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GROUPOID ENRICHED CATEGORIES AND HOMOTOPY THEORY

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Introduction. We are concerned in this paper with category-theoretic aspects of homotopy theory. Originally, category theory developed as a simplifying language in the context of algebraic topology and yet one primary example: the category $\prod$ of spaces and homotopy classes of maps admits only limited use of the language owing to the very sparse occurrence of limits. Of course, full use has been made of them nevertheless: limits and colimits exist in the case of products and coproducts, and in almost no other case; yet, from this we obtain the theory of Samelson products, Whitehead products, and Hopf invariants which can all be expressed in $\prod$ see [**8**]. In addition, there are hosts of adjoint functors and yet the outcome is disappointing because the language applies only to special cases rather than to the situation as a whole.

It is for this reason that we propose to study those concepts that arise from spaces, maps, and homotopy classes of homotopies of maps. As it stands, they comprise a special type of 2-category (see [**10**]) in which the morphism sets form a groupoid, viz., a "groupoid enriched category" (g.e. category).

The idea of doing homotopy theory using a category embellished with homotopies was originally suggested by Gabriel and Zisman [**9**] and they investigated several features of this. Before and since that time, 2-categories have been extensively studied (cf. [**10**, **12**, **20**, **21**]), but by authors who have had quite a different type of paradigm example in mind, namely, the category of categories.

Our concentration on spaces, maps, and homotopies suggests that one can view the approach not only as an extension of the study of $\prod$, but also as a truncation of the quite general type of homotopy theory that takes into account, what one may call, "higher homotopies": actual homotopies, homotopies of these homotopies, and so forth. Thus, in the sense that $\prod$

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truncates this general set-up at the bottom level, the appropriate g.e. category truncates it at the next level.

Although higher homotopies are essential in describing properties of H-spaces, for instance, the reasons they come into the picture here are due to the development of limits, in this setting, firstly by Mather [**14**], and then by Vogt [**22**, **23**], Walker [**24**], and others. The advantage of this approach is that the limits have an expected universality property in a homotopy sense; the disadvantages are, firstly, the incredible complexity of the situation and, secondly, the lack of clarity as to how one can abstract the situation to form what one would like to call a higher homotopy category. Such a definition would be essential if one wished to investigate, for example, the behaviour of limits under the chain-complex functor, or to develop a useable adjoint functor or a triple theory. In fact a preliminary look at the number of possibilities would suggest that there are difficulties in defining even the composition of the counterpart of natural transformations cf. [**22**]. (For another approach see [**2**].)

Regarded as a truncation of higher homotopies, the study of g.e. categories can be regarded as a preliminary probe: natural transformations are easy to handle, the representable functors of ordinary category theory have a simple counterpart, and there are useable adjoint functor theorems (as we shall see in examples). The counterpart of limits, however, fails to be universal in an interesting respect and, for this reason, are inferior to the limits of Mather [**14**] or the pseudo-limits in 2-category theory. By contrast, regarded as an extension of $\prod$, the g.e. language applied to CW complexes appears to recapture all of classical homotopy theory (including, of course, the homology and cohomology groups of CW complexes) with the exception of those features (such as triads, A_n-spaces and higher Toda brackets) that are specifically involved with higher homotopies. In particular, in contrast to the closed model theory of Quillen [**19**], the special role of fibrations and cofibrations disappears. We have no intention here of explaining how all this comes about although we shall give many specific cases as illustrative examples. More detail of certain aspects may be found in [**7**, **17**].

1. G. E. categories.

(1.1) *Notation.* Recall that a *groupoid* G is a small category whose morphisms are invertible. The set of objects (called *points*) and morphisms (called *paths*) will be written G_0, G_1. $\pi_0(G)$ denotes the isomorphism classes of points; $G(x, x)$ with identity e_x is written $\pi_1(x)$. Functors and natural transformations of groupoids are called *homomorphisms* and *homo-*

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topies. A homomorphism $f:G \to H$ induces $\pi_0 G \to \pi_0 H$, $\pi_1(x) \to \pi_1(f(x))$. If these are bijections we say that f is respectively a π_0-*equivalence*, π_1-*equivalence*. In the latter case we note that every path in H between objects in the image of f is the image of exactly one path in G (the "lifting property"). If $\pi_1(x) \to \pi_1(f(x))$ is surjective for each x, we say that f is π_1-*surjective*.

A g.e. *category* is a 2-category $\mathscr{C}$ for which $\mathscr{C}(A, B)$, which we denote $\mathbf{Hom}(A, B)$, is, for each A, B, a groupoid whose operation is written $\cdot$ and whose objects and morphisms (written $\alpha:f \xrightarrow{\sim} g$) are called maps and homotopies (rather than the more usual 2-cells which leads to confusion in the examples). Composition

$$\mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) \times \mathbf{Hom}(A, C)$$

is a functor and is denoted $f \circ g$ for morphisms, $f * \beta$, $\alpha * g$, etc. in the other cases. We refer to homotopic maps in the evident way; an *equivalence* is a map f with $g \circ f$, $f \circ g$ homotopic to the respective identities. We note that an ordinary category is a g.e. category in which the homotopies are trivial.

We observe that groupoids, with respect to homomorphisms and homotopies form a g.e. category which we denote $\mathscr{G}$ and we note without difficulty that $f:G \to H$ is an equivalence if and only if the induced $\pi_0(G) \to \pi_0(H)$, $\pi_1(x) \to \pi_1(f(x))$ are bijections (the Whitehead Lemma).

A *pseudo-functor* (p-functor) $\mathscr{T}:\mathscr{C} \to \mathscr{D}$ between g.e. categories is an assignment of objects, maps, and homotopies together with homotopies

$$\mathscr{T}(g, f):\mathscr{T}(g) \circ \mathscr{T}(f) \xrightarrow{\sim} \mathscr{T}(g \circ f)$$

called the cells of $\mathscr{T}$, given for composable f, g such that, for $\alpha:g \xrightarrow{\sim} h$, we have

$$(1)\begin{cases} \mathscr{T}(f * \alpha) = \mathscr{T}(f, h) \cdot (\mathscr{T}(f) * \mathscr{T}(\alpha)) \cdot \mathscr{T}(f, g)^{-1} \\ \mathscr{T}(\alpha * f) = \mathscr{T}(h, f) \cdot (\mathscr{T}(\alpha) * \mathscr{T}(f)) \cdot \mathscr{T}(g, f)^{-1} \\ \mathscr{T}((h \circ g, f) \circ (\mathscr{T}(h, g) * \mathscr{T}(f)) = \mathscr{T}(h, g \circ f) \circ (\mathscr{T}((h) * \mathscr{T}(g, f)) \\ \mathscr{T}(1_X) = 1_{\mathscr{T}(X)} \\ \mathscr{T}(\alpha \cdot \beta) = \mathscr{T}(\alpha) \cdot \mathscr{T}(\beta). \end{cases}$$

If the cells are trivial, $\mathscr{T}$ is called a *strict functor*. For a g.e. category $\mathscr{C}$, we denote by $\pi\mathscr{C}$ the category with the same objects as $\mathscr{C}$ and whose maps are homotopy classes of maps of $\mathscr{C}$; $\pi\mathscr{C}$ is called the *homotopy class category* of $\mathscr{C}$. The morphism sets of $\pi\mathscr{C}$ are denoted $\pi(X, Y)$ and we note that a

p-functor induces an ordinary functor $\pi\mathscr{C} \to \pi\mathscr{D}$. Observe, also, that $f \in \mathscr{C}$ is an equivalence if and only if cls (f) in $\pi\mathscr{C}$ is invertible.

If $\mathscr{T}:\mathscr{C} \to \mathscr{D}$ and $\mathscr{U}:\mathscr{D} \to \mathscr{E}$ are p-functors of g.e. categories, we define the *composition* $\mathscr{U} \circ \mathscr{T}:\mathscr{C} \to \mathscr{E}$ by composing the element assignments and putting

$$(\mathscr{U} \circ \mathscr{T})(f, g) = \mathscr{U}(\mathscr{T}(f, g)) \cdot \mathscr{U}(\mathscr{T}(f), \mathscr{T}(g)).$$

It is immediately verified that this satisfies the conditions for a p-functor.

If $\mathscr{S}$, $\mathscr{T}:\mathscr{C} \to \mathscr{D}$ are p-functors of g.e. categories, a *pseudo natural transformation* (p-natural transformation) $\tau:\mathscr{T} \to \mathscr{S}$ assigns to each object X of $\mathscr{C}$ a map $\tau_X:\mathscr{T}(X) \to \mathscr{S}(X)$ and for each map $f:X \to Y$ a homotopy

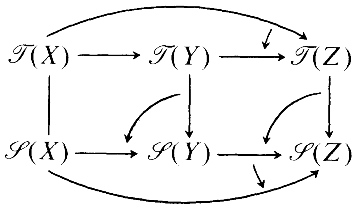
$$\tau_f:\tau_Y \circ \mathscr{T}(f) \overset{\sim}{\Rightarrow} \mathscr{S}(f) \circ \tau_X$$

such that

$$(2) \begin{cases} \tau_{g\circ f} = (\mathscr{S}(g, f) * \tau_X) \cdot (\mathscr{S}(g) * \tau_f) \cdot (\tau_g * \mathscr{T}(f)) \cdot (\tau_Z * \mathscr{T}(g, f)^{-1}), \\ (\mathscr{S}(\alpha) * \tau_X) \cdot \mathscr{T}_f = \tau_h \cdot (\tau_Y * \mathscr{T}(\alpha)). \end{cases}$$

A p-natural transformation with τ_f trivial for all f will be called *strict*.

N.B. One should note that when working with such conditions as (1) and (2) it is useful to draw diagrams. For example, the right hand side expression for $\tau_{g \circ f}$ can best be described as the composed homotopy in the diagram



(For detailed comments on the so-called pasting of diagrams in 2-categories, see [**10**, **12**, **21**].)

The *composition* of p-natural transformations $\tau:\mathscr{S} \to \mathscr{T}$, $\tau':\mathscr{T} \to \mathscr{U}$ with $\mathscr{S}$, $\mathscr{T}$, $\mathscr{U}:\mathscr{C} \to \mathscr{D}$ is $\tau' \circ \tau:\mathscr{S} \to \mathscr{U}$ defined by

$$(\tau' \circ \tau)_X = \tau'_X \circ \tau_X, \quad (\tau' \circ \tau)_f = (\tau'_f * \tau_X) \cdot (\tau'_Y * \tau_f).$$

A *homotopy* (instead of the familiar *modification*) of p-natural transformations $\alpha:\tau \overset{\sim}{\Rightarrow} \tau'$ with τ, $\tau':\mathscr{S} \to \mathscr{T}$, $\mathscr{S}$, $\mathscr{T}:\mathscr{C} \to \mathscr{D}$, assigns to each object of X of $\mathscr{C}$ a homotopy $\alpha_X:\tau_X \overset{\sim}{\Rightarrow} \tau'_X$ such that

$$(\mathscr{S}(f) * \alpha_X) \cdot \tau'_f = \tau_f \cdot (\alpha_Y^{-1} * \mathscr{T}(f)).$$

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We define $\alpha * \tau$, $\tau * \alpha$ for a homotopy α and a p-natural transformation τ by

$$(\alpha * \tau)_X = \alpha_X * \tau_X, \quad (\tau * \alpha)_X = \tau_X * \alpha_X.$$

With these definitions, the p-functors $\mathscr{C} \to \mathscr{D}$ for given $\mathscr{C}$, $\mathscr{D}$ along with their p-natural transformations and homotopies form a g.e. category Psfun(C, D) with morphism groupoids that we denote pnt(S, T). (This is in a slightly extended sense, since the groupoids involved are not necessarily small.)

A p-natural transformation that is an equivalence in this category is called a *pseudo* (*natural*) *equivalence*, whereas a p-functor $\mathscr{F}:\mathscr{C} \to \mathscr{D}$ is called a *pseudo equivalence* if there is a functor $\mathscr{U}$ with $\mathscr{U} \circ \mathscr{F}$, $\mathscr{F} \circ \mathscr{U}$ both pseudo equivalent to the respective identities; when such a functor exists, we say that the categories $\mathscr{C}$ and $\mathscr{D}$ are *pseudo equivalent*.

N.B. In ordinary category theory if a functor $\mathscr{F}$ is defined by choices, the accepted attitude is that it is to be regarded as unique if a functor obtained by different choices is naturally equivalent to it. In judging the examples it is convenient in the present situation, where different choices are merely pseudo equivalent, to describe the functor as "pseudo unique".

For a p-natural transformation $\tau:\mathscr{S} \to \mathscr{T}$, $\mathscr{S}, \mathscr{T}:\mathscr{C} \to \mathscr{D}$ and p-functors $\mathscr{U}:\mathscr{B} \to \mathscr{C}$, $\mathscr{V}:\mathscr{D} \to \mathscr{E}$, we define the p-natural transformations

$$\tau * \mathscr{U}:\mathscr{S} \circ \mathscr{U} \to \mathscr{T} \circ \mathscr{U} \quad \text{and} \quad \mathscr{V} * \tau:\mathscr{V} \circ \mathscr{S} \to \mathscr{V} \circ \mathscr{T}$$

by

$$(\tau * \mathscr{U})_X = \tau_{\mathscr{U}(X)}, \quad (\tau * \mathscr{U})_f = \tau_{\mathscr{U}(f)} \quad \text{and}$$
$$(\mathscr{V} * \tau)_X = \mathscr{V}(\mathscr{T}_{\mathscr{V}(X)}), (\mathscr{V} * \tau)_f$$
$$= \mathscr{V}(\mathscr{T}(f), \tau_X)^{-1} \cdot \mathscr{V}(\tau_f) \cdot \mathscr{V}(\tau_Y, (f)).$$

It is easy to check that these are indeed p-natural transformations and that they define p-functors

$$\text{Psfun}(\mathscr{C}, \mathscr{D}) \to \text{Psfun}(\mathscr{B}, \mathscr{D}), \quad \text{Psfun}(\mathscr{C}, \mathscr{D}) \to \text{Psfun}(\mathscr{C}, \mathscr{E}) .$$

Finally, contravariant p-functors and the opposite category $\mathscr{C}^{\text{opp}}$ are introduced in the evident way.

(1.2) *Lemmas.*

HOMOTOPY REPLACEMENT LEMMA. *If $\tau:\mathscr{S} \to \mathscr{T}$ is a p-natural transformation of p-functors and for each X there is a given homotopy*

$$\theta_X : \tau_X \xrightarrow{\sim} \tau'_X : \mathcal{S}(X) \to \mathcal{T}(X)$$

then there is a unique p-natural transformation $\tau' : \mathcal{S} \to \mathcal{T}$ *whose object components are* τ'_X *and for which* $\theta_X : \tau_X \xrightarrow{\sim} \tau'_X$ *defines a homotopy* $\tau \xrightarrow{\sim} \tau'$.

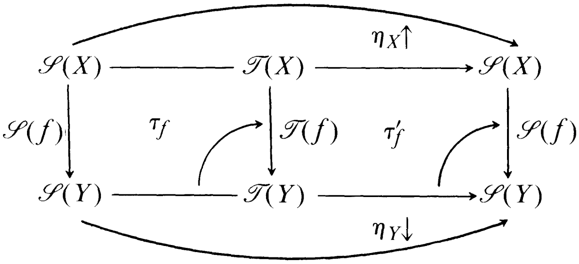
The proof of this lemma is immediate.

INDUCED EQUIVALENCE THEOREM. *If* $\mathcal{C}$ *is a g.e. category, a map* $f : X \to Y$ *is an equivalence if and only if the induced* $f^* : \mathrm{Hom}(Y, W) \to \mathrm{Hom}(X, W)$ *is an equivalence of groupoids for each* W *(resp. the induced* $\mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y)$ *is an equivalence for each* U*).*

Proof. Let $\alpha : g \circ f \xrightarrow{\sim} 1_X$, $\beta : f \circ g \xrightarrow{\sim} 1_Y$ define homotopies of $g^* \circ f^*$, $f^* \circ g^*$ to the identities so that if f is an equivalence, so is f^*. Conversely, if f^* is an equivalence the induced $\pi(X, W) \to \pi(Y, W)$ is bijective and the result follows by ordinary category theory. The other case is proved similarly.

PSEUDO EQUIVALENCE THEOREM. *A p-natural transformation* $\tau : \mathcal{S} \to \mathcal{T}$ *with* $\mathcal{S}, \mathcal{T} : \mathcal{C} \to \mathcal{D}$ *p-functors on a g.e. category is a pseudo equivalence if and only if each* $\tau_X : \mathcal{S}(X) \to \mathcal{T}(X)$ *is an equivalence.*

Proof. To establish this, choose a homotopy inverse τ'_X to τ_X for each X and homotopies $\eta_X : \tau'_X \circ \tau_X \xrightarrow{\sim} \mathrm{id}_X$. Construct τ'_f so that the composed homotopy $\mathcal{S}(X) \to \mathcal{S}(Y)$ represented by the following diagram is trivial:



Indeed by the Induced Equivalence Theorem,

$$\mathbf{Hom}(\mathcal{T}(X), \mathcal{S}(Y)) \to \mathbf{Hom}(\mathcal{S}(X), \mathcal{S}(Y))$$

is an equivalence so, by the lifting property, τ'_f exists and is unique. Also (via some mild diagram chasing) τ'_X, τ'_f define a natural transformation $\tau' : \mathcal{T} \to \mathcal{S}$, and η_X defines a homotopy $\tau' \circ \tau \xrightarrow{\sim} \mathrm{id}$. We similarly obtain τ'' with $\tau \circ \tau'' \xrightarrow{\sim} \mathrm{id}$ where, for elementary reasons, τ' is homotopic to τ''. We may, therefore, replace τ'' in its role by τ' and the result is established.

Remark. We note from the proof that the theorem can be improved somewhat by requiring τ'_X and η_X to satisfy additional requirements. For

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example, if τ_X is the identity whenever X belongs to a certain class α of objects, then we can insist that $\tau'_X = \mathrm{id}_X$ and that η_X is trivial when $X \in \alpha$.

THE EXTENSION LEMMA. *Let* $\mathcal{S}:\mathcal{C} \to \mathcal{D}$ *be a p-functor of g.e. categories,* $\mathcal{T}$ *be a class function sending objects and maps of* $\mathcal{C}$ *to objects and maps of* $\mathcal{D}$, η_X *be a pseudo equivalence* $\mathcal{S}(X) \to \mathcal{T}(X)$ *defined for each* X, *and* η_f *be a homotopy* $\mathcal{S}(X) \to \mathcal{T}(Y)$ *defined for each* $f:X \to Y$ *such that*

$$\eta_f:\eta_Y \cdot \mathcal{S}(f) \rightrightarrows \mathcal{T}(f) \cdot \eta_X.$$

Then $\mathcal{T}$ *extends to a functor for which* $\{\eta_X, \eta_f\}$ *defines a p-natural equivalence.*

Proof. By the Induced Equivalence Theorem and the lifting property we define unique homotopies $\mathcal{T}(f, g)$, $\mathcal{T}(\alpha)$ so that the equations in the definition of p-natural transformations are satisfied. We then check that these additional terms turn $\mathcal{T}$ into a functor for which we obtain the described p-natural transformation. By the Pseudo Equivalence Theorem, this is an equivalence.

THE LIFTING THEOREM. *A p-functor* $\mathcal{F}:\mathcal{C} \to \mathcal{D}$ *of g.e. categories is a pseudo equivalence if and only if* (i) *the induced functor* $\pi\mathcal{F}:\pi\mathcal{C} \to \pi\mathcal{D}$ *is an equivalence of categories* (*in the ordinary sense*), (ii) *for each* X, Y *in* $\mathcal{D}$ *the induced* $\mathbf{Hom}(X, Y) \to \mathbf{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$ *is an equivalence of groupoids.*

Proof. That the pseudo equivalence of $\mathcal{F}$ implies (i) and (ii) is clear. Let us establish the converse first for the case where $\mathcal{F}$ is the inclusion of a full subcategory. In each class A of equivalent objects of $\mathcal{D}$, choose an object Z_A in $\mathcal{C}$. For each $Z \in A$, define $\mathcal{H}(Z) = Z$ if $Z \in \mathrm{Im}\mathcal{F}$ and $\mathcal{H}(Z) = Z_A$ otherwise. Choose inverse equivalences $\phi_Z:Z \to Z_A$, $\psi_Z:Z_A \to Z$ with $\theta_Z:\mathrm{id} \to \psi_Z \cdot \phi_Z$, with the proviso that if $Z \in \mathcal{C}$ all these are trivial at the identity. For $f:Z \to W$ define

$$\mathcal{H}(f) = \phi_W \circ f \circ \psi_Z, \quad \eta_Z = \phi_Z \quad \text{and} \quad \eta_f = (\phi_W \circ f) * \theta_Z.$$

Then by the Extension Lemma, the special case follows.

In the general case let $\mathcal{D}'$ be the full subcategory generated by $\mathrm{Im}\mathcal{F}$; for each $Z \in \mathcal{D}'$ choose $\mathcal{H}(Z)$ with $\mathcal{F}\mathcal{H}(Z) = Z$ and let $\mathcal{C}'$ be the full subcategory generated by $\mathrm{Im}\mathcal{H}$. Clearly, the inclusions $\mathcal{C}' \hookrightarrow \mathcal{C}$ and $\mathcal{D}' \hookrightarrow \mathcal{D}$ are homotopy equivalences by the special case.

Define $\mathcal{H}$ on maps and homotopies of $\mathcal{D}'$ so that the induced

$$\mathcal{H}^*:\mathbf{Hom}(Z, W) \to \mathbf{Hom}(\mathcal{H}(Z), \mathcal{H}(W))$$

is a homotopy inverse of the appropriate

$$\mathscr{F}^*:\mathbf{Hom}(X, Y) \to \mathbf{Hom}(\mathscr{F}(X), \mathscr{F}(Y))$$

induced by $\mathscr{F}$ and choose $\eta_{Z,W}:\mathscr{F} \circ \mathscr{H} \overset{\sim}{\Rightarrow} \mathrm{Id}$. The image under $\eta_{Z,W}$ of $f \in \mathrm{Hom}_{\circ}(Z, W)$ will be denoted $\eta(f)$ so that

$$\alpha \cdot \eta(g) = \eta(h) \cdot \mathscr{F}\mathscr{H}(\alpha) \quad \text{for } \alpha:g \overset{\sim}{\Rightarrow} h$$

and we define $\mathscr{H}(g, f):\mathscr{H}(g) \circ \mathscr{H}(f) \to \mathscr{H}(g \circ f)$ to be the unique path satisfying

$$\eta(g \circ f) \cdot \mathscr{F}(\mathscr{H}(g, f)) \cdot \mathscr{F}(\mathscr{H}(g), \mathscr{H}(f)) = \eta(g) * \eta(f).$$

One then verifies by directly checking the definitions of p-functor and p-natural transformation that this prescription defines a p-functor $\mathscr{H}$ for which $\zeta:\mathscr{F} \circ \mathscr{H} \to \mathrm{Id}$ defined by $\zeta_Z = \mathrm{Id}$, $\zeta_f = \eta(f)$ is a p-natural transformation. That ζ is a pseudo equivalence follows by the Pseudo-Equivalence Theorem. Thus, the induced p-functor $\mathscr{C}' \to \mathscr{D}'$ and hence, again by repeated use of the Pseudo Equivalence Theorem, $\mathscr{F}:\mathscr{C} \to \mathscr{D}$ is itself a homotopy equivalence.

(1.3) *Examples.*

Example 1. Topological spaces with continuous maps and homotopy classes of homotopies of maps form a g.e. category as is readily seen; we shall refer to it as *Top*. We can restrict this in various ways. Thus, with pointed spaces and basepoint preserving maps and homotopies we obtain $Top^{\bullet}$. Similarly, with compactly generated spaces of the homotopy type of well pointed CW complexes, continuous basepoint preserving maps, and basepoint preserving homotopy classes of basepoint preserving homotopies, we obtain $CW^{\bullet}$. Despite the preponderant role played by $\hat{\mathscr{G}}$ in framing the theory, $CW^{\bullet}$ must be regarded as the paradigm case.

Example 2. Chain complexes (free, projective, or arbitrary) over a ring together with chain maps and chain homotopy classes of chain homotopies form a g.e. category in the evident way. The full subcategory produced from free positive $\mathbf{Z}$-complexes $\to C_n \to \dots \to C_0$ will be denoted *Ch*.

Example 3. For each connected X in $CW^{\bullet}$, let $\eta_X:X \to X^{[n]}$ be a map that induces isomorphisms in the homotopy groups π_i, $i \leqq n$ and such that $\pi_i(X^{[n]}) = 0$, $i > n$. This construction is certainly not unique but, for choices η_X, η_Y, each $f:X \to Y$ provides a map $f^{[n]}$ unique to within homotopy such that $f^{[n]} \circ \eta_X \simeq \eta_Y \circ f$. (This means, of course, that the induced maps define a unique functor $\pi CW^{\bullet} \to \pi CW^{\bullet}$.) Choose such an $f^{[n]}$ and a homotopy

$$\eta_f{:}f^{[n]} \circ \eta_X \overset{\sim}{\Rightarrow} \eta_Y \circ f \quad \text{for each } f.$$

We contend that these terms extend to give a p-functor $\mathscr{T}$ with $\mathscr{T}(X) = X^{[n]}$, $\mathscr{T}(f) = f^{[n]}$, and a p-natural transformation defined by the chosen η_f, η_X. Furthermore, we contend that different choices are pseudo equivalent.

Let us begin with a standard model $\mathscr{T}(X)$ for $X^{[n]}$ obtained by adding cells to X and let $\mathscr{T}(f)$ be chosen to extend f. To extend $\mathscr{T}$ to a p-functor, we define $\mathscr{T}(\alpha)$ for homotopies α by extension and $\mathscr{T}(f, g)$ as the class of a rel. X homotopy. One then verifies that the conditions for a p-functor are satisfied, that is, because all homotopies $\mathscr{T}(X) \to \mathscr{T}(Y)$ extending a given homotopy $X \to Y$ are homotopic. Clearly the inclusion defines a strict p-natural transformation. Any given $\{X^{[n]}, f^{[n]}, \eta_X\}$ can be used to construct, by choices, terms $\bar{\eta}_X{:}\mathscr{T}(X) \to X^{[n]}$ and $\bar{\eta}_f{:}\mathscr{T}(X) \to Y^{[n]}$ so that

$$\bar{\eta}_f{:}\bar{\eta}_Y \circ \mathscr{T}(f) \overset{\sim}{\Rightarrow} f^{[n]} \circ \bar{\eta}_X.$$

Since $\bar{\eta}_X$ induces isomorphisms of the homotopy groups, it is a homotopy equivalence and the assertions now follow immediately from the Extension Lemma.

Let $CW^{[n]}$ denote the full subcategory of $CW^{\bullet}$ formed of connected spaces whose homotopy groups vanish above the n^{th}. Thus the above considerations define p-functors $CW^{\bullet} \to CW^{[n]}$ that are all pseudo equivalent.

Example 4. We note that $CW^{[1]}$ is the full subcategory of Eilenberg-MacLane 1-spaces and we consider the functor $\pi_1{:}CW^{[1]} \to$ category of groups, defined by the fundamental group functor. We recall that π_1 induces a bijection between the homotopy classes $X \to Y$ and the homomorphisms $\pi_1(X) \to \pi_1(Y)$; we also note that all homotopies $\alpha{:}f \overset{\sim}{\Rightarrow} g$ (in the ordinary sense) of $f, g{:}X \to Y$, $X, Y \in CW^{[n]}$, are homotopic to one another. It then follows by the Lifting Theorem that π_1 is a pseudo equivalence of g.e. categories. A similar argument shows that the full subcategory of $CW^{\bullet}$ of spaces $K(\pi, n)$ for fixed n is pseudo equivalent to the category of abelian groups.

Example 5. Let CW' be the full subcategory of $CW^{\bullet}$ obtained from CW complexes, cellular maps, and classes of cellular homotopies. As a consequence of the Lifting Theorem, one sees that there is a p-functor $CW^{\bullet} \to CW'$ extending the identity and, from the Pseudo Equivalence Theorem, that such functors are pseudo unique.

The process of constructing the various simplicial-cellular-singular homology groups in $CW^{\bullet}$ consists essentially of choosing such a $CW^{\bullet} \to CW'$ and then composing with the cellular chain complex functor to Ch. The result is clearly pseudo unique.

Example 6. Let Sing:$Top \to CW^{\bullet}$ be, to begin with, the ordinary realized singular complex functor. This extends in the evident way to a strict p-functor of g.e. categories. Restricted to CW' this functor is a pseudo equivalence (because, Sing $X \cong X$ for $X \in CW'$) and, consequently the restriction to $CW^{\bullet}$ is also a pseudo equivalence.

2. h-limits.

(2.1) We recall [**6**] that a *graph* $\mathscr{C}$ is a class comprising *points* X and *arrows* $f:X \to Y$, and that a *diagram* $\mathscr{F}:\mathscr{C} \to \mathscr{D}$ of graphs is a function sending points to points and arrows to arrows such that if $f:X \to Y$ then $\mathscr{F}(f):\mathscr{F}(X) \to \mathscr{F}(Y)$. In the sequel, $\mathscr{C}$ will be small and $\mathscr{D}$ will be a g.e. category whose points and arrows are the objects and maps. We define a *pseudo-map* (p-map) $\tau:\mathscr{S} \to \mathscr{T}$ of diagrams $\mathscr{S}, \mathscr{T}:\mathscr{C} \to \mathscr{D}$ to consist of

$$\tau(X):\mathscr{S}(X) \to \mathscr{T}(X),$$

for each point X of $\mathscr{C}$ and

$$\tau(f):\tau(Y) \circ \mathscr{S}(f) \stackrel{\sim}{\Rightarrow} \mathscr{T}(f) \circ \tau(X),$$

for each arrow $f:X \to Y$ of $\mathscr{C}$. p-maps are composed in the evident way; that is, $\sigma:\mathscr{R} \to \mathscr{S}$, $\tau:\mathscr{S} \to \mathscr{T}$ gives $\tau \circ \sigma:\mathscr{R} \to \mathscr{T}$ by

$$\tau \circ \sigma(X) = \tau(X) \circ \sigma(X) \quad \text{and}$$
$$\tau \circ \sigma(f) = (\tau(f) * \sigma(X)) \cdot (\tau(Y) * \sigma(f)).$$

A homotopy $\theta:\sigma \stackrel{\sim}{\Rightarrow} \tau$, where $\sigma, \tau:\mathscr{S} \to \mathscr{T}$, consists of

$$\theta(X):\sigma(X) \stackrel{\sim}{\Rightarrow} \tau(X)$$

for each point $x \in \mathscr{C}$ such that

$$(\mathscr{T}(f) * \theta(x)) \cdot \sigma(f) = \tau(f) \cdot (\theta(Y) * \mathscr{S}(f)).$$

One then sees that for given $\mathscr{C}, \mathscr{D}$ the diagrams, p-maps, and homotopies form a g.e. category Diag $(\mathscr{C}, \mathscr{D})$; it is in this regard that we write **Hom**$(\mathscr{T}, \mathscr{S})$ for diagrams $\mathscr{T}, \mathscr{S}:\mathscr{C} \to \mathscr{D}$.

For a p-functor $\mathscr{V}:\mathscr{D} \to \mathscr{E}$ we define, for a diagram $\mathscr{S}:\mathscr{C} \to \mathscr{D}$, the diagram $\mathscr{V} \circ \mathscr{S}$ in the obvious termwise fashion and, as with p-natural transformations, for a p-map $\tau:\mathscr{S} \to \mathscr{T}$ of diagrams we define $\mathscr{V} * \tau:\mathscr{V} \circ \mathscr{S} \to \mathscr{V} \circ \mathscr{T}$ by

$$(\mathscr{V} * \tau)(X) = \mathscr{V}(\tau(\mathscr{V}(X))),$$
$$(\mathscr{V} * \tau)(f) = \mathscr{V}(T(f), \tau(X))^{-1} \cdot \mathscr{V}(\tau(f)) \cdot \mathscr{V}(\tau(Y), \mathscr{S}(f));$$

whence we define a p-functor $\mathscr{V}_*:\text{Diag}(\mathscr{C}, \mathscr{D}) \to \text{Diag}(\mathscr{C}, \mathscr{E})$.

For each $X \in \mathscr{D}$, we define the *constant diagram* $k_X:\mathscr{C} \to \mathscr{D}$ which assigns X and id_X and trivial homotopies to every point and arrow, respectively; k defines a strict functor in the evident way. From a given map $\theta:k_L \to \mathscr{T}$, we obtain a homomorphism of groupoids:

$$\theta^*_X:\mathbf{Hom}(X, L) \to \mathbf{Hom}(k_X, \mathscr{T}).$$

θ is called a *h-limit* if θ^*_X is a π_0-equivalence and a *pseudo limit* if θ^*_X is an equivalence. We refer to L as the *h-limit object* (respectively, *pseudo limit object*).

Remark 1. We should point out that the reason for isolating h-limits is that, in the main examples, pseudo limits are rare. The suggestion is that, if we had a notion of m-homotopy category, then limits could be expected to be pseudo universal only at the $(m\text{-}1)$-homotopy level.

Remark 2. It is easy to see that, in the terminology of (1.1) the map θ^*_X is π_1-surjective.

Remark 3. There appears to be no purpose in extending the above definition to include diagrams with homotopies.

Uniqueness and naturality properties of h-limits are as expected. Thus, if $\tau:\mathscr{T} \to \mathscr{S}$ is a p-map we obtain, for the limits, $\theta_T:k_L \to \mathscr{T}$, $\theta_S:k_M \to \mathscr{S}$, a map $f(\tau) = f:L \to M$ such that $\theta_S \circ k_f \simeq \tau \circ \theta_T$. Furthermore, homotopic p-maps induce homotopic maps but, owing to the π_0-nature of h-limits a homotopy $\mathscr{T} \to \mathscr{S}$ does not induce a homotopy $L \to M$ as such unless they are pseudo limits. Furthermore,

$$f(\tau \circ \sigma) \simeq f(\tau) \circ f(\sigma)$$

and so if τ is an equivalence in $\text{Diag}(\mathscr{C}, \mathscr{D})$ then so is $L \to M$; in particular, different limit objects of the same diagram are equivalent. Furthermore, if $\mathscr{U}:\mathscr{D} \to \mathscr{E}$ is a p-functor, a h-limit $\theta_T:k_L \to \mathscr{T}$ induces

$$\mathscr{U} * \theta_T:k_{\mathscr{U}(L)} \to \mathscr{U}(\mathscr{T})$$

whence, if $\mathscr{U}(\mathscr{T})$ admits a limit object M, we obtain

$$\mathscr{U}(\theta_T) = \theta_{\mathscr{U}(\mathscr{T})} \circ k_f$$

where $f:\mathscr{U}(L) \to M$ is unique to within homotopy. We describe f as an *induced map of h-limits under* $\mathscr{U}$. If $\mathscr{U}(\theta_T)$ is itself a h-limit (viz. if f is an equivalence) we say that $\mathscr{U}$ *preserves the h-limit* of $\mathscr{T}$. A simple case occurs when $\mathscr{U}$ is the inclusion of a full subcategory, where it is clear that if M lies in (or is equivalent to an object of) $\mathscr{D}$ then M also serves as a h-limit in $\mathscr{D}$; we refer to this fact as *full-subcategory reduction*.

Dually, for $\phi:\mathscr{T} \to k_L$, we obtain

$$\phi_X^*:\mathbf{Hom}(L, X) \to \mathbf{Hom}(\mathscr{T}, k_X) ;$$

ϕ is called a *h-colimit* if ϕ_X^* is a π_0-equivalence for each X and a *pseudo colimit* if it is an equivalence. The analogous results for h-colimits in the above discussion apply; in particular, we use the terms *colimit object* and *preservation of a h-colimit*.

Case 1. $\mathscr{C}$ consists of a set $\mathscr{P}$ with no arrows. The h-limit and h-colimit objects etc. will be called the *h-product*, *h-coproduct*, etc. and the defining maps $P_C:L \to C$, $E_C:C \to M$, in the two cases, will be called the *projections* and the *coprojections*. We note that $L \to C$, $C \in P$, defines a h-product if for $f_C:L \to C$, $C \in P$, there is a map $l:Z \to L$ and homotopies $\gamma_C:f_C \overset{\sim}{\rightarrow} p_C \circ l$ such that if $l':Z \to L$, $\gamma'_C:f_C \overset{\sim}{\rightarrow} p_C \circ l'$ is another solution there is a homotopy $\phi:l \overset{\sim}{\rightarrow} l'$ where

$$(p_C * \phi) \circ \gamma_C = \gamma'_C.$$

For a pseudo product, we also require that $p_C * \theta = p_C * \theta'$, $C \in P$, implies $\theta = \theta'$; in particular, the ϕ, in the previous sentence is unique. In *Top* and in *Top*$^\bullet$ the product and projections in the ordinary sense serve also to define pseudo products; the same is true in $CW^\bullet$ and *Ch* for finite products. In these two g.e. categories the pseudo coproduct is similarly provided by the ordinary coproduct (that is, wedge and direct sum, respectively).

Case 2. Consider the h-limit and h-colimit, respectively, where the graph $\mathscr{C}$ is given diagramatically by

$$\text{Span} = \cdot \longrightarrow \cdot \longleftarrow \cdot\ , \quad \text{Cospan} = \cdot \longleftarrow \cdot \longrightarrow \cdot$$

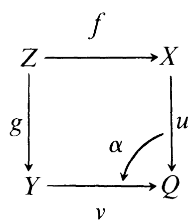


Figure 1

Let G_Z be the groupoid of p-maps from k_Z to the diagram

$$D:X \xrightarrow{u} Q \xleftarrow{v} Y$$

and let the *quintuplet groupoid* H_Z be the groupoid whose objects are quintuplets $(f, v; g, u; \alpha)$, as in Figure 1, and for which a path is a triple

$$(\beta, \gamma, q):q = (f, v; g, u; \alpha) \rightrightarrows (u, v) \ulcorner q = (f', v; g', u; \alpha')$$

where $\beta:f \rightrightarrows f'$, $\gamma:g \rightrightarrows g'$ are such that

$$(v * \gamma^{-1}) \cdot \alpha \cdot (u * \beta) = \alpha'.$$

We define $G_Z \to H_Z$ by sending $k_Z \to D$ given by

$$\alpha_1:u \circ f \rightrightarrows h, \quad \alpha_2:v \circ g \rightrightarrows h$$

to the quintuplet $(f, v; g, u; \alpha_2^{-1} \cdot \alpha_1)$. One then checks without difficulty that $G_Z \to H_Z$ is an equivalence. Thus, we may retrieve the h-limit of D via an object P and a quintuplet $q = (f, v; g, u; \alpha)$ in H_P such that the map $\mathbf{Hom}(Z, P) \to H_Z$ defined by

$$k \to k \setminus q = (f \circ k, v; g \circ k, u; \alpha * k)$$

is a π_0-equivalence. This contracted or substitute h-limit is called an h.p.b. for (u, v) with *lead object* P, and is specified by the condition that every quintuplet $q' = (f', v; g', u; \alpha')$ admits a decomposition $q' = (\beta, \gamma) \ulcorner (k \setminus q)$ which is semi-unique in the sense that if

$$(\beta, \gamma) \ulcorner (k \setminus q) = (\beta', \gamma') \ulcorner (k' \setminus q)$$

then there is a homotopy $\theta:k \rightrightarrows k'$ with

$$\beta = \beta' \cdot (f * \theta), \quad \gamma = \gamma' \cdot (g * \theta).$$

One may also verify that the original h-limit is a pseudo limit if and only if the θ here is unique. When this is the case we say that the h.p.b. is *strong* and we note that a necessary and sufficient condition for this is that if β is such that $f * \beta$, $g * \beta$ are trivial homotopies then β is itself trivial.

If $\mathcal{V}:\mathcal{D} \to \mathcal{E}$ is a p-functor, we define

$$\mathcal{V} * (f, v; g, u; \alpha) = (\mathcal{V}(f), \mathcal{V}(v); \mathcal{V}(g), \mathcal{V}(u); \mathcal{V}(v, g)^{-1} \cdot \mathcal{V}(\alpha) \cdot \mathcal{V}(u, f)).$$

We say that $\mathcal{V}$ *preserves* h.p.b.'s if, whenever a quintuplet q in $\mathcal{D}$ is an h.p.b., the image $\mathcal{V} * g$ is also an h.p.b. One can check that this is equivalent to the preservation by $\mathcal{V}$ of h-limits of Span-diagrams.

Dual considerations lead to h.p.o.'s, strong h.p.o.'s and preservation of h.p.o.'s.

Case 3. $\mathscr{C}$ consists of 2 points with a set A of arrows $X \to Y$. Choose a point $0 \in A$ and consider the diagram $\mathbf{f} = (f_i:X \to Y, i \in A)$. We consider the cone groupoid K_Z whose points are cones $C = (l, \mathbf{f}, \boldsymbol{\alpha})$ where $\boldsymbol{\alpha} = (\alpha_i), i \in A - \{0\}, \alpha_i \circ l \simeq f_0 \circ l$, k departs from Z and whose paths are pairs

$$\lceil(C, \theta):C = (l, \mathbf{f}, \boldsymbol{\alpha}) \overset{\to}{\to} \theta \lceil C = (l', \mathbf{f}, \boldsymbol{\alpha}'),$$

where

$$\alpha_i' = (f_0 * \theta) \cdot \alpha_i \cdot (f_i * \theta^{-1}).$$

We define $\mathbf{Hom}(k_Z, \mathbf{f}) \to K_Z$ by sending the p-map specified by $\beta_i:f_i \circ l \overset{\to}{\to} m$, $i \in A$, to the cone $(l, \mathbf{f}, \boldsymbol{\alpha})$ where $\alpha_i = \beta_0^{-1} \cdot \beta_i$. From this we may retrieve the h-limit of $\mathbf{f}$ via an object P and a cone $C = (l, \mathbf{f}, \boldsymbol{\alpha})$ in K_P such that the map $\mathbf{Hom}(Z, P) \to K_Z$ defined by

$$k \to (l \cdot k, \mathbf{f}, \boldsymbol{\alpha} * k) = k \setminus C$$

is a π_0-equivalence. This substitute h-limit is called a *h-equalizer for* $\mathbf{f}$ (relative to the base point 0) and P the *lead object*. It is specified by the condition that every cone $C' = (l', \mathbf{f}, \boldsymbol{\alpha}')$ admits a decomposition $C' = \theta \lceil (k \setminus C)$ which is semi-unique in the sense that if

$$\theta \lceil(k \setminus C) = \theta' \lceil(k' \setminus C)$$

then there is a homotopy $\phi:k \overset{\to}{\to} k'$ with $\theta' \cdot (l * \phi) = \theta$.

If $\mathscr{V}:\mathscr{D} \to \mathscr{E}$ is a p-functor, we define

$$\mathscr{V} * (l, \mathbf{f}, \boldsymbol{\alpha}) = (\mathscr{V}(l), \mathscr{V}(\mathbf{f}), \mathscr{V}(\boldsymbol{\alpha}))$$

where

$$\mathscr{V}(\mathbf{f})_i = \mathscr{V}(f_i), \quad \mathscr{V}(\boldsymbol{\alpha})_i = \mathscr{V}(f \circ l) \cdot \mathscr{V}(\alpha_i) \cdot \mathscr{V}(f_i \cdot l)^{-1}.$$

We say that $\mathscr{V}$ *preserves* h-equalizers if, whenever C is a h-equalizer, so also is $\mathscr{V} * C$. One checks that this is equivalent to the preservation by $\mathscr{V}$ of h-limits of diagrams from C.

Dual considerations (with the same $\mathscr{C}$) lead to *h-coequalizers* and the *preservation* of h-coequalizers.

Case 4. $\mathscr{C}$ is the graph whose points are the integers 1, 2, ... and whose arrows are $n + 1 \to n$. This serves to provide the sequential h-limits, whereas, the dual graph, $\mathscr{C}^{\text{opp}}$ provides sequential h-colimits. A map from a constant diagram k_Y to

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \leftarrow X_n \xleftarrow{f_n} \cdots$$

appears as a family of maps $\phi_n:Y \to X_n$ together with homotopies

$$\alpha_n:f_n \circ \phi_{n+1} \rightrightarrows \phi_n,$$

whereas a homotopy $(\phi_n, \alpha_n) \rightrightarrows (\phi'_n, \alpha'_n)$ is specified by $\gamma_n:\phi_n \to \phi'_n$ such that

$$\gamma_n \cdot \alpha_n = \alpha'_n \cdot (f_n * \gamma_{n+1}).$$

Thus, (ϕ_n, α_n) departing from L is a sequential h-limit if for (ϕ'_n, α'_n) departing from Y, say, there is a factorization (h, γ_n) with

$$\gamma_n:\phi'_n \circ h \rightrightarrows \psi_n \quad \text{and} \quad \gamma_n \cdot (\alpha_n * h) = \alpha'_n \cdot (f_n * \gamma_{n+1});$$

this is semiunique in the sense that if (h', γ'_n) defines a second factorization, then there is a homotopy $\delta:h \rightrightarrows h'$, not necessarily unique, with $\gamma'_n \cdot (\phi_n * \delta) = \gamma_n$. There is a similar reduction for the sequential h-colimits.

(2.2) *Examples.*

Example 7. Relative to Figure 1 one verifies in *Top* and in $Top^\bullet$ that if u or v is a fibration the ordinary pull-back provides a h.p.b. with trivial homotopy. Similarly, if f or g is a cofibration, the ordinary pushout provides a h.p.o. with trivial homotopy. Note that if $f, g \in CW'$ then $Q \in CW'$, whence, by full subcategory reduction, the same quintuplet is a h.p.o. for CW'. That the same construction holds generally for $CW^\bullet$ then follows by homotopy equivalence.

In general, a h.p.b. for u, v is provided in *Top* and in $Top^\bullet$, as one may easily verify, by taking P to be the subset of $X \times Y \times Q^I$ of (X, Y, θ) with $\theta(0) = u(x)$, $\theta(1) = v(y)$ and f, g, θ to be the projections. By the results of Milnor [**15**], P is in $CW^\bullet$ if X, Y, and Q are in $CW^\bullet$ where this construction with the product replaced by the compactly generated product serves as a h.p.b. in $CW^\bullet$. The general h.p.o. in both $Top^\bullet$ and $CW^\bullet$ is provided by the double mapping cylinder $Q = M(f, g)$ for which the homotopy α moves up the f mapping cylinder $M(f)$ and down $M(g)$. These cases are discussed by Mather [**14**].

In *Top* and in $Top^\bullet$ one verifies that a h-equalizer for $f_i:X \to Y$, $i \in I \cup \{0\}$, can be formed as the subset of $X \times Y^I$ comprising those $(X, (\phi_i))$ with $\phi_i(0) = f_i(x)$, $\phi_i(1) = f_0(x)$ and with the obvious projection maps. Correspondingly, a h-coequalizer in $Top^\bullet$ is formed by taking the mapping cylinders $M(f_i)$ and then identifying together all the tops and all the bases. If $f_i \in CW'$ this result is contained in CW' and so, by the same device as used for the h.p.o., above, the same construction holds for $CW^\bullet$ generally.

In $Top^\bullet$ one general form of the sequential h-colimit of $f_n:X_n \to X_{n+1}$ is obtained by joining together the $M(f_n)$ to form a telescope in the familiar fashion and one easily verifies that this construction also holds in $CW^\bullet$. Various special cases of the sequential h-colimit exist; for example, if $A \in CW'$ is the union of subcomplexes A_n, then A is the sequential h-colimit of the inclusion maps $A_n \hookrightarrow A_{n+1}$.

Remark. Because of the failure of h-limits to induce homotopies the familiar procedures in arbitrary categories of defining functors as limits of others does not in general extend. Thus, in $CW^\bullet$, although the suspension SX can be defined as a h.p.o. (Figure 2), this device displays SX merely as a functor in the homotopy class category.

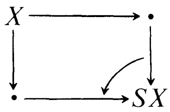


Figure 2

(2.3) *Limit Reduction.*

THE LIMIT REDUCTION THEOREM. *Let the g.e. category $\mathscr{C}$ admit pseudo products.*

(1) *The h-limit object of a 2-element h-equalizer is the h-limit object of a h.p.b. and vice versa.*

(2) *A h-limit object of a diagram $S : \Gamma \to C$ can be reconstructed as a h-equalizer of pseudo products.*

Proof. For a given diagram C of the type $X \rightrightarrows Y$ with maps f_1, f_2 we consider p-maps $k_Z \to C$ of the form $(l, m, (\alpha_i))$ where $l:Z \to X$, $m:Z \to Y$ and $\alpha_i:m \overset{\sim}{\to} f_i \circ l$, $i = 1, 2$. We denote the p-product of 2 copies of Y by $Y \times Y$. We construct $\Delta: Y \to Y \times Y$ together with homotopies $\delta_i:p_i \circ \Delta \overset{\sim}{\to} 1_Y$, $i = 1, 2$, and $h:X \to Y \times Y$ together with homotopies $\gamma_i:p_i \circ h \overset{\sim}{\to} f_i$, $i = 1, 2$. Since $X \times Y$ is a pseudo product there is a unique $\alpha':\Delta \circ m \overset{\sim}{\to} h \circ l$ with

$$(\gamma_i^{-1} * l) \cdot \alpha' \cdot (\delta_i * m) = \alpha_i, \quad i = 1, 2.$$

In this way, we obtain a bijective correspondence

$$(l, m, (\alpha_i)) \mapsto (m, h; l, \Delta; \alpha')$$

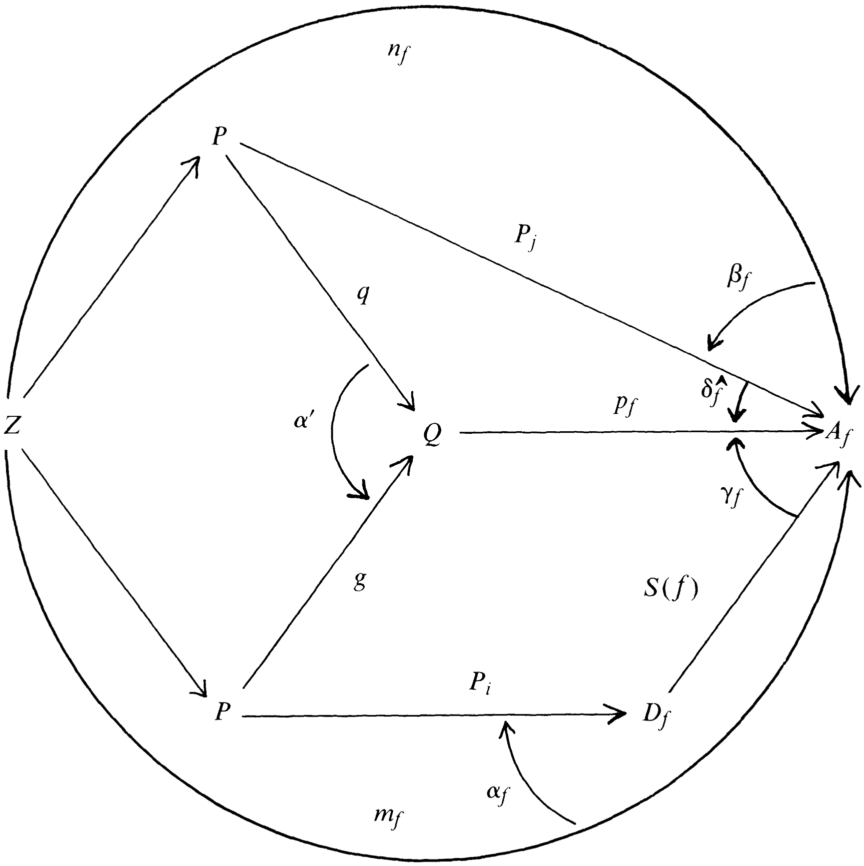
from maps $k_Z \to l$ to quintuplets extending the span $Y \xrightarrow{\Delta} Y \times Y \xleftarrow{h} X$. One verifies that this correspondence commutes (in the obvious sense) with homotopies and with composition by a map $k:Z' \to Z$. We conclude, therefore, that a h-limit object P of C is the lead object of a h.p.b. and vice-versa. If we now relate the h-limit of C with the h-equalizer, we obtain the proof of (1).

For the second part, let P denote the pseudo product of the family $S(i)$ labelled over the points of Γ and let Q denote the pseudo product of $A(f)$, labelled over the arrows of Γ, and where, if $f:i \to j$, $S(i)$, $S(j)$ are denoted by $D(f)$, $A(f)$. We construct maps q, $g:P \to Q$ together with homotopies

$$\delta_f:p_j \simeq p_f \circ q, \quad \gamma_f:S(f) \circ p_i \simeq p_f \circ g \quad \text{where } f:i \to j.$$

Consider a p-map $k_Z \to S$, written as ((l_i), (α_f)) where

$$l_i:Z \to S(i) \quad \text{and} \quad \alpha_f:l_j \simeq S(f) \cdot l_i \quad \text{for } f:i \to j.$$



Corresponding to (l_i), we chose $l:Z \to P$ together with homotopies $\phi_i:l_i \overset{\sim}{\to} p_i \circ l$ and for $f:i \overset{\sim}{\to} j$, we write: $l_j = n_f, \phi_j = \beta_f, l_i = m_f, \phi_i = \alpha_f$. With these choices, and since Q is a pseudo product, we can find a unique homotopy $\alpha':q \circ l \overset{\sim}{\to} g \circ l$ with

$$(S(f) * \alpha_f^{-1}) \cdot (\gamma_f^{-1} * l) \cdot (p_f * \alpha') \cdot (\delta_f * l) \cdot \beta_f = \alpha_f$$

and hence a cone $C = (l, (q, g), \alpha')$. We note the correspondence

$$((l_i), (\alpha_f)) \to (l, (q, g), \alpha')$$

satisfies the following conditions:

(a) for given (l_i), l, (ϕ_i) the correspondence is bijective (i.e., $\alpha \leftrightarrow \alpha'$),

(b) for homotopic $((l_i), (\alpha_f))$, $((l_i'), (\alpha_f'))$ we can take the same l and appropriate ϕ_i' so that the assigned cone is still C,

(c) if we choose l', (ϕ_i') instead of l, (ϕ_i) for the given (l_i) we shall, because P is a pseudo product, obtain a unique homotopy $\theta:l \overset{\sim}{\to} l'$ with $\phi_i' = (p_i * \theta) \cdot \phi_i$; the cone C is then replaced by $\theta \lceil C$,

(d) each cone $(l, (q, g), \alpha')$ comes from some $k_Z \to S$ on defining l_i and $\phi_i:l_i \overset{\sim}{\to} p_i \circ l$ in any manner and, with different choices, the different p-maps are homotopic,

(e) by the process of (d), homotopic cones give, in all manner, homotopic p-maps.

Since, finally, the correspondence clearly commutes with composition of maps $k:Z' \to Z$, it follows that Z is a h-equalizer object of $q, g:P \to Q$ if and only if Z is a h-limit object of S.

COROLLARY 1. *A g.e. category admits all h-limits if it admits pseudo products and h.p.b.'s* (*or* 2-*element h-equalizers*).

COROLLARY 2. *If $\mathcal{C}$ admits all h-limits and pseudo products, a functor $\mathcal{F}:C \to \mathcal{D}$ preserves h-limits if it preserves pseudo products and h.p.b.'s* (*or* 2-*element h-equalizers*).

We register the dual results without comment.

Example 8. The chain-complex analogue of the double mapping cylinder construction produces, for given f, g as in Figure 1, a quintuplet with $\phi_n = Z_{n-1} \oplus X_n \oplus Y_n$ and differential in Q given by

$$\begin{pmatrix} -d & \cdot & \cdot \\ -f & d & \cdot \\ g & \cdot & d \end{pmatrix}$$

u_n, v_n, and $\alpha_n:P_n \to Q_n$ are all defined by inclusion (in the notation of Figure 1). One checks that this satisfies the definition for a h.p.o.

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Each pointed CW complex X produces an augmented chain complex with unit $C(X) \rightleftarrows \mathbf{Z}$ and thence a reduced chain complex $\hat{C}(X)$. From this the cellular chain complex functor $\hat{C}:CW' \to Ch$ arises and it is clear that this preserves the double mapping cylinder construction (in the present sense). Consequently $\hat{C}$ preserves h.p.o.'s and, since it clearly preserves coproducts it is h-colimit preserving.

3. The special role of $\hat{\mathcal{G}}$.

(3.1) To examine h-limits in $\hat{\mathcal{G}}$ we choose a groupoid e comprising a single element. For a diagram $F:\Gamma \to G$, we consider the groupoid $\mathbf{Hom}(k_e, F)$ which we denote by $L(F)$. We decompose the elements of $L(F)$ into families of objects and points in the various $\mathbf{Hom}(e, F(i))$ through which we note that a point of $L(F)$ appears as a family $\mathbf{x} = (x_i, x_\alpha)$, where $x_i \in F(i)$ for each point of Γ and where x_α is a path $x_j \to F(\alpha)(x_i)$ in $F(j)$ for each arrow $\alpha:i \to j$ of Γ. Similarly, a path $\eta:\mathbf{x} \to \mathbf{y}$ of $L(F)$ appears as a family $\eta = (\eta_i)$, where $\eta_i:x_i \to y_i$ is a path in $F(i)$ subject to

$$y_\alpha \cdot \eta_j = F(\alpha)(\eta_i) \cdot x_\alpha$$

for each arrow $\alpha:i \to j$ in Γ.

As a first application, we consider a canonical groupoid isomorphism

$$\zeta:\mathbf{Hom}(G, L(F)) \approx \mathbf{Hom}(k_G, F)$$

in which the object $g \mapsto (x_i(g), x_\alpha(g))$ of the first term goes to

$$x_i:G \to F(i), \quad x_\alpha:x_j \xrightarrow{\sim} F(\alpha)(x_i),$$

and the path

$$g \mapsto \eta_i(g):(x_i(g), x_\alpha(g)) \to (y_i(g), y_\alpha(g))$$

goes to $\eta_i:x_i \xrightarrow{\sim} y_i$. By the existence of ζ we see that $L(F)$ appears as an actual limit (in the 2-category sense) of F and, consequently, that pseudo limits exist in $\hat{\mathcal{G}}$.

As a second application, let $\mathcal{F}:\Gamma \to \mathcal{C}$ be a diagram in a g.e. category. Then, by decomposing the elements of $\mathbf{Hom}(k_X, F)$, as we did $L(F)$, this groupoid is seen to be canonically isomorphic to $L(\mathbf{Hom}(X, F))$, where $\mathbf{Hom}(X, F)$ denotes, in evident fashion, the image of the diagram through the functor $\mathbf{Hom}(X, \cdot)$. A map $k_L \to F$ is then a h-limit (respectively, pseudo limit of F) by definition, provided the composition of induced maps

$$\mathbf{Hom}(X, L) \to \mathbf{Hom}(k_X, F) \xrightarrow{\approx} L(\mathbf{Hom}(X, F)) = \mathcal{M}$$

is a π_0-equivalence (resp. equivalence). This composition is readily seen to be an induced map of h-limits under the functor $\mathbf{Hom}(X, \cdot)$, in the sense of Section 2.

More generally, for a diagram $\mathscr{F}:\Gamma \to \mathscr{C}$ and a functor $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$, we say that $\mathscr{T}$ *π_0-preserves the h-limit of* $\mathscr{F}$ if the induced map of limits $\mathscr{T}(L) \to \mathscr{M}$ is a π_0-equivalence and that $\mathscr{T}$ is *π_0-limit preserving* if it π_0-preserves all h-limits. By a *pseudo-representable functor* $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$, we mean a p-functor that is pseudo equivalent to one of the form $\mathbf{Hom}(X, \cdot)$. It follows from the above analysis that pseudo-representable functors are all π_0-limit preserving. By taking the opposite category to $\mathscr{C}$, one may also consider pseudo-representability and π_0-limit preservation for a contravariant p-functor $\mathscr{U}:\mathscr{C} \to \hat{\mathscr{G}}$; this is pseudo-representable if it is pseudo equivalent to a functor of the form $\mathbf{Hom}(\cdot, X)$ in which case it is π_0-limit preserving in the sense that if L is the k-colimit object of a diagram, then the induced $\mathscr{U}(L) \to \mathscr{M}$ to the h-limit object M of $\mathscr{T}(\mathscr{F})$ is a π_0-equivalence.

(3.2) *Substitute limits.* Along with genuine h-limits, we also consider h.p.b.'s and h-equalizers. For $H \overset{u}{\to} G \overset{v}{\leftarrow} K$, we consider the groupoid $P(u, v)$ whose objects are (h, k, ξ) with $\xi:u(h) \to v(k)$ and paths

$$(\theta, \phi):(h, k, \xi) \to (h', k', \xi')$$

where $\theta:h \to h'$, $\phi:k \to k'$ with

$$\xi' \cdot u(\theta) = v(\phi) \cdot \xi.$$

We define $i:P \to H$, $j:P \to K$, $\alpha:u \circ f \overset{\sim}{\to} v \circ g$ by evaluating on (h, k, ξ) as h, k, ξ, respectively, whence we construct $q = (i, v; j, u; \alpha)$ through which we obtain, for an arbitrary quintuplet $q' = (f, v; g, u; \beta)$, a decomposition $q' = k \setminus q$ where

$$k(z) = (f(z), g(z), \beta(z))$$

and we easily check from this that q is an h.p.b.

Similarly, for maps $\mathbf{f}:(f_i:G \to H;\ i \in A)$ with base-point $0 \in A$, we construct a groupoid $C(\mathbf{f})$ whose objects are $(x, (\xi(i)))$ with

$$\xi_i:f_i(x) \to f_0(x), \quad i \in A - \{0\},$$

and paths $\theta:(x, (\xi_i)) \to (x', (\xi_i'))$ where

$$\theta:x \to x' \text{ and } \quad f_0(\theta) \cdot \xi_i = \xi_i' \cdot f_i(\theta).$$

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We define $l:C(\mathbf{f}) \to G$ and homotopies $\alpha_i:f_i \circ l \overset{\sim}{\Rightarrow} f_0 \circ l$ by evaluating on $(x, (\xi_i))$ as x, ξ_i, respectively. In this way, we construct the cone $C = (l, \mathbf{f}, \boldsymbol{\alpha})$, whence for an arbitrary cone $C' = (l', \mathbf{f}, \boldsymbol{\alpha}')$ we decompose as $C' = k \setminus C$ where

$$k(z) = (l'(z), \boldsymbol{\alpha}'(z)) \in C(\mathbf{f})$$

and we check from this that C is an h-equalizer.

By relating h.p.b.'s and h-equalizers with genuine h-limits, we conclude that, if $\mathscr{F}:\mathscr{C} \to \hat{\mathscr{G}}$ is π_0-limit preserving, then, if q is an h.p.b. in $\mathscr{C}$ with lead object P, q' an h.p.b. in $\hat{\mathscr{G}}$ with $\mathscr{F} * q = k \setminus q'$ and with lead object Q then $k:\mathscr{F}(P) \to Q$ is a π_0-equivalence, whereas, if C is an h-equalizer cone in $\mathscr{C}$ with lead object E, C' an h-equalizer cone in $\hat{\mathscr{G}}$ with lead object E' and with $\mathscr{F} * C = k \setminus q'$, then $F(E) \to E'$ is a π_0-equivalence.

For later reference, we give:

THE π_0-LIMIT REDUCTION LEMMA. *If $\mathscr{C}$ admits pseudo products and if $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$ is a p-functor that preserves all pseudo products and π_0-preserves h.p.b.'s or 2-element h-equalizers then it is π_0-limit preserving.*

This follows from the earlier Limit Reduction Theorem.

4. Pseudo-representability. Let $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$ be a p-functor on a g.e. category. A p-natural transformation $\tau:\mathbf{Hom}(K, \cdot) \to \mathscr{T}$ is termed a *Yoneda transformation* if, for each $g:K \to Y$, and each Y in $\mathscr{C}$, the homotopy

$$\tau_g(\mathrm{id}_K):\tau_Y \circ \mathbf{Hom}(K, g) \overset{\sim}{\Rightarrow} \mathscr{T}(g) \circ \tau_K:\mathbf{Hom}(K, K) \to \mathscr{T}(Y)$$

is trivial. For a given K and chosen point $\eta \in \mathscr{T}(K)$, we verify that there is a Yoneda transformation specified by $\tau_X(f) = \mathscr{T}(f)(\eta)$, $\tau_X(\alpha) = \mathscr{T}(\alpha)(\eta)$ for each map and homotopy $f, \alpha:K \to X$ and with

$$\tau_g(f) = \mathscr{T}(g, f)^{-1}(\eta).$$

We refer to this p-natural transformation as the *Yoneda transformation defined by* η. We note, in passing, that all Yoneda transformations are of this type.

We may also verify that a homotopy $\theta:\tau \overset{\sim}{\Rightarrow} \tau'$ of the Yoneda transformations determined by points $\eta, \eta' \in \mathscr{T}(K)$ is determined by a path $\pi:\eta \to \eta'$ in $\mathscr{T}(K)$ in the sense that for $f:X \to K$,

$$\theta_X(f) = \mathscr{T}(f)(\pi).$$

The following result plays the same role as the Yoneda lemma in ordinary category theory [**18**].

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THE YONEDA REDUCTION LEMMA. *Any p-natural transformation* $\tau:\mathbf{Hom}(K, \cdot) \to \mathscr{T}$ *is homotopic to the Yoneda transformation* $\bar{\tau}$ *defined by*

$$\eta = \tau\,(\mathrm{id}_K) \in \mathscr{T}(K).$$

The proof consists of taking the homotopy

$$\tau_X(f) \overset{\sim}{\to} \bar{\tau}_X(f) = \mathscr{T}(f)(\eta)$$

to be $\tau_f(\mathrm{id}_K)$.

For the next result we say that a p-functor $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$ is π_0*-represented* by the object K in $\mathscr{C}$ (or π_0*-representable* when there exists such a K) if there is a p-natural transformation $\tau:\mathbf{Hom}(K, \cdot) \to \mathscr{T}$ which induces, for each X, a π_0-equivalence $\mathbf{Hom}(K, X) \to \mathscr{T}(X)$.

THE BROWN COMPLEMENT THEOREM. *If all h-limits exist in* $\mathscr{C}$ *and if* $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$ *is* π_0*-limit preserving and* π_0*-representable, then it is pseudo-representable.*

Proof. It suffices to prove that if the Yoneda transformation $\mathscr{Y}$ defined by $\eta \in \mathscr{T}(K)$ induces π_0-equivalences $\mathbf{Hom}(K, X) \to \mathscr{T}(X)$, then for each $f:K \to X$, the induced

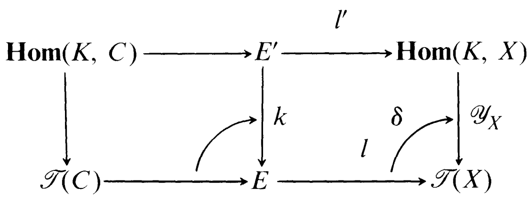
$$\pi_1(f) \to \pi_1(\mathscr{T}(f)(\eta))$$

is an isomorphism of groups. We establish the surjectivity first then the injectivity.

Take an indexing set $A + \{0\}$ with base-point 0 and the diagram defined by the family of maps $X \to X$ indexed over $A + \{0\}$ each one of which is the identity map. Construct h-equalizer cones:

$$(\lambda, (1_X), (\beta_i)), (l', (l_{\mathscr{H}(X)}), (\alpha'_i), (l, (l_{\mathscr{T}(X)}), (\alpha_i))$$

(with lead objects: C, E', E) of this family and its images under the p-functors $\mathscr{H}(X) = \mathbf{Hom}(K, X)$ and $\mathscr{T}(X)$; we construct the last two as in (3.2). Consider the diagram of induced maps:



where the two horizontal components are homotopic to $\mathscr{T}(\lambda)$, $\mathscr{H}(\lambda)$. Since the two horizontal maps on the left and the left-vertical are π_0-

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equivalence, it follows that k is also. Put $x = \mathcal{T}(f)(\eta)$ and take a family of homotopies $\gamma(i):x \to x$ indexed over A and consider the element

$$(x, (\gamma(i))) \in C(1_{\mathcal{T}(X)})) = E.$$

Then, there is an element y in E' and a path

$$\epsilon:k(y) \to (x, (\gamma(i)));$$

that is,

$$k(y) = (x', (\gamma'(i)))) \quad \text{where } \gamma'(i) = \epsilon^{-1} \cdot \gamma(i) \cdot \epsilon .$$

If $y = (f', (\xi(i)))$ then the image of $\xi(i)$ through the Yoneda transformation $\mathcal{Y}_X$ is

$$\delta(Y) \cdot \gamma'(i) \cdot \delta^{-1}(y) = \delta' \cdot \gamma(i) \cdot \gamma'^{-1} \quad \text{where } \delta' = \delta(y) \cdot \epsilon^{-1}.$$

If we now take A to be the set $\pi_1(X)$ with $\gamma_\beta = \beta$ for $\beta \in \pi_1(x)$, the required surjectivity follows.

In a similar way, to show the injectivity, we construct the h-equalizer of a mere pair of maps $X \to X$ equal to the identity and of the images of these pairs under $\mathcal{H}, \mathcal{T}$. If, as in the previous case, the lead objects are denoted C, E', E and we define k, l, l', δ in a corresponding way, with (l', α'), (l, α) the last two h-equalizers, we obtain a diagram written the same as the one above with k a π_0-equivalence. Thus, for $\xi:f \stackrel{\sim}{\Rightarrow} f$ in $\mathbf{Hom}(K, X)$, we take $y = (f, \xi) \in E'$ whence

$$\mathcal{Y}_X(\xi) = (\delta \cdot (\alpha * k) \cdot \delta^{-1})(y).$$

Thus, if $\mathcal{Y}_X(\xi)$ is trivial, so is $(\alpha * k)(y) = \alpha(k(y))$; that is, $k(y) = (z, e_z)$ where z is homotopic to f. On the other hand if $y' = (f, e_f)$ then $k(y') = (w, e_w)$ with w homotopic to f. Hence, there is a path $k(y) \to k(y')$. However, since k is a π_0-equivalence, this implies a path $(f, \xi) \to (f, e_f)$ and hence that $\xi = e_f$. This establishes the injectivity.

The motivation of this theorem (and especially its name) comes from the well known representability theorem of E. H. Brown [**3**, **4**]. We recall that this refers to a contravariant functor (in the ordinary sense) from the homotopy class category of $CW^\bullet$ to the category of sets that sends wedge unions to products and has an equalizer property to the effect that, if $x \in T(X)$ and $T(f)(x) = T(g)(x)$ for $f, g:A \to X$, then there is a map $h:X \to Z$ with $h \circ f \simeq h \circ g$ and an element $z \in T(Z)$ such that $T(h)(z) = x$. Brown's theorem then states that T is representable in the sense that there is a natural equivalence of set valued functors of the form $[\cdot, K] \to T$.

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If we now apply Brown's theorem to a contravariant p-functor $\mathscr{T}:\mathscr{C} \to \hat{\mathscr{G}}$, we see that if $\mathscr{T}$ is π_0-limit preserving (in the sense of h-colimits to h-limits) then the conditions of the theorem apply to the induced set-valued functor $\pi_0\, \mathscr{T}$. Consequently, there is an object K and an element $\eta \in \mathscr{T}(K)$ such that the p-natural transformation defined (via Yoneda's lemma) by η gives a bijection $\pi_0\, \mathbf{Hom}(X, K) \to \pi_0\, \mathscr{T}(X)$ for each X. Thus, the Yoneda transformation defined by η is a π_0-equivalence and so, by the dual of the theorem just proved, is a pseudo equivalence. We shall apply this result (which we shall refer to as the refined Brown's theorem) to the study of fibrations in [**8**].

5. The construction of adjoint functors.

(5.1) Let $\mathscr{L}:\mathscr{C} \to \mathscr{D}$, $\mathscr{R}:\mathscr{D} \to \mathscr{C}$ be two p-functors of g.e. categories $\mathscr{C}$, $\mathscr{D}$. We say that these p-functors are *adjoints* if there is a natural homotopy equivalence

$$\eta_{X,Y}:\mathbf{Hom}(\mathscr{L}(X),\ Y) \approx \mathbf{Hom}(X, \mathscr{R}(Y))$$

which is natural in the sense that for each Y, $\eta_{X,Y}$ together with homotopies $\eta_{f,Y}$ determine a p-natural equivalence in X and for each X, $\eta_{X,Y}$ together with homotopies $\eta_{X,g}$ determine a p-natural equivalence in Y. We also observe that through η, $\mathbf{Hom}(\mathscr{L}(X),\ Y)$ is a p-representable functor of Y, whereas $\mathbf{Hom}(X, \mathscr{R}(Y))$ is a p-representable functor of X.

Under these conditions, let $\mathscr{F}:\Gamma \to \mathscr{C}$ be a diagram with M a h-limit object, N the h-limit object of $\mathscr{R} \circ \mathscr{F}$ and P the h-limit object of $\mathbf{Hom}(X, \mathscr{R} \circ \mathscr{F}(\cdot))$. We then obtain a canonical map $\lambda:\mathscr{F}(M) \to \mathrm{N}$ and induced maps

$$\mathbf{Hom}(\mathscr{L}(X),\ M) \xrightarrow{\cong} \mathbf{Hom}(X, \mathscr{R}(M)) \xrightarrow{\lambda_*} \mathbf{Hom}(X,\ N) \xrightarrow{\mu} P.$$

Since p-representable functors are π_0-limit preserving, μ and the whole composition are π_0-equivalences. Consequently, λ^* is π_0-limit preserving whereas, by the Induced Equivalence Theorem, λ is an equivalence. Thus, $\mathscr{R}$ preserves h-limits and, similarly, $\mathscr{L}$ preserves h-colimits.

Example 9. Consider a model of the p-functor $X \to X^{[n]}$ of Example 3 with the property that $X^{[n]} = X, f^{[n]} = f$, and $\alpha^{[n]} = \alpha$ for X, f, α in $CW^{[n]}$. The functor serves to define a transformation

$$\eta_{X,Y}:\mathbf{Hom}(X,\ Y) \to \mathbf{Hom}(X^{[n]},\ Y)$$

where $i:CW^{[n]} \to CW^{\bullet}$ is the inclusion functor. This is clearly pseudo-natural in X and Y. We check, by elementary homotopy theory, that $\eta_{X,Y}$ is an equivalence for each X, Y whence we conclude that $X \to X^{[n]}$ and the inclusion functor are adjoints.

As a consequence of this, $X \to X^{[n]}$ preserves h-colimits which shows that the g.e. category $CW^{[n]}$ admits all h-colimits (and also that the p-functor provides a device for computing them). If we apply this to the case $n = 1$ and bear in mind the comments of Example 4, we obtain a simple proof of the classical van Kampen theorem, c.f. [**13**] Chapter 4. Indeed this theorem states, in the present terminology, that the fundamental group functor from $CW^{\bullet}$ to the ordinary category of groups preserves h-colimits.

Example 10. Let $\mathscr{T}:Top^{\bullet} \to CW^{\bullet}$ stand for the realized singular complex functor, that is, $\mathscr{T}(Y) = |\text{Sing } Y|$. Then, if $i:CW^{\bullet} \to Top$ is the inclusion functor, the map induced by the singular projection $|\text{Sing } Y| \to Y$ produces a homomorphism

$$\eta_{X,Y}:\mathbf{Hom}(X, |\text{Sing } Y|) \to \mathbf{Hom}(iX, Y)$$

which is natural with respect to X and Y and, by a traditional argument, is a π_0-equivalence. Then, since the inclusion functor preserves h-colimits, $\eta_{X,Y}$ is a pseudo equivalence by the Brown Complement Theorem. It then follows that i, $\mathscr{T}$ are adjoints. Thus, $Y \to |\text{Sing } Y|$ preserves h-limits and since $X \cong |\text{Sing } X|$ when $X \in CW$ it follows that the g.e. category $CW^{\bullet}$ admits all h-limits. It also follows, for example, that a model for the product in $CW^{\bullet}$ is obtained by

$$(X_i) \to \text{Sing } |\textstyle\prod(X_i)|$$

whence, through $\eta_{X,Y}$, we see that the n^{th} homotopy group of this h-product is canonically isomorphic to $\prod(\pi_n(X_i))$.

Incidentally, one also checks, without difficulty, that the $\mathscr{R}$ of an adjoint functor pair sends pseudo limits to pseudo limits. Consequently, the h-products in $CW^{\bullet}$ are pseudo products.

(5.2) For the first stage in the construction of adjoint functors we suppose that $\mathscr{L}:\mathscr{C} \to \mathscr{D}$ is given and that we wish to construct an adjoint. In the following lemma, we show that the necessary condition that $\mathbf{Hom}(\mathscr{L}(X), Y)$ is a p-representable functor of X for all Y is also sufficient.

THE NATURALITY LEMMA. *If for each Y, $\mathbf{Hom}(\mathscr{L}(X), Y)$ is a p-representable functor of X represented by an object $\mathscr{R}(Y)$, say, then the term $\mathscr{R}$ extends to maps and homotopies to provide a p-functor $\mathscr{R}:\mathscr{D} \to \mathscr{C}$ adjoint to $\mathscr{L}$.*

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Proof. Let

$$\eta_{\cdot,Y}:\mathbf{Hom}(\mathscr{L}(\cdot), Y) = A_Y \rightarrow \mathbf{Hom}(\cdot , \mathscr{R}(Y)) = H_{\mathscr{R}(Y)}$$

be a p-natural equivalence and $\zeta_{\cdot,Y}$ be a pseudo-inverse with

$$w_Y:\mathrm{Id} \rightrightarrows \zeta_{\cdot,Y} \circ \eta_{,Y}.$$

For $f:Y \rightarrow Z$, the p-natural transformation

$$\theta(f) = \eta_{\cdot,Z} \circ \mathbf{Hom}(L(\cdot), f) \circ \zeta_{\cdot,Y}:H_{\mathscr{R}(Y)} \rightarrow H_{\mathscr{R}(Z)}$$

is homotopic, by $\theta_f:\sigma(f) \rightrightarrows \sigma'(f)$, say, to a Yoneda transformation whose generator we denote $\mathscr{R}(f):\mathscr{R}(Y) \rightarrow \mathscr{R}(Z)$.

$\sigma'(f)$ together with:

$$\eta_{X,f} = (\theta_f * \eta_{\cdot,Y}) \cdot ((\eta_{\cdot,Z} \circ \mathbf{Hom}(\mathscr{L}(\cdot), f)) * \omega_Y):\eta_{\cdot,Z} \circ \mathbf{Hom}(\mathscr{L}(\cdot), f) \rightrightarrows \sigma'(f) \circ \eta_{\cdot,Y}$$

provide, through the Extension Lemma, a p-functorial structure on $H'_X = \mathbf{Hom}(X, \mathscr{R}(\cdot))$ which extends

$$H'_X(Y) = H_{\mathscr{R}(Y)}(X), \quad H'_X(f) = \sigma'(f) = \mathbf{Hom}(X, \mathscr{R}(f))$$

and for which $\{\eta_{X,Y}\ \eta_{X,f}\}$ defines a p-natural transformation: $\mathbf{Hom}(\mathscr{L}(X), \cdot) \rightarrow H'_X$.

For a homotopy $\alpha:f \rightrightarrows g$, we define

$$\sigma(\alpha) = \eta_{\cdot,Z} * \mathbf{Hom}(\mathscr{L}(\cdot), \alpha) * \zeta_{\cdot,Y}$$

which is a homotopy $\sigma(\alpha):\sigma(f) \rightrightarrows \sigma(g)$ of p-natural transformations, whence the homotopy

$$\theta_g \cdot \sigma(\alpha) \cdot \theta_f^{-1}:\sigma'(f) \rightrightarrows \sigma'(g)$$

is a homotopy of Yoneda transformations and so it is generated by a homotopy $\mathscr{R}(\alpha):\mathscr{R}(f) \xrightarrow{\eta} \mathscr{R}(g)$. We shall show that

$$H'_X(\alpha) = \mathbf{Hom}(X, \mathscr{R}(\alpha)).$$

To do this, we note that since $\eta_{X,Y}$ is an equivalence $H'_X(\alpha)$ is the unique homotopy ξ satisfying

$$\xi \cdot \eta_{X,f} = \eta_{X,g} \cdot \mathbf{Hom}(\mathscr{L}(X), \alpha).$$

We now check that $\xi = \mathbf{Hom}(X, \mathscr{R}(\alpha))$ satisfies this equation.

We next check that there is a homotopy

$$\mathscr{R}(f, g):\mathscr{R}(f) \circ \mathscr{R}(g) \rightrightarrows \mathscr{R}(f \circ g)$$

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such that

$$\mathbf{Hom}(X, \mathscr{R}(f, g)) = H'_X(f, g).$$

Indeed the right hand member is the unique homotopy satisfying

$$(\xi * \eta_{X,Y}) \cdot (H'_X(f) * \eta_{X,g}) \cdot (\eta_{X,f} * \mathbf{Hom}(\mathscr{L}(X), g)) = \eta_{X,f \circ g}.$$

This implies that $\xi * \eta_{X,Y}$ is the evaluation at X of a certain homotopy χ of p-natural transformations. From this we see that ξ, itself, is the evaluation at X of the homotopy:

$$(\mathbf{Hom}(\cdot, \mathscr{R}(f \circ g)) * \omega_Y^{-1}) \cdot (\chi * \zeta_{\cdot, Y}) \cdot (\omega * (\mathbf{Hom}(\cdot, \mathscr{R}(f)) \cdot \mathbf{Hom}(\cdot, \mathscr{R}(g)))$$

which, being a homotopy of Yoneda transformation takes the form **Hom** $(\cdot, \mathscr{R}(f, g))$ as required.

One now verifies easily from the fact that $H'_X(f)$, $H'_X(\alpha)$, $H'_X(f, g)$ define a p-functor and that the $\eta_{X,Y}$ are equivalences, that the terms $\mathscr{R}(f)$, $\mathscr{R}(\alpha)$, $\mathscr{R}(f, g)$ also serve to define a p-functor. This p-functor obviously satisfies the required properties and this completes the proof.

Example 11. Let $C:CW' \to Ch$ be the cellular chain complex strict functor. C preserves h-colimits and so, by the Refined Brown's Theorem, $\mathbf{Hom}(C(X), D)$ for each D in Ch is a p-representable functor of X. Consequently, there is an adjoint p-functor $\mathscr{P}$, that is,

$$\mathbf{Hom}(X, \mathscr{P}(D)) \cong \mathbf{Hom}(C(X), D).$$

We remark in passing (without going into details) that $\mathscr{P}C(X)$ is a thinly disguised form of the Dold-Thom symmetric product functor [**5**].

(5.3) In order to use the Naturality Lemma in cases other than that of h-colimit preserving p-functors in $CW^\bullet$ (where one makes use of Brown's theorem), we examine certain smallness type conditions under which p-representability can be independently established. Since the roles of $\mathscr{L}$ and $\mathscr{R}$ can be interchanged to suit comprehension, we shall suppose that $\mathscr{D}$ is a g.e. category possessing all h-limits, that $\mathscr{R}:\mathscr{D} \to \mathscr{C}$ is a given p-functor preserving h-limits and that it satisfies the following two conditions.

(1) For each X, there is a family $f_i:X \to \mathscr{R}(Y_i)$, $i \in I_X$, such that every map of the form $f:X \to \mathscr{R}(Y)$ for some Y in $\mathscr{D}$ is homotopic to one of the form $\mathscr{R}(h) \circ f_i$.

(2) For each map $f:X \to \mathscr{R}(Y)$ with X in $\mathscr{C}$ and Y in $\mathscr{D}$, there is a family I_f of pairs $g_i, h_i: Y \to Z_i$ with $\mathscr{R}(g_i) \circ f \simeq \mathscr{R}(h_i) \circ f$ subject to the condition that if $g, h: Y \to Z$ are such that $\mathscr{R}(g) \circ f \simeq \mathscr{R}(h) \circ f$, then there is a pair

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$g_i, h_i{:}Y \to Z_i$ in I_f and a map $k{:}Z_i \to Z$ such that $g = k \circ g_i$ and $h = k \circ h_i$.

Before we apply this, it is necessary to prove the following result which has some independent interest.

LEMMA. *If $\mathscr{C}$ is a g.e. category admitting sequential h-colimits and $r{:}K \to K$ is a map of $\mathscr{C}$ with $r \circ r \simeq r$, then the image of the natural transformations (in X) $\pi(K, X) \to \pi(K, X)$ induced by r is a representable functor from $\pi\,\mathscr{C}$ to the category of sets.*

Proof. We consider the sequential h-colimit of $f_n{:}X_n \to X_{n+1}$, $n = 0, 1, 2, \ldots$ where $f_n = r$ for each n. If this diagram is denoted by $\mathscr{F}$, the h-map $\mathscr{F} \to k_X$ determined by $(\phi_1, \phi_2, \ldots; \alpha_1, \alpha_2, \ldots)$ is homotopic to $(\phi_2 * r, \phi_3 * r \ldots; \alpha_1, \alpha_2 * r, \alpha_3 * r, \ldots)$ whence we may find, by induction, the terms of a homotopy that makes this homotopic to $(f, f, \ldots; \alpha, \alpha \ldots)$ where $f = \phi_2 * r$, $\alpha = \phi_2 * \xi$ and $\xi{:}r \circ r \rightrightarrows$ is given. We refer to this as the h-map (f, α).

One may also see that the maps $(\phi \circ r, \phi * \xi)$ and $(\psi \circ r, \psi * \xi)$ are homotopic if and only if $\phi \circ r \rightrightarrows \psi \circ r$. Indeed, the "only if" part of the proof is clear. Conversely, by repeating the above argument we see that

$$(\phi \circ r, \phi * \xi) \simeq (\phi \circ r \circ r, (\phi \circ r) * \xi)$$

whence it suffices to show that if $\theta{:}g' \rightrightarrows \psi'$, then $\theta * r$ determines, termwise, a homotopy

$$(\phi' \circ r, \phi' * \xi) \rightrightarrows (\psi' \circ r, \psi' * \xi).$$

The condition for this is that

$$(\psi' * \xi) \cdot (\theta * (r \circ r)) = (\theta * r) \cdot (\psi' * \xi)$$

and this follows immediately. Thus, the image

$$\pi(K, X) \to \pi(K, X)$$

is in canonical bijection with $\pi_0\,\mathbf{Hom}(\mathscr{F}, k_X)$ and hence with $\pi(L, X)$, where L is the h-colimit object of $\mathscr{F}$.

Remark. The lemma implies via the ordinary Yoneda lemma the existence of L with maps $i{:}L \to K$, $p{:}K \to L$ such that $i \circ p \simeq \mathrm{id}_L$. One deduces from this that the image of the natural transformation $\pi(X, K) \to (X, K)$, induced by r, is also representable. Consequently, by dualizing the lemma and this remark, we see that the pseudo representability of the image of $\pi(K, X) \to \pi(K, X)$ will also follow from the admission of sequential h-limits.

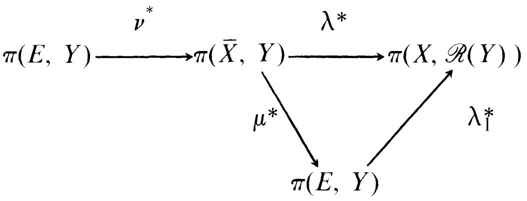
THE ADJOINT p-FUNCTOR THEOREM. *If the conditions* (1) *and* (2) *and their preamble are satisfied, the p-functor* $\mathscr{R}$ *possesses a* (*left*) *adjoint.*

Proof. Let $X \in \mathscr{C}$. If we compound the maps f_i, we obtain, by limit preservation, a map $\lambda:X \to \mathscr{R}(\bar{X})$, where $\bar{X}$ is the product of $(Y_i)_{i \in I_X}$. This has the property that any map $f:X \to R(Y)$ is homotopic to one of the form $\mathscr{R}(g) \circ \lambda$; in other words, we have reduced the family I_X to one member.

Now consider the graph Γ whose points are 0 and the elements of I_λ (as in (2)) and whose arrows comprise $i(1)$, $i(2):0 \to i$, for each $i \in I_\lambda$. Let $\mathscr{F}:\Gamma \to \mathscr{D}$ be the diagram with $\mathscr{F}(0) = \bar{X}, \mathscr{F}(i) = Z_i, \mathscr{F}(i(1)) = g_i, \mathscr{F}(i(2)) = h_i$. Let E be the limit object of this diagram and let $\mu:E \to \bar{X}$ be the initial map of the defining $k_E \to \mathscr{F}$. By the limit preservation of $\mathscr{R}$, the map λ factorizes to within homotopy as

$$\mathscr{R}(\mu) \circ \lambda:X \to \mathscr{R}(E) \to \mathscr{R}(\bar{X}).$$

If $\nu:\bar{X} \to E$ is a map such that $\lambda_1 = \mathscr{R}(\nu) \circ \lambda$, we consider the following commutative diagram in which the maps induced by λ, μ , . . . are denoted λ^*, μ^*, ...



Then, λ^* is surjective, whereas μ^* identifies any pair identified by λ^*. Since $\lambda_1^* = \lambda^* \circ \nu^*$, it follows that λ_1^* restricted to the image of $\mu^* \circ \nu^*:\pi(E, Y) \to \pi(E, Y)$ displays a natural bijection with $\pi(X, \mathscr{R}(Y))$. Thus, by the above lemma, $\pi(X, \mathscr{R}(Y))$ is pseudo-representable, that is, **Hom**$(X, \mathscr{R}(Y))$ is π_0-representable. The result now follows by applying the Brown Complement Theorem and the Naturality Lemma.

6. Example 12: localization. We recall (c.f. [**2**, **11**, **16**]) that if **P** is a family of primes, a group G is called **P**-*local* if, for every prime p not in **P** and every $g \in G$, there is a unique y with $g = y^p$. Let $\mathscr{G}$, $\mathscr{G}_{\mathbf{P}}$ denote the category of groups and the full subcategory of **P**-local groups, respectively. We call an object X in $CW^\bullet$ **P**-*local* if all its homotopy groups are **P**-local and we denote by $CW^\bullet_{\mathbf{P}}$ the full subcategory of $CW^\bullet$ comprising **P**-local spaces.

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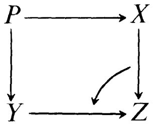
We observe that a product of groups or spaces (c.f. Example 10) that are **P**-local is also **P**-local. In addition to this, the h.p.b. object of homomorphisms $A \to C \to B$ in $\mathcal{G}_{\mathbf{P}}$ is obviously **P**-local, whereas the h.p.b. of maps $X \to Z \to Y$ in $CW^{\bullet}_{\mathbf{P}}$ is again **P**-local by virtue of the evident Mayer-Vietoris sequence of homotopy groups associated with an h.p.b.



It follows, by the π_0-Limit Reduction Lemma that the inclusions $\mathcal{G}_{\mathbf{P}} \to \mathcal{G}$, $CW^{\bullet}_{\mathbf{P}} \to CW^{\bullet}$ preserve all *h*-limits. We proceed to show that these inclusions possess adjoints; we observe that, on the π_0 level, this will imply that to every object X in $CW^{\bullet}$ there is what is best described as a localization $X_{\mathbf{P}}$ with the property that there is a defining map $\lambda:X \to X_{\mathbf{P}}$ such that every map $X \to Y$ of X to a **P**-local space factorizes uniquely to within homotopy through λ.

The first affair is to show that for a given X there is a set of maps $X \to Y_i$ into **P**-local spaces with the property that every such map $X \to Y$ is obtained to within homotopy by a composition $X \to Y_i \to Y$. For this purpose, we observe that, for a cardinal **a**, the homotopy equivalence classes of spaces in the full subcategory $CW^{\bullet}_{\mathbf{a}}$ of CW complexes whose set of cells has cardinality $\leqq \mathbf{a}$ form a set and that, consequently, the homotopy classes of maps $X \to Y$ with Y in $CW^{\bullet}_{\mathbf{a}}$ may all be obtained from a set by composition with a homotopy equivalence. It therefore suffices to show that there is a cardinal **a** depending on X such that every map from X to a **P**-local space factorizes through a **P**-local space in $CW^{\bullet}_{\mathbf{a}}$. Similarly, to check the corresponding condition for groups, it suffices to display, for a given G, a cardinal **a** such that every homomorphism from G to a **P**-local group factorizes through one with cardinality $\leqq \mathbf{a}$.

To establish this first for groups (c.f. [**1**]), we consider a homomorphism $f:G \to H$ with $H \in \mathcal{G}_{\mathbf{P}}$. For any subset $A \subset H$, we refer to the set of $y \in H$ with $p \in A$ for some prime p not in **P** as the set of roots of A. We define, inductively: $H_0 = f(G)$, H_n the set of roots of $\bar{H}_{n-1}$ and $\bar{H}_n$ the subgroup of H generated by H_n. It is clear that $H_\infty = \cup (\bar{H}_n)$ is **P**-local. Furthermore, if the cardinality of a set A is less than or equal to an infinite cardinal **a**, the set of roots of A and the subgroup generated by A both clearly have cardinality $\leqq \mathbf{a}$. If Card $G \leqq \mathbf{a} \geqq \aleph_0$ then, by induction, H_n, $\bar{H}_n$, and hence H_∞ itself, all have cardinality $\leqq \mathbf{a}$.

Next, we take a map $f:X \to Y$ with $X \in CW^{\bullet}$, $Y \in CW^{\bullet}_{\mathbf{P}}$. We may suppose that Y is a CW complex, whence Y_0 denotes the smallest subcomplex containing the image of f. Take $\pi = H_\infty$ as above for the homomorphism $\pi_*:\pi_1(Y_0) \to \pi_1(Y)$ induced by inclusion, and choose generators (α_i) of π. Attach copies of S^1 to Y_0 corresponding to each α_i to form the space Y'_0. Then map this to Y_1 to extend the inclusion of Y_0 so that the element $\alpha'_i \in \pi_1(Y'_0)$ from the i^{th} attachment is mapped to α_i under π_*. Attach 2-cells to Y'_0 to form Y_1 and map $Y_1 \to Y$ to induce a π_1-injection. Now adjust Y, via mapping cylinders, to remain of the same homotopy type but to contain Y_1. We now proceed inductively with the assumption that we have $Y_n \subset Y$ (where Y, but not its homotopy type, has changed its original form) with the induced $\pi_i(Y_n) \to \pi_i(Y)$ an injection of **P**-local groups for $i \leqq n$. As in the case $n = 0$, we take H_∞ corresponding to $\pi_{n+1}(Y_n) \to \pi_{n+1}(Y)$, attach $n + 1$ cells to Y_n to form Y'_n, attach $n + 2$ cells to produce Y_{n+1}, then produce a map providing an injection

$$\pi_{n+1}(Y_{n+1}) \to \pi_{n+1}(Y)$$

and, finally, adjust Y so that $Y_{n+1} \subset Y$. We observe that the adjustment is made by additions not interfering with Y_{n+1}, whence we may define $Y_\infty = \cup(Y_n) \subset Y$. This is clearly **P**-local and there is a factorization $X \to Y_\infty \to Y$ to within homotopy. If **a** is an infinite cardinal with X in $CW^\bullet_{\mathbf{a}}$ then, clearly, $Y_0 \in CW^\bullet_{\mathbf{a}}$. Thus the cardinality of π is less than or equal to **a** and so $Y_1 \in CW^\bullet_{\mathbf{a}}$. Then, by induction, π_n has cardinality $\leqq$ **a** and so $Y_n \in CW^\bullet_{\mathbf{a}}$. Finally, then, $Y_\infty \in CW^\bullet_{\mathbf{a}}$.

To establish the second condition, suppose $g, h: Y \to Z$ are in $CW^\bullet_{\mathbf{P}}$ and $f:X \to Y$ is such that $g \circ f \simeq h \circ f$. It suffices to show that there is a cardinal **b** and maps $g', h': Y \to Z', Z' \in CW^\bullet_{\mathbf{b}}$, with $g' \circ f \simeq h' \circ f$ and $g = k \circ g', h = k \circ h'$ for an appropriate k. For this it suffices to take a homotopy $F:g \circ f \simeq h \circ f$ and perform the construction Z_∞ above for the evident map $X \times I \vee Y \to Z$. That is, we may take **b** to be any infinite cardinal with $X, Y \in CW^\bullet_{\mathbf{b}}$. A similar but simpler observation holds for the case of groups.

From this and the preceding theory, the existence of adjoints follows.

Remark. When X is nilpotent, it would be of interest to know if the localization $\lambda:X \to X_{\mathbf{P}}$ agrees with the well-known nilpotent localization (see for example [**11**]), $\lambda':X \to X_{(\mathbf{P})}$; we recall that λ' is universal to nilpotent **P**-local spaces only and it is not certain whether $X_{\mathbf{P}}$ and $X_{(\mathbf{P})}$ are homotopy equivalent. On the other hand, this is clearly so when X is simply connected. Indeed, $\lambda \simeq \mu \circ \tilde{\lambda}$, where $\tilde{\lambda}:X \to \tilde{X}_{\mathbf{P}}$ is a map to the universal cover of $X_{\mathbf{P}}$. By universality, $\tilde{\lambda} \simeq \nu \circ \lambda$, say, and $\mu \circ \nu \simeq \text{id}$, whence it follows that $X_{\mathbf{P}}$ is itself simply connected and the assertion follows immediately.

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