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ERGODICITY AND DIFFERENCES OF FUNCTIONS ON SEMIGROUPS

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Abstract

Iseki [11] defined a general notion of ergodicity suitable for functions $\varphi : J \to X$ where J is an arbitrary abelian semigroup and X is a Banach space. In this paper we develop the theory of such functions, showing in particular that it fits the general framework established by Eberlein [9] for ergodicity of semigroups of operators acting on X. Moreover, let $\mathscr{A}$ be a translation invariant closed subspace of the space of all bounded functions from J to X. We prove that if $\mathscr{A}$ contains the constant functions and φ is an ergodic function whose differences lie in $\mathscr{A}$ then $\varphi \in \mathscr{A}$. This result has applications to spaces of sequences facilitating new proofs of theorems of Gelfand and Katznelson-Tzafriri [12]. We also obtain a decomposition for the space of ergodic vectors of a representation $T : J \to L(X)$ generalizing results known for the case $J = \mathbb{Z}^+$. Finally, when J is a subsemigroup of a locally compact abelian group G, we compare the Iseki integrals with the better known Cesàro integrals.

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1. Introduction

In a successful attempt to unify and extend the growing collection of ergodic theorems, Eberlein [9] introduced systems of almost invariant integrals for semigroups of continuous linear transformations on locally convex spaces. A semigroup possessing such a system he called ergodic, and for such semigroups he proved a very general mean ergodic theorem ([9, Theorem 3.1]). Since that time many more ergodic theorems have appeared and many have been revealed as special cases of Eberlein's classical theorem. See for example [17].

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In a different direction, Iseki [11] introduced the notion of ergodicity of functions $\varphi : J \to X$ where J is a semigroup and X is a locally convex space. With it he was able to show that every such function which is almost periodic in the sense of Maak is necessarily ergodic.

Ruess and Summers [18] considered asymptotically almost periodic functions $\varphi : \mathbb{R}^+ \to X$. They showed that if the indefinite integral Φ of φ is weakly almost periodic in the sense of Eberlein, then Φ is asymptotically almost periodic. Subsequently Basit [3] observed that weak almost periodicity could be replaced by the more general property of ergodicity, that is the Cesàro integrals of Φ converge uniformly to a constant. Moreover, he replaced asymptotically almost periodic functions by large classes of functions. Ruess and Phóng [16] independently obtained some of these results.

Basit also observed that the integral problem discussed above is closely related to the *difference problem*: if $\varphi \in C_b(J, X)$ and $\Delta_t \varphi \in \mathscr{A} \subseteq C_b(J, X)$ for all $t \in J$, find conditions that ensure $\varphi \in \mathscr{A}$. Basit investigated this problem for the cases $J = \mathbb{R}^+$ or $\mathbb{R}$ and gave applications to the solutions of certain integro-differential difference equations [3] and to the abstract Cauchy problem [4]. Once again ergodicity of φ played an important role.

In the present paper we develop the theory of (Iseki) ergodic functions $\varphi : J \to X$ where J is an arbitrary semigroup and X is a Banach space. For the sake of simplicity and clarity, we restrict ourselves to the case of abelian J. In particular, we show how this theory fits into the framework established by Eberlein. Our main result concerns the difference problem and its relationship with ergodicity. This is in Section 2.

In Section 3 we apply our results to spaces of sequences. Among other things we obtain new proofs of theorems of Gelfand and Katznelson-Tzafriri on power bounded elements of Banach algebras. Section 4 deals with representations of semigroups on Banach spaces. We obtain a decomposition for the subspace of ergodic vectors generalizing known results for the case $J = \mathbb{Z}^+$.

Finally, in section 5 we exhibit a large class of semigroups J for which one can take limits of Cesàro integrals of functions φ in $C_{ub}(J, X)$. We show that these limits, when they exist, are identical to the Iseki means. Similarly, when G is a locally compact abelian group, we show that the means studied by Argabright [2] and Datry and Muraz [7] for $\varphi \in C_b(G, X)$ are identical to the Iseki means. We conclude by giving a simple condition on the Beurling spectrum of a function $\varphi \in C_{ub}(G, X)$ that ensures φ is ergodic.

2. Ergodicity

Throughout this paper, J will denote an abelian semigroup and X a Banach space over $\mathbb{R}$ or $\mathbb{C}$. By $B(J, X)$ we denote the space of bounded functions $\varphi : J \to X$, endowed with the norm $\|\varphi\|_\infty = \sup_{t\in J} \|\varphi(t)\|$. For such a function, φ_s and $\Delta_s\varphi$ will denote the translate and difference by s of φ, defined by $\varphi_s(t) = \varphi(t+s)$ and $\Delta_s\varphi = \varphi_s - \varphi$ for $s, t \in J$. The closed subspaces of $B(J, X)$ consisting of continuous and uniformly continuous functions respectively are denoted $C_b(J, X)$ and $C_{ub}(J, X)$. We will use the same symbol, say x, for an element of X and for the function in $B(J, X)$ taking the constant value x.

Following Iseki [11, I] we say that a function $\varphi : J \to X$ is *ergodic* if $\varphi \in B(J, X)$ and there exists $M_\varphi \in X$ such that for each $\varepsilon > 0$ there are elements $t_1, \dots, t_n \in J$ with $\|(1/n)\sum_{i=1}^{n}(\varphi_{t_i} - M_\varphi)\|_\infty < \varepsilon$. The element M_φ, clearly unique, is called the (Iseki) *mean* of φ and the class of all such ergodic functions is denoted $E(J, X)$. We define $M : E(J, X) \to X$ by $M(\varphi) = M_\varphi$.

PROPOSITION 2.1. *The space $E(J, X)$ is a translation invariant closed subspace of $B(J, X)$ containing all the constant functions. Moreover, $M : E(J, X) \to X$ is a bounded linear map.*

PROOF. Let $\varphi, \psi \in E(J, X)$. By the definition of ergodicity, for each $\varepsilon > 0$ there exist elements $s_1, \dots, s_m, t_1, \dots, t_n \in J$ such that $\|(1/m)\sum_{i=1}^{m}(\varphi_{s_i} - M_\varphi)\|_\infty < \varepsilon$ and $\|(1/n)\sum_{j=1}^{n}(\psi_{t_j} - M_\psi)\|_\infty < \varepsilon$. Since $\|\varphi_t\|_\infty \le \|\varphi\|_\infty$ for all $t \in J$, we obtain $\|(1/nm)\sum_{i=1}^{m}\sum_{j=1}^{n}(\varphi_{s_i+t_j} + \psi_{s_i+t_j} - M_\varphi - M_\psi)\|_\infty < 2\varepsilon$. Hence $\varphi + \psi \in E(J, X)$ and $M(\varphi + \psi) = M(\varphi) + M(\psi)$. The rest of the proposition is proved similarly.

The following result shows that there are many ergodic functions. Further examples will be provided later.

PROPOSITION 2.2. *If $\varphi \in B(J, X)$ and $s \in J$ then $\Delta_s\varphi \in E(J, X)$ and $M(\Delta_s\varphi) =$ 0.*

PROOF. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $\|(1/n)\varphi\|_\infty < \varepsilon/2$. Since $(\Delta_s\varphi)_t = \Delta_{s+t}\varphi - \Delta_t\varphi$, we have $\|(1/n)\sum_{j=1}^{n}(\Delta_s\varphi)_{js}\|_\infty < \varepsilon$. This proves the proposition.

The following alternative characterization of ergodic functions will be useful. For this we set $\mathscr{F}(J) = \{F \subseteq J : |F| < \infty\}$ where $|F|$ is the cardinality of F. Then $\mathscr{F}(J)$ becomes a directed set if we define $F_1 \le F_2$ whenever there exists $F \in \mathscr{F}(J)$ such that $F_2 = F_1 + F$.

PROPOSITION 2.3. *Let $\varphi \in B(J, X)$. Then $\varphi \in E(J, X)$ if and only if there exists $y \in X$ such that $\lim_{F\in\mathscr{F}(J)} ((1/|F|)\sum_{t\in F}\varphi_t) = y$. In this case, $y = M_\varphi$.*

PROOF. Let $\varphi \in E(J, X)$. For each $\varepsilon > 0$ there is a set $F_\varepsilon \in \mathscr{F}(J)$ such that $\|(1/|F_\varepsilon|)\sum_{t\in F_\varepsilon}(\varphi_t - M_\varphi)\|_\infty < \varepsilon$. If $F \in \mathscr{F}(J)$ satisfies $F \geq F_\varepsilon$, that is $F = F_\varepsilon + H$ for some $H \in \mathscr{F}(J)$, then

$$\left\|\frac{1}{|F|}\sum_{u\in F}(\varphi_u - M_\varphi)\right\|_\infty = \left\|\frac{1}{|F_\varepsilon|}\cdot\frac{1}{|H|}\sum_{t\in F_\varepsilon}\sum_{s\in H}(\varphi_{t+s} - M_\varphi)\right\|_\infty < \varepsilon,$$

showing that $\lim_{F\in\mathscr{F}(J)}(1/|F|)\sum_{t\in F}\varphi_t = M_\varphi$. The converse is clear.

Our next task is to set Iseki ergodicity in the framework of Eberlein. For this, let $\mathscr{S}$ be a sub-semigroup under composition of the Banach algebra $L(E)$ of all bounded operators $A : E \to E$ where E is a Banach space. The *orbit* of $x \in E$ under $\mathscr{S}$ is $\mathrm{orb}_\mathscr{S}(x) = \{Sx : S \in \mathscr{S}\}$. A net $(A_\alpha)_{\alpha\in\Lambda}$ in $L(E)$ is called a *system of invariant integrals* for $\mathscr{S}$ if

(2.1) $A_\alpha x \in \overline{\mathrm{co}}\,\mathrm{orb}_\mathscr{S}(x)$ for all $x \in E$ and $\alpha \in \Lambda$,

(2.2) $\sup_{\alpha\in\Lambda}\|A_\alpha\| < \infty$,

(2.3) $\lim_{\alpha\in\Lambda}\|(A_\alpha S - A_\alpha)x\| = \lim_{\alpha\in\Lambda}\|(SA_\alpha - A_\alpha)x\| = 0$ for all $x \in E$ and $S \in \mathscr{S}$.

If (2.1), (2.2) hold but (2.3) only holds at $x_0 \in E$ then we say (A_α) is a *system of invariant integrals for $\mathscr{S}$ at x_0*.

For $\varphi \in B(J, X)$, $F \in \mathscr{F}(J)$ and $s \in J$, define $R_F\varphi = (1/|F|)\sum_{t\in F}\varphi_t$, interpreted as 0 if $F = \emptyset$, and $R_s = R_{\{s\}}$. Hence $R_F, R_s \in L(E)$ where $E = B(J, X)$.

PROPOSITION 2.4. *The net $(R_F)_{F\in\mathscr{F}(J)}$ is a system of invariant integrals for the translation semigroup $\mathscr{R} = \{R_s : s \in J\}$.*

PROOF. For $\varphi \in B(J, X)$, $(R_F R_s - R_F)\varphi = R_F(\Delta_s\varphi)$. By Proposition 2.2, $M(\Delta_s\varphi) = 0$ and so by Proposition 2.3, $\lim_{F\in\mathscr{F}(J)}(R_F R_s - R_F)\varphi = 0$. Hence (2.3) follows, and (2.1), (2.2) are obvious.

By Eberlein's mean ergodic theorem [9, Theorem 3.1] we have immediately

COROLLARY 2.5. *For $\varphi \in B(J, X)$ the following are equivalent*

(1) *$\varphi \in E(J, X)$ and $M(\varphi) = y$,*
(2) *the net $(R_F\varphi)_{F\in\mathscr{F}(J)}$ converges to y,*
(3) *some subnet of $(R_F\varphi)_{F\in\mathscr{F}(J)}$ converges weakly to y,*
(4) *$y \in \overline{\mathrm{co}}\,\mathrm{orb}_\mathscr{R}(\varphi)$ with y a constant function.*

Recall that the space $W(J, X)$ of *Eberlein weakly almost periodic functions* consists of the bounded functions $\varphi : J \to X$ for which $\text{orb}_{\mathscr{R}}(\varphi)$ is weakly relatively compact. From Corollary 2.5 we obtain

COROLLARY 2.6. *$W(J, X)$ is a closed linear subspace of $E(J, X)$.*

Note that $M : E(J, X) \to X$ is a (translation) invariant mean in the sense of [6, p.79] for scalar X and [21] for general X. The latter proved the existence of an invariant mean on $W(J, X)$ for certain non-abelian semigroups J [21, Theorem 8.7]. However, the invariant means in these references are not given explicitly.

To conclude this section we prove our main result for ergodic functions. With the additional assumption that $\mathscr{A}$ contains the constant functions, this theorem provides a solution of the difference problem.

THEOREM 2.7. *Let $\mathscr{A}$ be a translation invariant closed subspace of $B(J, X)$. If $\varphi \in E(J, X)$ and $\Delta_t \varphi \in \mathscr{A}$ for all $t \in J$, then $\varphi - M(\varphi) \in \mathscr{A}$.*

PROOF. For each non-empty $F \in \mathscr{F}(J)$ we have $\varphi - R_F \varphi = -(1/|F|) \sum_{t \in F} \Delta_t \varphi \in \mathscr{A}$. The theorem follows from Corollary 2.5 by taking the limit over F in $\mathscr{F}(J)$.

3. Sequence spaces

In this section we give some applications of our results to spaces of sequences. Here we take $J = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$ and use the condition

(3.1) $\mathscr{A}$ is a closed subspace of $B(J, X)$ such that $\psi_t|_J \in \mathscr{A}$ whenever $\psi \in B(\mathbb{Z}, X)$, $t \in \mathbb{Z}$ and $\psi|_J \in \mathscr{A}$.

Examples of such subspaces $\mathscr{A}$ include $E(J, X)$, the space $C_0(J, X)$ of functions convergent to 0 at infinity, the space $AP(\mathbb{Z}, X)$ of almost periodic functions and the space $WAP(J, X)$ of Eberlein weakly almost periodic functions.

Following [3, Definition 4.1.2] we define the *spectrum with respect to $\mathscr{A}$* of a function $\varphi \in B(\mathbb{Z}, X)$ by $\text{sp}_{\mathscr{A}}(\varphi) = \{\gamma \in \widehat{\mathbb{Z}} : \hat{f}(\gamma) = 0 \text{ for all } f \in I_{\mathscr{A}}(\varphi)\}$ where $\widehat{\mathbb{Z}}$ is the (unitary) character group of $\mathbb{Z}$, $\hat{f} : \widehat{\mathbb{Z}} \to \mathbb{C}$ is the Fourier transform of f, and $I_{\mathscr{A}}(\varphi) = \{f \in L^1(\mathbb{Z}) : (\varphi * f)|_J \in \mathscr{A}\}$.

The following proposition is well-known for the case $\mathscr{A} = \{0\}$ and $J = \mathbb{Z}$, in which case $\text{sp}_{\mathscr{A}}(\varphi) = \text{sp}(\varphi)$, the *Beurling spectrum* of φ.

PROPOSITION 3.1. *Let $\varphi, \psi \in B(\mathbb{Z}, X)$, $f \in L^1(\mathbb{Z})$, $\gamma \in \widehat{\mathbb{Z}}$ and $\mathscr{A}$ satisfy condition* (3.1).

(i) $\text{sp}_{\mathscr{A}}(\varphi) = \text{sp}_{\mathscr{A}}(\varphi_t)$ *for all $t \in \mathbb{Z}$.*

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(ii) $\mathrm{sp}_{\mathscr{A}}(\varphi * f) \subseteq \mathrm{sp}_{\mathscr{A}}(\varphi) \cap \mathrm{supp}(\hat{f})$.
(iii) $\mathrm{sp}_{\mathscr{A}}(\varphi + \psi) \subseteq \mathrm{sp}_{\mathscr{A}}(\varphi) \cup \mathrm{sp}_{\mathscr{A}}(\psi)$.
(iv) $\mathrm{sp}_{\mathscr{A}}(\gamma\varphi) = \gamma + \mathrm{sp}_{\mathscr{A}}(\varphi)$.
(v) $\mathrm{sp}_{\mathscr{A}}(\varphi) = \emptyset$ *if and only if* $\varphi|_J \in \mathscr{A}$.

PROOF. The arguments are the same as for the Beurling spectrum. See for example [8, part II, p.988] or [5]. We present a proof for (v). If $\varphi|_J \in \mathscr{A}$ then by (3.1), $\varphi_t|_J \in \mathscr{A}$ for all $t \in \mathbb{Z}$. Hence for $f \in L^1(\mathbb{Z})$, $(\varphi * f)|_J = \sum_{n\in\mathbb{Z}} f(n)\varphi_{-n}|_J \in \mathscr{A}$. So $I_{\mathscr{A}}(\varphi) = L^1(\mathbb{Z})$ and $\mathrm{sp}_{\mathscr{A}}(\varphi) = \emptyset$. Conversely, if $\mathrm{sp}_{\mathscr{A}}(\varphi) = \emptyset$ then $I_{\mathscr{A}}(\varphi) = L^1(\mathbb{Z})$. Choose $f_n \in L^1(\mathbb{Z})$ such that $\varphi * f_n \to \varphi$ in $B(\mathbb{Z}, X)$. Since $f_n \in I_{\mathscr{A}}(\varphi)$, $(\varphi * f_n)|_J \in \mathscr{A}$ and since $\mathscr{A}$ is closed, $\varphi|_J \in \mathscr{A}$.

In the sequel we denote the elements of $\widehat{\mathbb{Z}}$ by γ_λ or λ, where $\lambda \in \mathbb{T}$ the circle group and $\gamma_\lambda(n) = \lambda^n$ for $n \in \mathbb{Z}$. Hence γ_1 or 1 is the unit in $\widehat{\mathbb{Z}}$.

PROPOSITION 3.2. *Suppose $\mathscr{A}$ satisfies* (3.1), $\varphi \in B(J, X)$, $\varphi|_J \in E(J, X)$ *and* $\mathrm{sp}_{\mathscr{A}}(\varphi) \subseteq \{1\}$. *Then* $\varphi|_J - M(\varphi|_J) \in \mathscr{A}$.

PROOF. By Wiener's tauberian theorem [15, 7.2.5] the condition $\mathrm{sp}_{\mathscr{A}}(\varphi) \subseteq \{1\}$ is equivalent to $I_{\mathscr{A}}(\varphi) \supseteq \{f \in L^1(\mathbb{Z}) : \hat{f}(1) = 0\}$. For $t \in \mathbb{Z}$, $g \in L^1(\mathbb{Z})$ and $\lambda \in \mathbb{T}$ we have $(\Delta_t g)\widehat{\ }(\lambda) = (\gamma_\lambda(t) - 1)\hat{g}(\lambda)$. Hence $\Delta_t g \in I_{\mathscr{A}}(\varphi)$. In other words, $(\Delta_t\varphi * g)|_J = (\varphi * \Delta_t g)|_J \in \mathscr{A}$. Setting $g = \chi_{\{0\}}$, the characteristic function of $\{0\}$ in $\mathbb{Z}$ we have $\Delta_t\varphi = \Delta_t\varphi * g$ and so $\Delta_t\varphi|_J \in \mathscr{A}$. By Theorem 2.7, $\varphi|_J - M(\varphi|_J) \in \mathscr{A}$.

As a consequence we have the following application of spectra to the difference problem.

THEOREM 3.3. *Suppose $\mathscr{A}$ satisfies* (3.1) *and* $\varphi \in B(\mathbb{Z}, X)$. *Then* $\mathrm{sp}_{\mathscr{A}}(\varphi) \subseteq \{1\}$ *if and only if* $\Delta_t\varphi|_J \in \mathscr{A}$ *for all* $t \in J$.

PROOF. Let $\Delta_t\varphi|_J \in \mathscr{A}$ for all $t \in J$. If $g \in L^1(\mathbb{Z})$ then by (3.1), $(\varphi * \Delta_t g)|_J = \sum_{n\in\mathbb{Z}} g(n)\,(\Delta_t\varphi)_{-n}|_J \in \mathscr{A}$. So $I_{\mathscr{A}}(\varphi) \supseteq \{\Delta_t g : t \in J,\ g \in L^1(\mathbb{Z})\}$. But $(\Delta_t g)\widehat{\ }(\lambda) = (\gamma_\lambda(t) - 1)\hat{g}(\lambda)$ is zero for all $t \in J$ and $g \in L^1(\mathbb{Z})$ only when $\lambda = 1$. So $\mathrm{sp}_{\mathscr{A}}(\varphi) \subseteq \{1\}$. Conversely, let $\mathrm{sp}_{\mathscr{A}}(\varphi) \subseteq \{1\}$. By Proposition 2.2, $\Delta_t\varphi|_J \in E(J, X)$ and $M(\Delta_t\varphi|_J) = 0$ for each $t \in J$. By Proposition 3.2, $\Delta_t\varphi|_J \in \mathscr{A}$.

In order to apply Theorem 3.3, we first prove the following result. In it, $\sigma(x)$ denotes the Banach algebra spectrum of x.

THEOREM 3.4. *Let X be a unital Banach algebra. Suppose $\mathscr{A} \subseteq B(J, X)$ satisfies* (3.1) *and in addition $y\mathscr{A} \subseteq \mathscr{A}$ for all $y \in X$. Let $\varphi : \mathbb{Z} \to X$ be a bounded solution of the recurrence equation $\varphi(n + 1) = x\varphi(n) + \psi(n)$ for some $x \in X$ and $\psi \in C_b(\mathbb{Z}, X)$. If $\psi|_J \in \mathscr{A}$ then* $\mathrm{sp}_{\mathscr{A}}(\varphi) \subseteq \sigma(x) \cap \mathbb{T}$.

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PROOF. Let $\lambda_0 \in \mathbb{T}\backslash\sigma(x)$. Choose $\delta > 0$ such that $B_\delta(\lambda_0) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\} \subseteq \mathbb{C}\backslash\sigma(x)$. Take $f \in L^1(\mathbb{Z})$ with $\hat{f}(\lambda_0) = 1$ and $\operatorname{supp}(\hat{f}) \subseteq B_{\delta/4}(\lambda_0)$. Let $\xi = \varphi * f$. It suffices to prove $\xi|_J \in \mathscr{A}$, for then $f \in I_{\mathscr{A}}(\varphi)$ and $\lambda_0 \notin \operatorname{sp}_{\mathscr{A}}(\varphi)$.

To do this, let $g \in L^1(\mathbb{Z})$ be such that $\hat{g}(\lambda) = 1$ for $\lambda \in B_{\delta/2}(\lambda_0) \cap \mathbb{T}$, $\operatorname{supp}(\hat{g}) \subseteq B_\delta(\lambda_0)$ and $\hat{g} \in C^1(\mathbb{T})$. Define $h : \mathbb{T} \to X$ by $\hat{h}(\lambda) = \hat{g}(\lambda)(\lambda e - x)^{-1}$, interpreted as 0 outside $B_\delta(\lambda_0)$, where e is the unit in X. Then $\hat{h} \in C^1(\mathbb{T}, X)$ so $\hat{h}(\lambda) = \sum_{n=-\infty}^{\infty} h(n)\lambda^{-n}$ for some $h \in L^1(\mathbb{Z}, X)$ with $h(n)x = xh(n)$ for all $n \in \mathbb{Z}$. Moreover, if $\eta_\lambda(n) = \gamma_\lambda(n+1)e - \gamma_\lambda(n)x$, where $\gamma_\lambda(n) = \lambda^n$ and $\lambda \in B_{\delta/2}(\lambda_0) \cap \mathbb{T}$, then $h * \eta_\lambda = \gamma_\lambda$. Indeed,

$$\begin{aligned} h * \eta_\lambda(n) &= \sum_j h(j)(\lambda^{n+1-j}e - \lambda^{n-j}x) = \lambda^n(\lambda e - x)\sum_j h(j)\lambda^{-j} \\ &= \lambda^n(\lambda e - x)\hat{g}(\lambda)(\lambda e - x)^{-1} = \lambda^n. \end{aligned}$$

Now $\xi = \varphi * f \in B(\mathbb{Z}, X)$ and $\operatorname{sp}(\xi) \subseteq \operatorname{supp}(\hat{f}) \subseteq B_{\delta/4}(\lambda_0)$, so there is a sequence of trigonometric polynomials $\pi_m \in B(\mathbb{Z}, X)$ converging pointwise to ξ and with $\operatorname{sp}(\pi_n) \subseteq B_{\delta/2}(\lambda_0)$. Let $\eta_m(n) = \pi_m(n+1)e - x\pi_m(n)$. Then $h * \eta_m = \pi_m$.

From the recurrence equation, $\eta_m(n) \to \xi(n+1) - x\xi(n) = \psi * f(n)$ for each $n \in \mathbb{Z}$. Hence $\xi = h * \psi * f$. Since $\xi = \sum_{n\in\mathbb{Z}} h(n)(\psi * f)_{-n}$ and $y\mathscr{A} \subseteq \mathscr{A}$ for each $y \in X$, it follows from (3.1) that $\xi|_J \in \mathscr{A}$ as required.

As a consequence we easily obtain the following two results. The first was proved by Gelfand (see [12]) and the second by Katznelson and Tzafriri [12]. Recall that an element x of a unital Banach algebra X is called *power bounded* if $\{x^n : n \in \mathbb{Z}^+\}$ is bounded and *doubly power bounded* if $\{x^n : n \in \mathbb{Z}\}$ is bounded.

COROLLARY 3.5. *Let x be a doubly power bounded element of a unital Banach algebra X. If $\sigma(x) = \{1\}$ then $x = e$.*

PROOF. We may apply Theorem 3.4 with $\mathscr{A} = \{0\}$, $J = \mathbb{Z}$, $\psi = 0$ and $\varphi(n) = x^n$. So $\operatorname{sp}(\varphi) \subseteq \sigma(x) \cap \mathbb{T} = \{1\}$. By Theorem 3.3, $\Delta_t\varphi = 0$ for all $t \in \mathbb{Z}$ and hence $x = e$.

COROLLARY 3.6. *Let x be a power bounded element of a unital Banach algebra X. If $\sigma(x) \cap \mathbb{T} \subseteq \{1\}$ then $\|x^{n+1} - x^n\| \to 0$ as $n \to \infty$.*

PROOF. Apply Theorem 3.4 with $\mathscr{A} = C_0(J, X)$, $J = \mathbb{Z}^+$ and φ, ψ as follows. For $n \geq 0$ set $\varphi(n) = x^n$, $\psi(n) = 0$ and for $n < 0$ set $\varphi(n) = e$, $\psi(n) = e - x$. So $\operatorname{sp}_{\mathscr{A}}(\varphi) \subseteq \{1\}$ and by Theorem 3.3, $\Delta_t\varphi|_J \in \mathscr{A}$ for all $t \in J$. This gives the corollary.

In a subsequent paper we will use ergodicity and the difference problem to obtain generalizations of these last two results.

4. Ergodic vectors of representations

Throughout this section J will denote an abelian semigroup and $T : J \to L(X)$ a *representation.* That is, T is a semigroup homomorphism mapping J into the semigroup under composition $L(X)$. The dual representation $T^* : J \to L(X^*)$ is defined by $\langle x, T^*(t)\varphi\rangle = \langle T(t)x, \varphi\rangle$ for $x \in X$, $t \in J$ and $\varphi \in X^*$.

The space of *fixed points* of T is $N = N(T) = \bigcap_{t\in J} \ker(T(t) - I)$ and its *complementary space* is $R = R(T) = \text{span}\{T(s)x - x : x \in X,\ s \in J\}$. The closure of R is denoted $\overline{R} = \overline{R}(T)$. The set of *ergodic vectors* of T is $X_{\text{erg}} = X_{\text{erg}}(T) = \{x \in X : T(\cdot)x \in E(J, X)\}$.

Next let $T(J)$ be the range of T in $L(X)$ and for $F \in \mathscr{F}(J)$ define $T_F \in L(X)$ by $T_F x = (1/|F|)\sum_{t\in F} T(t)x$, again interpreted as 0 if $F = \emptyset$. Finally, the *orbit* under T of an element $x \in X$ is $\text{orb}_T(x) = \text{orb}_{T(J)}(x)$.

PROPOSITION 4.1. *If $T : J \to L(X)$ is a representation and* $\text{orb}_T(x)$ *is bounded for some $x \in X$, then the set $(T_F)_{F\in\mathscr{F}(J)}$ is a system of invariant integrals for the semigroup $T(J)$ at x.*

PROOF. Let $s \in J$. The function $T(\cdot)x : J \to X$ is bounded and hence by Proposition 2.2, $\Delta_s T(\cdot)x \in E(J, X)$ and $M(\Delta_s T(\cdot)x) = 0$. By Corollary 2.5, $\lim_F R_F \Delta_s T(\cdot)x = 0$ and in particular $\lim_F \|R_F \Delta_s T(t)x\| = 0$ for each $t \in J$. But $R_F \Delta_s T(t)x = (R_{F+t}T(s) - R_{F+t})x$ and so $\lim_F \|(R_F T(s) - R_F)x\| = 0$. Condition (2.3) follows for this x. Since (2.1) and (2.2) are clear the proposition is proved.

COROLLARY 4.2. *If $T : J \to L(X)$ is a representation and* $\text{orb}_T(x)$ *is bounded for some $x \in X$ then the following are equivalent*

(i) *$x \in X_{\text{erg}}(T)$ and $M(T(\cdot)x) = y$,*
(ii) *$(T_F x)_{F\in\mathscr{F}(J)}$ converges to y,*
(iii) *some subnet of $(T_F x)_{F\in\mathscr{F}(J)}$ converges weakly to y.*
(iv) *$y \in N(T) \cap \overline{\text{co}}\,\text{orb}_T(x)$.*

PROOF. By Eberlein's mean ergodic theorem (see Theorem 3.1 in [9] and the remark following it) we conclude that (ii), (iii) and (iv) are equivalent. Let $\kappa = \sup\{\|z\| : z \in \text{orb}_T(x)\}$. Then for each $t \in J$ and $F \in \mathscr{F}(J)$ we have $\|T_{F+t}x - y\| = \|R_F T(t)x - y\| \le \|R_F T(\cdot)x - y\|_\infty \le \kappa\|T_F x - y\|$. Hence $(T_F x) \to y$ in X if and only if $(R_F T(\cdot)x) \to y$ in $B(J, X)$. By Corollary 2.5, (ii) is equivalent to (i).

PROPOSITION 4.3. *If $T : J \to L(X)$ is a bounded representation, then X_{erg} is a closed linear subspace of X. Moreover, $X_{\text{erg}} = N \oplus \overline{R}$.*

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PROOF. Since $E(J, X)$ is a linear space, so too is X_{erg}. The closedness of X_{erg} follows from the boundedness of T and the closedness of $E(J, X)$ in $B(J, X)$. If $x \in N$ then $T(t)x = x$ for all $t \in J$. Hence $T(\cdot)x \in E(J, X)$ and $M(T(\cdot)x) = x$, showing $N \subseteq X_{\text{erg}}$. If $z \in R$ then there exist $t_1, \ldots, t_n \in J$ and $x_1, \ldots, x_n \in X$ such that $z = \sum_{j=1}^{n}(T(t_j)x_j - x_j)$. Hence $T(\cdot)z = \sum_{j=1}^{n} \Delta_{t_j} T(\cdot)x_j$. By Proposition 2.2, $T(\cdot)z \in E(J, X)$ and $M(T(\cdot)z) = 0$. By Proposition 2.1, the same is true for $z \in \overline{R}$. Hence $\overline{R} \subseteq X_{\text{erg}}$ and moreover, $N \cap \overline{R} = \{0\}$.

Finally we show $X_{\text{erg}} \subseteq N + \overline{R}$. If $y \in X_{\text{erg}}$ then by Corollary 4.2, $M(T(\cdot)y) \in N$. Setting $z = y - M(T(\cdot)y)$ we show $z \in \overline{R}$. Indeed, for each $\varepsilon > 0$ there exist $t_1, \ldots, t_n \in J$ such that $\|(1/n)\sum_{j=1}^{n}[T(t)T(t_j)y - M(T(\cdot)y)]\| < \varepsilon$ for all $t \in J$. Now $z_\varepsilon = (1/n)\sum_{j=1}^{n}[z - T(t + t_j)z] \in R$ and $\|z - z_\varepsilon\| < \varepsilon$, so $z \in \overline{R}$. Hence $y \in N + \overline{R}$ and the proposition is proved.

The following two results provide examples of ergodic vectors.

COROLLARY 4.4. *Let $T : J \to L(X)$ be a representation and $x \in X$. If $\text{orb}_T(x)$ is weakly relatively compact then $x \in X_{\text{erg}}(T)$.*

PROOF. Since $\text{orb}_T(x)$ is weakly relatively compact, it is bounded and by Proposition 4.1, (T_F) is a system of invariant integrals for $T(J)$ at x. Moreover, $\text{co}\,\text{orb}_T(x)$ is weakly relatively compact so $(T_F x)$ has a weak limit point y. By Corollary 4.2, $x \in X_{\text{erg}}(T)$.

PROPOSITION 4.5. *Let $T : J \to L(X)$ be a bounded representation. If X is reflexive, or more generally if $N + R$ is dense in X, then $X_{\text{erg}} = X$.*

PROOF. Since $N + R \subseteq X_{\text{erg}} \subseteq X$ we conclude that $X_{\text{erg}} = X$ whenever $N + R$ is dense in X. It remains to prove that $N + R$ is dense in X if X is reflexive. For $S \subseteq X$ let $S^\perp = \{\varphi \in X^* : \langle x, \varphi \rangle = 0 \text{ for all } x \in S\}$. It is easy to check that $R^\perp = N(T^*)$. Hence for reflexive X, $R(T^*)^\perp = N(T^{**}) = N$. Further, $N^\perp = R(T^*)^{\perp\perp} = \overline{R}(T^*)$. Hence $(N + R)^\perp = N^\perp \cap R^\perp = \overline{R}(T^*) \cap N(T^*) = \{0\}$, showing that $N + R$ is dense in X.

As an application we present the following

PROPOSITION 4.6. *Given $A \in L(X)$ define $T : \mathbb{Z}^+ \to L(X)$ by $T(n) = A^n$. If $x \in X_{\text{erg}}(T)$ and $A^{n+1}x - A^n x \to 0$ as $n \to \infty$ then $A^n x \to y$ for some $y \in X$ with $Ay = y$.*

PROOF. We apply Theorem 2.7 with $\mathscr{A} = C_0(J, X)$, $J = \mathbb{Z}^+$ and $\varphi(n) = A^n x$. Since $\Delta_t \varphi \in \mathscr{A}$ for all $t \in J$ and $\varphi \in E(J, X)$ we conclude that $\varphi - M_\varphi \in \mathscr{A}$. So $A^n x \to y$ where $y = M_\varphi$.

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REMARK 4.7. If $A \in L(X)$ and $T : \mathbb{Z}^+ \to L(X)$ is given by $T(n) = A^n$ then $N(T) = \ker(A - I)$ and $R(T) = \text{range}(A - I)$. If A is power bounded then T is a bounded representation and if the Cesàro sums $A_n x = (1/n)\sum_{j=1}^n A^j x$ converge weakly for some $x \in X$ then $T(\cdot)x$ is ergodic. If in addition X is reflexive then by Propositions 4.1 and 4.5, $X = N \oplus \overline{R}$. This special case may be found in [20, p.214]. Also see [10].

5. Cesàro and other means

Throughout this section we will assume that J is a measurable sub-semigroup of a locally compact abelian group G carrying a fixed Haar measure μ. Let $\mathscr{K}(G)$ denote the set of compact neighbourhoods of 0 in G and set $\mathscr{K}(J) = \{V \cap J : V \in \mathscr{K}(G)$ and $\mu(V \cap J) \neq 0\}$. We shall call a net $(K_\alpha)_{\alpha \in \Lambda}$ in $\mathscr{K}(J)$, a *Følner net* if

$$\lim_{\alpha \in \Lambda} \frac{\mu(K_\alpha \Delta (K_\alpha + s))}{\mu(K_\alpha)} = 0 \quad \text{for all } s \in J, \tag{5.1}$$

where Δ denotes symmetric difference.

Condition (5.1) was introduced by Følner (see [6, p.80]). As an example, let $G = \mathbb{R}^2$ and $J = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \le m(x_1 - a)\}$ where $a \ge 0$ and $m > 0$. If $K_r = \{x \in J : |x| \le r\}$ then $K_r \in \mathscr{K}(J)$, $\mu(K) \sim r^2$ and $\mu(K_r \Delta (K_r + s)) \sim r$ for fixed $s \in J$. Hence $(K_r)_{r>a}$ is a Følner net.

We define the *Cesàro integrals* of functions $\varphi \in C_b(J, X)$ by $C_K\varphi(t) = (1/\mu(K)) \int_K \varphi(t+s)\,d\mu(s)$ for $K \in \mathscr{K}(J)$, $t \in J$.

PROPOSITION 5.1. *If* (5.1) *holds then* $(C_{K_\alpha})_{\alpha \in \Lambda}$ *is a system of invariant integrals for the translation semigroup* $\mathscr{R}$ *acting on* $C_{ub}(J, X)$.

PROOF. Let $K \in \mathscr{K}(J)$ and $\varphi \in C_{ub}(J, X)$. Given $\varepsilon > 0$ choose $V \in \mathscr{K}(G)$ such that $\|\varphi_s - \varphi_t\|_\infty < \varepsilon$ for all $t \in J$ and all $s \in (t+V) \cap J$. Since $\|C_K\varphi(s) - C_K\varphi(t)\| \le \|\varphi_s - \varphi_t\|_\infty$ we conclude that $C_K\varphi \in C_{ub}(J, X)$. Moreover, $C_K \in L(C_{ub}(J, X))$. Next, by the compactness of K we can choose $t_1, \dots, t_m \in K$ such that $K \subseteq \bigcup_{j=1}^m (t_j + V)$. Set $\pi_1 = (t_1 + V) \cap K$ and for $2 \le j \le m$, $\pi_j = (t_j + V) \cap K \setminus \bigcup_{i=1}^{j-1} \pi_i$. Then $K = \bigcup_{j=1}^m \pi_j$ and the π_j are disjoint measurable sets. Since

$$\left\| C_K\varphi - \sum_{j=1}^m \frac{\mu(\pi_j)}{\mu(K)} \varphi_{t_j} \right\| < \varepsilon$$

we conclude that $C_K\varphi \in \overline{\text{co}}\,\text{orb}_{\mathscr{R}}(\varphi)$, thereby proving (2.1).

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For (2.3), let $s \in J$. Then

$$\begin{aligned}\|(C_K R_s - C_K)\varphi\|_\infty &= \sup_{t\in J}\left\|\frac{1}{\mu(K)}\int_K [\varphi(t+s+u) - \varphi(t+u)]d\mu(u)\right\| \\ &= \sup_{t\in J}\left\|\frac{1}{\mu(K)}\int_{K\Delta(K+s)} \varphi(t+u)d\mu(u)\right\| \\ &\leq \|\varphi\|_\infty \frac{\mu(K\Delta(K+s))}{\mu(K)}\end{aligned}$$

and (2.3) follows from (5.1). Since (2.2) is clear, the proposition is proved.

COROLLARY 5.2. *If* $\varphi \in C_{ub}(J, X)$ *and* (5.1) *holds, then the following are equivalent*

(i) $\varphi \in E(J, X)$ *and* $M(\varphi) = y$,
(ii) *the net* $(C_{K_\alpha}\varphi)_{\alpha\in\Lambda}$ *converges to* y,
(iii) *some subnet of* $(C_{K_\alpha}\varphi)_{\alpha\in\Lambda}$ *converges weakly to* y.

PROOF. By Corollary 2.5 and Eberlein's mean ergodic theorem again, each of these conditions is equivalent to $y \in \overline{\text{co}}\,\text{orb}_{\mathscr{R}}(\varphi)$ with y a constant function.

We come to our final system of invariant integrals. Let $\mathscr{P} = \{f \in L^1(G) : f \geq 0$ and $\hat{f}(0) = 1\}$. Reiter [14, p.113] has proved the existence of a net $(f_\alpha)_{\alpha\in\Lambda}$ in $\mathscr{P}$ satisfying $\lim_{\alpha\in\Lambda}\|R_s f_\alpha - f_\alpha\|_1 = 0$ for all $s \in G$. For $\varphi \in C_{ub}(G, X)$ we can define $A_\alpha\varphi \in C_{ub}(G, X)$ by $A_\alpha\varphi = \varphi * f_\alpha$. So $\|A_\alpha\varphi\|_\infty \leq \|\varphi\|_\infty$ and $A_\alpha \in L(C_{ub}(G, X))$.

PROPOSITION 5.3. *The net* $(A_\alpha)_{\alpha\in\Lambda}$ *is a system of invariant integrals for the translation semigroup* $\mathscr{R} = (R_s)_{s\in G}$ *acting on* $C_{ub}(G, X)$.

PROOF. Given $V \in \mathscr{K}(G)$ and $\varphi \in C_{ub}(G, X)$ let $f_V = (1/\mu(V))\chi_{-V}$ where χ_{-V} is the characteristic function of $-V$. Then $f_V \in \mathscr{P}$ and since $\varphi * f_V = (1/\mu(V))\int_V \varphi_s d\mu(s) = C_V\varphi$, it follows from Proposition 5.1 that $\varphi * f_V \in \overline{\text{co}}\,\text{orb}_{\mathscr{R}}(\varphi)$. It is easy to check that $\mathscr{P} \subseteq \overline{\text{co}}\{f_V : V \in \mathscr{K}(G)\}$. Hence, $\varphi * \mathscr{P} \subseteq \overline{\text{co}}\,\text{orb}_{\mathscr{R}}(\varphi)$, proving (2.1). Since $\|A_\alpha\| \leq 1$, (2.2) holds. Finally, for $s \in G$ we have

$$\|(A_\alpha R_s - A_\alpha)\varphi\|_\infty = \|(R_s\varphi - \varphi) * f_\alpha\|_\infty = \|\varphi * (R_s f_\alpha - f_\alpha)\|_\infty \leq \|\varphi\|_\infty \|R_s f_\alpha - f_\alpha\|_1.$$

From the definition of (f_α), (2.3) follows and the proposition is proved.

As for Corollary 5.2 we obtain

COROLLARY 5.4. *For* $\varphi \in C_{ub}(G, X)$ *the following are equivalent*

(i) *$\varphi \in E(G, X)$ and $M(\varphi) = y$,*
(ii) *the net $(A_\alpha \varphi)_{\alpha \in \Lambda}$ converges to y,*
(iii) *some subnet of $(A_\alpha \varphi)_{\alpha \in \Lambda}$ converges weakly to y.*

Argabright [2] used the Reiter nets (f_α) to prove an ergodic limit for scalar-valued Eberlein weakly almost periodic functions on G. Datry and Muraz [7] also used them to introduce ergodicity in Banach $L^1(G)$-modules.

We conclude with two more examples, firstly of some ergodic functions and secondly of a non-ergodic one. Recall that for a function $\varphi \in C_b(G, X)$ the set $I(\varphi) = \{f \in L^1(G) : \varphi * f = 0\}$ is a closed ideal of $L^1(G)$. Let $\widehat{G}$ denote the character group of G, 0 the unit of $\widehat{G}$, and $\hat{f} : G \to \mathbb{C}$ the Fourier transform of f. The *Beurling spectrum* of φ is $\mathrm{sp}(\varphi) = \{\gamma \in \widehat{G} : \hat{f}(\gamma) = 0 \text{ for all } f \in I(\varphi)\}$.

THEOREM 5.5. *If $\varphi \in C_{ub}(G, X)$ and $0 \notin \mathrm{sp}(\varphi)$ then $\varphi \in E(G, X)$.*

PROOF. Take $V \in \mathscr{K}(\widehat{G})$ with $V \cap \mathrm{sp}(\varphi) = \emptyset$ and $f \in L^1(G)$ with $\hat{f}(0) = 1$ and $\mathrm{supp}(\hat{f}) \subseteq V$. Then $\mathrm{sp}(\varphi * f) = \emptyset$ so $\varphi * f = 0$. Moreover, f is continuous. Now, given $\varepsilon > 0$, choose a compact set K in G such that $\int_{G \setminus K} |f(t)| d\mu(t) < \varepsilon/(1 + 2\|\varphi\|_\infty)$. For $s \in G$ define $g(s) = (\varphi - \varphi_{-s}) f(s)$. Hence $\int_G g(s) d\mu(s) = \varphi - \varphi * f = \varphi$. Moreover, by Proposition 2.2, $g(s) \in E(G, X)$ and since φ is uniformly continuous, $g : G \to E(G, X)$ is continuous. Since K is compact, $g|_K$ is separably-valued and hence Bochner integrable. Therefore $\int_K g(s) d\mu(s) \in E(G, X)$. But $\|\varphi - \int_K g(s) d\mu(s)\| \le \|\int_{G \setminus K} g(s) d\mu(s)\| < \varepsilon$ and so $\varphi \in E(G, X)$ as claimed.

EXAMPLE 5.6. Define $\varphi : \mathbb{R} \to c_0$ by $\varphi(t) = (\sin(t/n))_{n=1}^{\infty}$. One easily checks that $\varphi \in C_{ub}(\mathbb{R}, c_0)$. Now the range of φ is not relatively compact in c_0. For, if it were, then given $0 < \varepsilon < 1/4$ there would exist $t_1, \dots, t_m \in \mathbb{R}$ such that $\inf_j \|\varphi(t) - \varphi(t_j)\| < \varepsilon$ for all $t \in \mathbb{R}$. In particular we would have $|\sin(t/n)| < 2\varepsilon$ for all $n > N(\varepsilon)$ and all $t \in \mathbb{R}$, which is false. It follows that φ is not almost periodic. On the other hand φ' is almost periodic (see [1, p. 53]) and so $\varphi \notin E(\mathbb{R}, c_0)$. For otherwise, by Levitan [13] or Basit [3, Theorem 3.1.1] it would follow that φ is almost periodic. From Theorem 5.5 we conclude that $0 \in \mathrm{sp}(\varphi)$.

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