *If H is a Hall-p'-subgroup of a p-solvable group G, then $G' \cap N_G(H) = H'$ if and only if G has no nonlinear irreducible character whose degree is a power of p.*

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