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Envelope Approach to Degenerate Complex Monge–Ampère Equations on Compact Kähler Manifolds

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Abstract. We use the classical Perron envelope method to show a general existence theorem to degenerate complex Monge–Ampère type equations on compact Kähler manifolds.

1 Introduction

Let (X,ω) be a compact Kähler manifold of complex dimension n. Recall that a $(1,1)$-cohomology class is *big* if it contains a Kähler current that is a positive closed current that dominates a Kähler form. Fix $\alpha \in H^{1,1}(X,\mathbb{R})$ a big class. Assume that α admits a smooth closed real $(1,1)$-form representative θ which is semi-positive. An θ-plurisubharmonic function (θ-psh for short) is an upper semi-continuous function φ on X such that $\theta + dd^c\varphi$ is nonnegative in the sense of currents. We let $PSH(X,\theta)$ denote the set of all such functions. In this note we consider equations of complex Monge–Ampère type

$$(\theta + dd^c\varphi)^n = F(\varphi,\cdot)d\mu, \tag{1.1}$$

where μ denotes a non-negative Radon measure, $F: \mathbb{R}\times X \to [0,+\infty)$ is a measurable function, and the (unknown) function φ is θ-psh.

It is well known that we cannot make sense to the left-hand side of (1.1). But according to [4] (see also [8,9,15]), we can define the non pluripolar product $(\theta + dd^c\varphi)^n$ as the limit of $\mathbf{1}_{(\varphi>-j)}(\theta+dd^c(\max(\varphi,-j))^n$. It was shown in [9] that its trivial extension is nonnegative closed current and

$$\int_X (\theta + dd^c u)^n \le \int_X \theta^n.$$

Denote by $\mathcal{E}(X,\theta)$ the set of all θ-psh with full non-pluripolar Monge–Ampère measure *i.e.*, θ-psh functions for which the last inequality becomes equality.

Equation (1.1) has been extensively studied by various authors; see, for example, [1,2,5,9,14,17–21] and reference therein. In this note, we prove the following result.

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Main Theorem *Assume that $F: \mathbb{R} \times X \to [0, +\infty)$ is a measurable function such that the following hold:*

(i) *for all $x \in X$ the function $t \mapsto F(t, x)$ is continuous and nondecreasing;*
(ii) *$F(t, \cdot) \in L^1(X, d\mu)$ for all $t \in \mathbb{R}$;*
(iii) *for all $x \in X$, $\lim_{t\to+\infty} F(t, x) = +\infty$ and $\lim_{t\to-\infty} F(t, x) = 0$.*

Then there exists a unique (up to additive constant) θ-psh function $\phi \in \mathcal{E}(X, \theta)$ solution to the equation

$$(\theta + dd^c\phi)^n = F(\phi, \cdot)d\mu.$$

Note that a similar result was proved recently in [5] by using fixed point theory. Our main objective here is to give an alternative proof by using the classical Perron upper envelope. Therefore, the solution ϕ is given by the following upper envelope of all sub-solutions

$$\phi = \sup\left\{ u; u \in \mathcal{E}(X, \theta) \text{ and } (\theta + dd^c u)^n \geq F(u, \cdot)\mu \right\}.$$

2 Proof

We start the proof with a global version of Demailly's inequality.

Lemma 2.1 *Let $u, v \in \mathcal{E}(X, \theta)$. Then*

$$(\theta + dd^c \max(u, v))^n \geq \mathbf{1}_{\{u\geq v\}}(\theta + dd^c u)^n + \mathbf{1}_{\{u<v\}}(\theta + dd^c v)^n.$$

For the convenience of the reader, we include a proof using the same idea as in [12] in the local context.

Proof It is enough to show the inequality on the set $\{u \geq v\}$. Let $K \subset \{u \geq v\}$ be compact.

First, we assume that u and v are bounded and non-positive. By the quasicontuinity (see [16, Corollary 3.8]), we have for any $\varepsilon > 0$ there exists an open subset $G \subset X$ such that $Cap_X(G) < \varepsilon$ and u and v are continuous $X \setminus G$. Here $\mathrm{Cap}_X(U)$ denotes the capacity of the open set U given by

$$\mathrm{Cap}_X(U) = \sup\left\{ \int_U (\theta + dd^c\varphi)^n, \varphi \in \mathcal{E}(X, \theta) \text{ and } -1 \leq \varphi \leq 0 \right\}.$$

Let $u_j, v_j \in \mathcal{E}(X, \theta)$ be two nonincreasing sequences of continuous functions converging towards u and v, respectively. Then for every $\delta > 0$ there exists an open neighbourhood U of K such that $u_j + \delta \geq v_j$ on $U \setminus G$ for j larger than some j_0. Then

$$\begin{aligned}
\int_K (\theta + dd^c u)^n &\leq \liminf_{j\to\infty} \int_U (\theta + dd^c u_j)^n \\
&\leq (\sup_X |u| + 1)^n \varepsilon + \liminf_{j\to\infty} \int_{U\setminus G} (\theta + dd^c u_j)^n \\
&\leq (\sup_X |u| + 1)^n \varepsilon + \liminf_{j\to\infty} \int_{U\setminus G} \left(\theta + dd^c \max(u_j + \delta, v_j)\right)^n.
\end{aligned}$$

Now let $\varepsilon \to 0$ and $j \to +\infty$ to get

$$\int_K (\theta + dd^c u)^n \leq \int_L \left(\theta + dd^c \max(u + \delta, v)\right)^n,$$

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where $L \supset U$ is compact. Therefore,

$$\int_K (\theta + dd^c u)^n \le \int_K \Big(\theta + dd^c \max(u+\delta, v)\Big)^n,$$

and the inequality follows if we let $\delta \to 0$.

Now, if u and v are not bounded, we consider the sequences $u^j := \max(u, -j)$ and $v^j := \max(v, -j)$. Let $K \subset X$ be compact. Then we have

$$\begin{aligned}
&\int_K (\theta + dd^c \max(u,v))^n \\
&\quad = \lim_{j\to\infty} \int_{K\cap\{\max(u,v)>-j\}} (\theta + dd^c \max(u,v,-j))^n \\
&\quad \ge \liminf_{j\to\infty} \int_{K\cap\{u^j\ge v^j\}\cap\{\max(u,v)>-j\}} (\theta + dd^c u^j)^n \\
&\qquad + \liminf_{j\to\infty} \int_{K\cap\{u^j< v^j\}\cap\{\max(u,v)>-j\}} (\theta + dd^c v^j)^n \\
&\quad = \liminf_{j\to\infty} \Big(\int_{K\cap\{u^j\ge v^j\}\cap\{u>-j\}} (\theta + dd^c u^j)^n + \int_{K\cap\{u^j< v^j\}\cap\{v>-j\}} (\theta + dd^c v^j)^n \Big) \\
&\quad \ge \lim_{j\to\infty} \Big(\int_{K\cap\{u\ge v\}\cap\{u>-j\}} (\theta + dd^c u^j)^n + \int_{K\cap\{u< v\}\cap\{v>-j\}} (\theta + dd^c v^j)^n \Big) \\
&\quad = \int_{K\cap\{u\ge v\}} (\theta + dd^c u)^n + \int_{K\cap\{u<v\}} (\theta + dd^c v)^n. \qquad \blacksquare
\end{aligned}$$

Proof of the Main Theorem Consider the set

$$\mathcal{H} := \Big\{ \varphi \in \mathcal{E}(X,\theta); (\theta + dd^c\varphi)^n \ge F(\varphi, \cdot)\mu \Big\}$$

of all sub-solutions of the Monge–Ampère equation (1.1).

Claim 1. *$\mathcal{H}$ is not empty.*

Indeed, by condition (ii) in the theorem, there exists a real $t_0 \in \mathbb{R}$ such that

$$\int_X F(t_0, x)\, d\mu(x) = \int_X \theta^n.$$

Then, by [7] (see also [9]) there exists a function $u_0 \in \mathcal{E}(X,\theta)$ such that $\max_X u_0 = 0$ and

$$(\theta + dd^c u_0)^n = F(t_0, \cdot)\, d\mu.$$

Hence,

$$\Big(\theta + dd^c(u_0 + t_0)\Big)^n = (\theta + dd^c u_0)^n = F(t_0, \cdot)\, d\mu \ge F(u_0 + t_0, \cdot)\, d\mu.$$

Therefore, $\varphi_0 := u_0 + t_0 \in \mathcal{H}$.

Let $\mathcal{H}_0$ denote $\{\varphi \in \mathcal{H}; \varphi \ge \varphi_0\}$.

Claim 2. *$\mathcal{H}_0$ is stable under taking the maximum.*

Indeed, let $\varphi_1, \varphi_2 \in \mathcal{H}_0$. It is clear that $\max(\varphi_1, \varphi_2) \ge \varphi_0$. Since $\mathcal{E}(X,\theta)$ is stable by taking the maximum, then $\max(\varphi_1,\varphi_2) \in \mathcal{E}(X,\theta)$. On the other hand, by Lemma 2.1,

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we have

$$\begin{aligned}(\theta + dd^c \max(\varphi_1, \varphi_2))^n &\geq \mathbf{1}_{(\varphi_1 \geq \varphi_2)}(\theta + dd^c \varphi_1)^n + \mathbf{1}_{(\varphi_1 < \varphi_2)}(\theta + dd^c \varphi_2)^n \\ &\geq \mathbf{1}_{(\varphi_1 \geq \varphi_2)} F(\varphi_1, \cdot) d\mu + \mathbf{1}_{(\varphi_1 < \varphi_2)} F(\varphi_2, \cdot) d\mu \\ &\geq F(\max(\varphi_1, \varphi_2), \cdot) d\mu,\end{aligned}$$

which implies that $\max(\varphi_1, \varphi_2) \in \mathcal{H}_0$.

Claim 3. *$\mathcal{H}_0$ is compact in $L^1(X)$.*

First, we prove that the functions of $\mathcal{H}_0$ are uniformly bounded from above on X. Let

$$m := \sup_{\varphi \in \mathcal{H}_0} \sup_{x \in X} \varphi(x).$$

Then

$$m = \lim_{j \to \infty} \sup_{x \in X} \varphi_j(x),$$

where φ_j is a sequence in $\mathcal{H}_0$.

Since $\mathcal{H}_0$ is stable under taking the maximum, we can assume that $(\varphi_j)_j$ is non-decreasing. The sequence $(\varphi_j - \sup_X \varphi_j)$ is relatively compact in $L^1(X)$. Let $\widetilde{\varphi}$ be a cluster point of $(\varphi_j - \sup_X \varphi_j)$. Then $\widetilde{\varphi} \in PSH(X, \theta)$. After extracting a subsequence, we can assume that $(\varphi_j - \sup_X \varphi_j)$ converges to $\widetilde{\varphi}$ point-wise on $X \setminus A$, where A is a pluripolar subset of X. By Fatou's lemma, we have

$$\begin{aligned}Vol(\alpha) = \int_X (\theta + dd^c \varphi_j)^n &= \lim_{j \to +\infty} \int_X (\theta + dd^c \varphi_j)^n \\ &\geq \liminf_{j \to +\infty} \int_X F(\varphi_j, \cdot) d\mu \\ &\geq \int_X \liminf_{j \to +\infty} F(\varphi_j - \sup_X \varphi_j + \sup_X \varphi_j, \cdot) d\mu \\ &\geq \int_X F(\widetilde{\varphi} + m, \cdot) d\mu,\end{aligned}$$

which proves that $m < \infty$.

To complete the proof of the claim, it is enough to prove that $\mathcal{H}_0$ is closed. Let $\varphi_j \in \mathcal{H}_0$ be a sequence converging towards a function $\varphi \in PSH(X, \theta)$. The limit function is given by $\varphi = (\limsup_{j \to \infty} \varphi_j)^* = \lim_{j \to \infty} (\sup_{k \geq j} \varphi_k)^*$. Hence, $\varphi \geq \varphi_0$ and therefore $\varphi \in \mathcal{E}(X, \theta)$. Now observe that the sequence $(\sup_{k \geq j} \varphi_k)^*$ decreases towards φ and for any $j \in \mathbb{N}$, the sequence $(\max_{l \geq k \geq j} \varphi_k)_{l \in \mathbb{N}}$ increases towards $(\sup_{k \geq j} \varphi_k)^*$. Thus, the continuity of the complex Monge–Ampère operator along monotonic sequences and Lemma 2.1 yield

$$\begin{aligned}(\theta + dd^c \varphi)^n &= \lim_{j \to +\infty} (\theta + dd^c (\sup_{k \geq j} \varphi_k)^*)^n \\ &= \lim_{j \to +\infty} \lim_{l \to +\infty} (\theta + dd^c \max_{l \geq k \geq j} \varphi_k)^n \\ &\geq \lim_{j \to +\infty} \lim_{l \to +\infty} F(\max_{l \geq k \geq j} \varphi_k, \cdot) d\mu \\ &\geq F(\varphi, \cdot) d\mu.\end{aligned}$$

Therefore, $\varphi \in \mathcal{H}_0$.

Now consider the following upper envelope

$$\phi(x) := \sup\{v(x);\ v \in \mathcal{H}_0\}, \quad \forall x \in X.$$

Notice that in order to get a θ-psh function ϕ we should a priori replace ϕ by its upper semi-continuous regularization $\phi^*(z) := \limsup_{\zeta \to z} \phi(\zeta)$, but since $\phi^* \in \mathcal{H}_0$, ϕ^* contributes to the envelope and therefore $\phi = \phi^*$.

Claim 4. *ϕ is the solution to Monge–Ampère equation* (1.1).

Indeed, by Choquet's lemma there exists a sequence $\phi_j \in \mathcal{H}_0$ such that

$$\phi = \big(\limsup_{j \to +\infty} \phi_j\big)^*.$$

Since $\mathcal{H}_0$ is stable under taking the maximum, we can assume that the sequence $\phi_j \in \mathcal{H}_0$ is nondecreasing.

Let B_1 be a local chart such that $\theta = dd^c\rho$, where ρ is smooth in B_1. Fix $B \Subset B_1$ to be a small ball. For $j \geq 1$, the sequence $h_j^k := \max(\phi_j, -k) \in \mathcal{E}(X,\theta)$ and decreases to ϕ_j. Now the function $f_j^k := \rho + \phi_j^k$ is *bounded psh* on B. Denote the set

$$\mathcal{G}(B) = \Big\{ u \in \mathcal{E}(B); \limsup_{z \to \partial B} u(z) \leq \widetilde{f_j^k} \text{ and } (dd^c u)^n \geq \mathbf{1}_B F(u - \rho, \cdot)d\mu \Big\},$$

where $\widetilde{f_j^k}$ denotes the smallest maximal function above f_j^k (*cf.* [10] for the general definition), but in our context, it can be defined by

$$\widetilde{f_j^k}(z) := \sup\Big\{ v(z); \limsup_{z \to \partial B} v(z) \leq f_j^k \text{ on } \partial B, v \in PSH(B) \Big\}, \quad \forall z \in B,$$

where $\mathcal{E}(B)$ denotes the largest subset of $PSH(B)$ where the (local) complex Monge–Ampère is well defined (*cf.* [11] for more details).

Consider the function

$$H_j^k(z) = \sup\Big\{ u(z);\ u \in \mathcal{G}(B) \Big\}, \quad \forall z \in B.$$

It follows from [6] that $(dd^c H_j^k)^n$ is well defined as a nonnegative measure and

$$(dd^c H_j^k)^n = \mathbf{1}_B F(H_j^k - \rho, \cdot)d\mu.$$

Let ψ_j^k be the function given by $H_j^k - \rho$ on B and extended on the complementary of B by h_j^k. Then ψ_j^k is a *global* θ-psh and decreasing with respect to k. Denote $\psi_j := \lim_{k \to +\infty} \psi_j^k$. This is a θ-psh function on X and equal to ϕ_j on $X \setminus B$. On B we have

$$(\theta + dd^c\psi_j)^n = \lim_{k \to +\infty} (dd^c H_j^k)^n = \mathbf{1}_B \lim_{k \to +\infty} F(H_j^k - \rho, \cdot)d\mu.$$

Hence $\psi_j \in \mathcal{H}$ and $\psi_{j+1} \geq \psi_j \geq \varphi_j$. Then $\phi = \lim_{j \to \infty} \psi_j$. The continuity of the complex Monge–Ampère operator along monotonic sequences imply that ϕ is a solution of (1.1) on B and therefore on X, since B was arbitrary chosen.

Uniqueness follows in a classical way from the comparison principle [3] and its generalizations [9, 13]. Indeed, assume that there exist two solutions φ_1 and φ_2 in $\mathcal{E}(X,\theta)$ such that

$$(\theta + dd^c\varphi_i)^n = F(\varphi_i, \cdot)d\mu, \quad i = 1, 2.$$

Then

$$\int_{(\varphi_1<\varphi_2)} F(\varphi_1,\cdot)d\mu \le \int_{(\varphi_1<\varphi_2)} F(\varphi_2,\cdot)d\mu = \int_{(\varphi_1<\varphi_2)} (\theta+dd^c\varphi_2)^n$$
$$\le \int_{(\varphi_1<\varphi_2)} (\theta+dd^c\varphi_1)^n = \int_{(\varphi_1<\varphi_2)} F(\varphi_1,\cdot)d\mu.$$

Therefore,

$$F(\varphi_1,\cdot)d\mu = F(\varphi_2,\cdot)d\mu \quad \text{on } (\varphi_1<\varphi_2).$$

In the same way, we get the equality on $(\varphi_1>\varphi_2)$ and then on X. Hence,

$$(\theta+dd^c\varphi_1)^n = (\theta+dd^c\varphi_2)^n.$$

It follows from [13, Theorem 1.2] that $\varphi_1-\varphi_2$ is constant which completes the proof. ∎

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