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Topological Games and Alster Spaces

Dedicated to Ofelia T. Alas on the occasion of her 70th birthday

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Abstract. In this paper we study connections between topological games such as Rothberger, Menger, and compact-open games, and we relate these games to properties involving covers by G_δ subsets. The results include the following: (1) If TWO has a winning strategy in the Menger game on a regular space X, then X is an Alster space. (2) If TWO has a winning strategy in the Rothberger game on a topological space X, then the G_δ-topology on X is Lindelöf. (3) The Menger game and the compact-open game are (consistently) not dual.

1 Topological Games

We start by recalling some definitions. The following properties were introduced in studies of strong measure zero and σ-compact metric spaces, respectively.

Definition 1.1 (Rothberger [26]) A topological space X is said to be a *Rothberger space* if for every sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X there is a sequence $(U_n)_{n\in\omega}$ satisfying $X = \bigcup_{n\in\omega} U_n$ with $U_n \in \mathcal{U}_n$ for all $n \in \omega$.

Definition 1.2 (Hurewicz [15]) A topological space X is said to be a *Menger space* if for every sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X there is a sequence $(\mathcal{F}_n)_{n\in\omega}$ satisfying $X = \bigcup\bigcup_{n\in\omega} \mathcal{F}_n$ with $\mathcal{F}_n \in [\mathcal{U}_n]^{<\aleph_0}$ for all $n \in \omega$.

The following topological games are naturally associated with the above properties.

Definition 1.3 (Galvin [12]) The *Rothberger game* in a topological space X is played according to the following rules. In each inning $n \in \omega$, ONE chooses an open cover $\mathcal{U}_n$ of X, and then TWO chooses $U_n \in \mathcal{U}_n$. The play is won by TWO if $X = \bigcup_{n\in\omega} U_n$; otherwise, ONE is the winner.

Definition 1.4 (Telgársky [37]) The *Menger game* in a topological space X is played as follows. In each inning $n \in \omega$, ONE chooses an open cover $\mathcal{U}_n$ of X, and then

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TWO chooses a finite subset $\mathcal{F}_n$ of $\mathcal{U}_n$. TWO wins the play if $\bigcup_{n\in\omega}\mathcal{F}_n$ is a cover of X; otherwise, ONE is the winner.

It is easy to see that if ONE does not have a winning strategy in the Rothberger (resp. Menger) game in a topological space X, then X is a Rothberger (resp. Menger) space. The following theorems show that these properties can in fact be expressed in terms of such games.

Theorem 1.5 (Pawlikowski [23]) *A topological space X is Rothberger if and only if ONE does not have a winning strategy in the Rothberger game on X.*

Theorem 1.6 (Hurewicz [15]) *A topological space X is Menger if and only if ONE does not have a winning strategy in the Menger game on X.*

A more systematic study of combinatorial properties in topological spaces was initiated by M. Scheepers in [29]. Scheepers introduced a framework for investigating some classes of properties and their naturally associated games in greater generality, which has originated the subject of *selection principles*. One such general selection principle and its associated game are defined as follows.

Definition 1.7 (Scheepers [29]) Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty families. Then $\mathrm{S}_1(\mathcal{A},\mathcal{B})$ denotes the following statement:

For every sequence $(A_n)_{n\in\omega}$ of elements of $\mathcal{A}$, there is a sequence $(B_n)_{n\in\omega}$ such that $B_n \in A_n$ for each $n\in\omega$ and $\{B_n : n\in\omega\}\in\mathcal{B}$.

Definition 1.8 (Scheepers [30]) Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty families with $\varnothing\notin\mathcal{A}$. The game $\mathrm{G}_1(\mathcal{A},\mathcal{B})$ is played as follows. In each inning $n\in\omega$, ONE chooses $A_n\in\mathcal{A}$, and then TWO chooses $B_n\in A_n$. TWO wins the play if $\{B_n : n\in\omega\}\in\mathcal{B}$; otherwise, ONE is the winner.

Thus the Rothberger property is the particular case $\mathrm{S}_1(\mathcal{O},\mathcal{O})$ of Definition 1.7, where $\mathcal{O}$ denotes the family of all open covers of the space; more explicitly, $\mathrm{S}_1(\mathcal{O}_X,\mathcal{O}_X)$ means that X is a Rothberger space, where $\mathcal{O}_X = \{\mathcal{U}\subseteq\tau_X : X=\bigcup\mathcal{U}\}$. It is also clear that X is a Menger space if and only if $\mathrm{S}_1(\mathcal{O}^*_X,\mathcal{O}_X)$ holds, where

$$\mathcal{O}^*_X = \{\mathcal{U}\in\mathcal{O}_X : \mathcal{U}\text{is closed by finite unions}\}.$$

Similarly, the Rothberger and Menger games can be regarded as the games $\mathrm{G}_1(\mathcal{O},\mathcal{O})$ and $\mathrm{G}_1(\mathcal{O}^*,\mathcal{O})$, respectively.[1]

The implication that was already observed in these particular cases holds in general. Namely, the nonexistence of a winning strategy for ONE in the game $\mathrm{G}_1(\mathcal{A},\mathcal{B})$ implies $\mathrm{S}_1(\mathcal{A},\mathcal{B})$. The converse, which holds in the particular cases previously considered (Theorems 1.5 and 1.6), is not always true; see [31, Example 3].

We now turn to (what appears to be) another game.

[1]Although the rules of $\mathrm{G}_1(\mathcal{O}^*,\mathcal{O})$ and the Menger game are not quite the same, it is easy to see that these games are *equivalent*; *i.e.*, a player has a winning strategy in $\mathrm{G}_1(\mathcal{O}^*,\mathcal{O})$ if and only if that player has a winning strategy in the Menger game.

Definition 1.9 (Galvin [12]) The *point-open game* in a topological space X is defined by the following rules. In each inning $n \in \omega$, ONE picks a point $x_n \in X$, and then TWO chooses an open set $U_n \subseteq X$ with $x_n \in U_n$. The play is won by ONE if $X = \bigcup_{n\in\omega} U_n$; otherwise, TWO is the winner.

In [12], F. Galvin showed that the point-open game is essentially the same as the Rothberger game, in the following sense. We say that two games G and G′ are *dual* if

- ONE has a winning strategy in G if and only if TWO has a winning strategy in G′; and
- TWO has a winning strategy in G if and only if ONE has a winning strategy in G′.

Theorem 1.10 (Galvin [12]) *The Rothberger game and the point-open game are dual.*

We may then wonder how the point-open game could be modified to produce a similar game that is dual to the Menger game. The following is a natural candidate.

Definition 1.11 (Telgársky [35]) The *compact-open game* in a topological space X is defined as follows: in each inning $n \in \omega$, ONE chooses a compact subset K_n of X, and then TWO chooses an open subset U_n of X such that $K_n \subseteq U_n$. The play is won by ONE if $X = \bigcup_{n\in\omega} U_n$; otherwise, TWO is the winner.

In [37, Corollary 3], R. Telgársky proved that ONE has a winning strategy in the compact-open game if and only if TWO has a winning strategy in the Menger game. Telgársky also observes (in Proposition 1 of the same paper) that ONE having a winning strategy in the Menger game implies TWO having a winning strategy in the compact-open game, and then asks the following question.

Problem 1.12 (Telgársky [37]) Does the converse hold, *i.e.,* are the Menger game and the compact-open game dual?

As we shall see later, in Examples 3.12 and 3.13, this may not always be the case. The relationship between these two games may be more clearly understood by considering the following definition.

Definition 1.13 Let X be a topological space. An open cover $\mathcal{U}$ of X is said to be a *k-cover* of X if for every compact subset K of X there is $U \in \mathcal{U}$ such that $K \subseteq U$. The family of all k-covers of X will be denoted by $\mathcal{K}_X$.

The next result is a particular case of [36, Theorem 6.2].

Proposition 1.14 (Galvin, Telgársky [36]) *The game* $\mathrm{G}_1(\mathcal{K}, \mathcal{O})$ *and the compact-open game are dual.*

Problem 1.12 can then be rewritten as follows.

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Problem 1.15 (Telgársky [37]) Does the existence of a winning strategy for ONE in $G_1(\mathcal{K}, \mathcal{O})$ imply the existence of a winning strategy for ONE in the game $G_1(\mathcal{O}^*, \mathcal{O})$?

Note that $\mathcal{O}^* \subseteq \mathcal{K}$; thus, a counterexample to Problem 1.12 (*i.e.*, Problem 1.15) must be a space in which these two classes of open covers are, in a certain sense, very far from each other. These problems shall be further discussed in Section 3.

As has already been observed, if ONE does not have a winning strategy in $G_1(\mathcal{K}, \mathcal{O})$ (*i.e.*, TWO does not have a winning strategy in the compact-open game), then $S_1(\mathcal{K}, \mathcal{O})$ holds. The question of whether the converse holds remains unsettled.

Problem 1.16 Is $S_1(\mathcal{K}, \mathcal{O})$ equivalent to ONE not having a winning strategy in the game $G_1(\mathcal{K}, \mathcal{O})$ (*i.e.*, TWO not having a winning strategy in the compact-open game)?

2 Alster Spaces

We now turn to properties of covers of topological spaces by G_δ subsets. The main object of our interest is the *Alster property*; see Definition 2.2.

Definition 2.1 Let X be a topological space. A cover $\mathcal{W}$ of X by G_δ subsets is said to be an *Alster cover* if every compact subset of X is included in some element of $\mathcal{W}$. The set of all Alster covers of X will be denoted by $\mathscr{A}_X$.

Definition 2.2 (Alster [1]) A topological space X is an *Alster space* if for every $\mathcal{U} \in \mathscr{A}_X$ there is a countable $\mathcal{V} \subseteq \mathcal{U}$ with $X = \bigcup \mathcal{V}$.

Alster spaces were introduced in [1] in an attempt to characterize the class of *productively Lindelöf spaces*, *i.e.*, the class of topological spaces X such that $X \times Y$ is Lindelöf whenever Y is a Lindelöf space.

Theorem 2.3 (Alster [1]) *Alster spaces are productively Lindelöf. Assuming the Continuum Hypothesis, productively Lindelöf regular spaces of weight not exceeding $\aleph_1$ are Alster.*

An internal characterization of productive Lindelöfness — a problem attributed to H. Tamano in [24, Problem 5] — is still unknown. The following problem, implicitly raised in [1] (see also [6]), remains open.

Problem 2.4 (Alster [1], Barr–Kennison–Raphael [6]) Is is true that every productively Lindelöf space is an Alster space?

After observing that both the properties "X is an Alster space" and "TWO has a winning strategy in the Menger game on X" are implied by "X is σ-compact"[2] and imply "X is Lindelöf and every continuous image of X in a separable metrizable space is σ-compact", F. Tall asks the following question in [34, Problem 5].

[2]A space is *σ-compact* if it is a countable union of compact subsets.

Problem 2.5 (Tall [34]) Is there any implication between the Alster property and TWO having a winning strategy in the Menger game?

We shall provide a complete answer to Problem 2.5 by showing the following:

- If TWO has a winning strategy in the Menger game on a regular space X, then X is Alster (Corollary 2.13).
- The regularity hypothesis in the above result is essential (Example 3.6).
- The converse implication does not hold (Example 3.5).

In what follows, we will denote by $\mathcal{O}_X^\delta$ the family of all covers of a topological space X by G_δ subsets. We start by proving a characterization of the Alster property in terms of the selection principle S_1.[3]

Proposition 2.6 *A topological space X is an Alster space if and only if* $\mathsf{S}_1(\mathscr{A}_X, \mathcal{O}_X^\delta)$ *holds.*

Proof The converse is clear. For the direct implication, suppose that X is an Alster space and let $(\mathcal{U}_n)_{n\in\omega}$ be a sequence in $\mathscr{A}_X$. Let $S = \prod_{n\in\omega}\mathcal{U}_n$ and, for each $f \in S$, define $V_f = \bigcap_{n\in\omega} f(n)$. It follows that $\{V_f : f \in S\}$ is an Alster cover of X; therefore, there is $\{f_n : n \in \omega\} \subseteq S$ such that $X = \bigcup_{n\in\omega} V_{f_n}$. Now, for every $n \in \omega$, define $A_n = f_n(n)$. Then

$$X = \bigcup_{n\in\omega} V_{f_n} = \bigcup_{n\in\omega}\bigcap_{k\in\omega} f_n(k) \subseteq \bigcup_{n\in\omega} f_n(n) = \bigcup_{n\in\omega} A_n,$$

and since $A_n \in \mathcal{U}_n$ for all $n \in \omega$, we are done. ■

Corollary 2.7 *Every Alster space satisfies* $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

Proof This is immediate in view of Proposition 2.6, since $\mathcal{K} \subseteq \mathscr{A}$. ■

Let us now consider a natural modification of the compact-open game.

Definition 2.8 (Telgársky [36]) The *compact-G_δ game* in a topological space X is defined in the same way as the compact-open game with the difference that now TWO is allowed to play G_δ subsets of X.

The proof of the following result is analogous to the proof of Proposition 1.14; see [36, Theorem 6.2].

Proposition 2.9 *The game* $\mathsf{G}_1(\mathscr{A}, \mathcal{O}^\delta)$ *and the compact-G_δ game are dual.*

Propositions 2.6 and 2.9 yield the following corollary.

Corollary 2.10 *If TWO does not have a winning strategy in the compact-G_δ game on a topological space X, then X is an Alster space.*

[3] Proposition 2.6 has also been obtained (independently) by L. Babinkostova, B. Pansera, and M. Scheepers in [4].

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As usual, the characterization of the selective property in terms of its naturally associated game is of interest.

Problem 2.11 Is the Alster property equivalent to TWO not having a winning strategy in the compact-G_δ game?

Finally, [36, Theorem 5.1] and [37, Corollary 3] can be combined to yield the following theorem.

Theorem 2.12 (Telgársky [36, 37]) *Consider the following statements about a topological space X:*

(i) *ONE has a winning strategy in the compact-G_δ game on X;*
(ii) *ONE has a winning strategy in the compact-open game on X;*
(iii) *TWO has a winning strategy in the Menger game on X.*

Then (i) $\leftrightarrow$ (ii) $\rightarrow$ (iii). *Furthermore, if X is regular, then the three statements are equivalent.*

This allows us to relate the Menger game to the Alster property.

Corollary 2.13 *If TWO has a winning strategy in the Menger game on a regular space X, then X is an Alster space.*

Proof This is proved by Theorem 2.12 and Corollary 2.10. ■

In [37, Corollary 4], Telgársky showed the following theorem.

Theorem 2.14 (Telgársky [37]) *If X is a metrizable space, then TWO has a winning strategy in the Menger game on X if and only if X is σ-compact.*

In [5], T. Banakh and L. Zdomskyy noted that Telgársky's argument would follow with "regular hereditarily Lindelöf" in place of "metrizable". Since hereditarily Lindelöf regular spaces have the property that every compact subset is a G_δ (a condition that clearly implies σ-compactness in the presence of the Alster property), Corollary 2.13 extends the Banakh–Zdomskyy version of Theorem 2.14.

Telgársky's proof of the equivalence between (ii) and (iii) in Theorem 2.12 is rather indirect. Inspired by [28], we can give a more straightforward proof of Corollary 2.13, which does not depend on the aforementioned equivalence.

An alternative proof of Corollary 2.13 Let $\sigma\colon {}^{<\omega}\mathcal{O}_X\setminus\{\varnothing\} \to [\tau_X]^{<\aleph_0}$ be a winning strategy for TWO in the Menger game on X. Now let $\mathcal{W}$ be an Alster cover of X. Our task is to find a countable subset of $\mathcal{W}$ that covers X.

The following claim is taken from [28].

Claim 1. For every $s \in {}^{<\omega}\mathcal{O}_X$, the set $K_s = \bigcap\{\overline{\bigcup \sigma(s^\frown\mathcal{U})} : \mathcal{U} \in \mathcal{O}_X\}$ is compact.

Indeed, let $\mathcal{V}$ be a cover of K_s by open subsets of X. For each $x \in K_s$, pick an open neighbourhood U_x of X such that $\overline{U_x}$ is included in some element of $\mathcal{V}$; now, for each $x \in X \setminus K_s$, pick an open neighbourhood U_x of X with $\overline{U_x} \subseteq X \setminus K_s$. Consider then $\mathcal{U}_0 = \{U_x : x \in X\} \in \mathcal{O}_X$, and let $F \in [X]^{<\aleph_0}$ be such that $\sigma(s^\frown\mathcal{U}_0) = \{U_x : x \in F\}$.

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Note that $K_s \subseteq \overline{\bigcup \sigma(s^\frown \mathcal{U}_0)} = \bigcup\{\overline{U_x} : x \in F\}$; thus, if for each $x \in F \cap K_s$ we pick a $V_x \in \mathcal{V}$ with $\overline{U_x} \subseteq V_x$, we will have $K_s \subseteq \bigcup\{V_x : x \in F \cap K_s\}$. This proves Claim 1.

For each $s \in {}^{<\omega}\mathcal{O}_X$, we can then fix a $W_s \in \mathcal{W}$ with $K_s \subseteq W_s$.

Claim 2. For every $s \in {}^{<\omega}\mathcal{O}_X$, there is a countable $\mathcal{C}_s \subseteq \mathcal{O}_X$ such that

$$K_s \subseteq \bigcap\Big\{\overline{\bigcup \sigma(s^\frown \mathcal{U})} : \mathcal{U} \in \mathcal{C}_s\Big\} \subseteq W_s.$$

Since $K_s \subseteq W_s$, the set $\{X \setminus \overline{\bigcup \sigma(s^\frown \mathcal{U})} : \mathcal{U} \in \mathcal{O}_X\}$ is an open cover of $X \setminus W_s$. But $X \setminus W_s$ is an F_σ-subset of X. Since our hypothesis implies that X is a Lindelöf space, it follows that $X \setminus W_s$ is Lindelöf as well, whence there is a countable $\mathcal{C}_s \subseteq \mathcal{O}_X$ such that $X \setminus W_s \subseteq \bigcup\{X \setminus \overline{\bigcup \sigma(s^\frown \mathcal{U})} : \mathcal{U} \in \mathcal{C}_s\}$. This proves Claim 2.

Now define recursively $\mathcal{A}_0 = \mathcal{C}_\varnothing$ and $\mathcal{A}_{n+1} = \mathcal{A}_n \cup \bigcup\{\mathcal{C}_s : s \in {}^{n+1}\mathcal{A}_n\}$ for all $n \in \omega$. Let $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \in \omega\}$. We will show that the countable subset $\mathcal{W}_0 = \{W_s : s \in {}^{<\omega}\mathcal{A}\}$ of $\mathcal{W}$ is a cover of X.

Suppose, to the contrary, that there is $p \in X \setminus \bigcup \mathcal{W}_0$. Since $p \notin W_\varnothing$, there is some $\mathcal{U}_0 \in \mathcal{C}_\varnothing$ such that $p \notin \overline{\bigcup \sigma((\mathcal{U}_0))}$. We also have $p \notin W_{(\mathcal{U}_0)}$, so there is $\mathcal{U}_1 \in \mathcal{C}_{(\mathcal{U}_0)}$ such that $p \notin \overline{\bigcup \sigma((\mathcal{U}_0, \mathcal{U}_1))}$. By proceeding in this fashion ($p \notin W_{(\mathcal{U}_0,\mathcal{U}_1)}$ and so on), we obtain a play

$$\big(\mathcal{U}_0, \sigma((\mathcal{U}_0)), \mathcal{U}_1, \sigma((\mathcal{U}_0, \mathcal{U}_1)), \mathcal{U}_2, \sigma((\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2)), \mathcal{U}_3, \dots\big)$$

of the Menger game on X such that $p \notin \overline{\bigcup \sigma((\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_k))}$ for all $k \in \omega$. But this is a contradiction, since TWO follows the winning strategy σ in this play. ■

A similar argument shows that, if "Menger" is replaced by "Rothberger" in Proposition 2.13, the conclusion can be replaced by "X_δ is Lindelöf". Here, X_δ is the set X endowed with the topology generated by the G_δ subsets from its original topology. But in this case we can avoid the requirement of any separation axioms by making use of Theorem 1.10 (*cf.* [35, Theorem 6.1] and [12, Theorem 2]).

Proposition 2.15 *If TWO has a winning strategy in the Rothberger game on a topological space X, then X_δ is a Lindelöf space.*

Proof By Theorem 1.10, this hypothesis is equivalent to the existence of a winning strategy for ONE in the point-open game. Let then $\sigma\colon {}^{<\omega}\tau \to X$ be such a strategy, where τ is the topology of X.

Now let $\mathcal{W}$ be a cover of X by G_δ subsets. For each $W \in \mathcal{W}$, fix a sequence $(U(W,n))_{n\in\omega}$ of open sets with $W = \bigcap_{n\in\omega} U(W,n)$. Proceeding by induction on $n \in \omega$, we shall assign to each $s \in {}^n\omega$ an element W_s of $\mathcal{W}$ as follows.

First, pick $W_\varnothing \in \mathcal{W}$ such that $\sigma(\varnothing) \in W_\varnothing$. Now let $n \in \omega$ be such that $W_s \in \mathcal{W}$ has already been defined for all $s \in {}^n\omega$. For each $s \in {}^n\omega$ and each $k \in \omega$, choose $W_{s^\frown k} \in \mathcal{W}$ satisfying $\sigma(t_{s,k}) \in W_{s^\frown k}$, where $t_{s,k} \in {}^{n+1}\tau$ is the sequence defined by $t_{s,k}(i) = U(W_{s\restriction i}, s(i))$ for all $i < n$ and $t_{s,k}(n) = U(W_s, k)$.

We claim that $\{W_s : s \in {}^{<\omega}\omega\} \subseteq \mathcal{W}$ is a cover of X. Suppose not, and fix $p \in X \setminus \bigcup_{s\in{}^{<\omega}\omega} W_s$. For $n \in \omega$, we can recursively pick $k_n \in \omega$ with $p \notin U(W_{(k_i)_{i<n}}, k_n)$.

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But then we get a contradiction from the fact that

$$(\sigma(\varnothing), U(W_\varnothing, k_0), \sigma(U(W_\varnothing, k_0)), U(W_{(k_0)}, k_1), \\ \sigma(U(W_{(k_0)}, k_1)), U(W_{(k_0,k_1)}, k_2), \dots)$$

is a play of the point-open game in which ONE plays according to σ and loses. ■

We shall see later, in Example 3.6, that the regularity hypothesis in Corollary 2.13 is essential.

The following diagram summarizes the connections between the properties considered in this paper. We will now quote some results from which some of the implications in the diagram follow.

For the first result (proved in [35, Theorem 9.3]), recall that a topological space X is *scattered* if every nonempty subspace $Y \subseteq X$ has an isolated point (relative to Y).

Theorem 2.16 (Telgársky [35]) *If a regular space X is Lindelöf and scattered, then TWO has a winning strategy in the Rothberger game on X.*

The next result is attributed to F. Galvin in [13]; see [32, Theorem 47]. Recall that a *P-space* is a topological space in which every G_δ subset is open.

Proposition 2.17 (Galvin) *A P-space is Lindelöf if and only if it is Rothberger.*

Finally, recall that a *Michael space* is a Lindelöf space X such that $X \times \omega^\omega$ is not Lindelöf. Michael spaces have been constructed with the aid of several set-theoretical hypotheses; see *e.g.*, [18, 20, 21]. In [25, Proposition 3.1], it was shown:[4]

Theorem 2.18 (Repovš-Zdomskyy [25]) *If there is a Michael space, then every productively Lindelöf space is Menger.*

Each arrow in Figure 1 has the number of the result from which the implication follows as well as the number of the counterexample (in Section 3) showing that the implication cannot be reversed or that the regularity assumption is necessary, if this is the case.

3 Counterexamples

We shall now see that unless the question of whether the converse implication holds is indicated, the implications in Figure 1 cannot be reversed (at least consistently); moreover, the regularity assumptions that appear in Figure 1 cannot be dropped (again, at least consistently). This will follow from the examples listed below.

Example 3.1 A regular nonscattered space on which TWO has a winning strategy in the Rothberger game.

This is just the space $\mathbb{Q}$ of rational numbers with the usual topology.

[4]We thank Lyubomyr Zdomskyy for bringing this result to our attention.

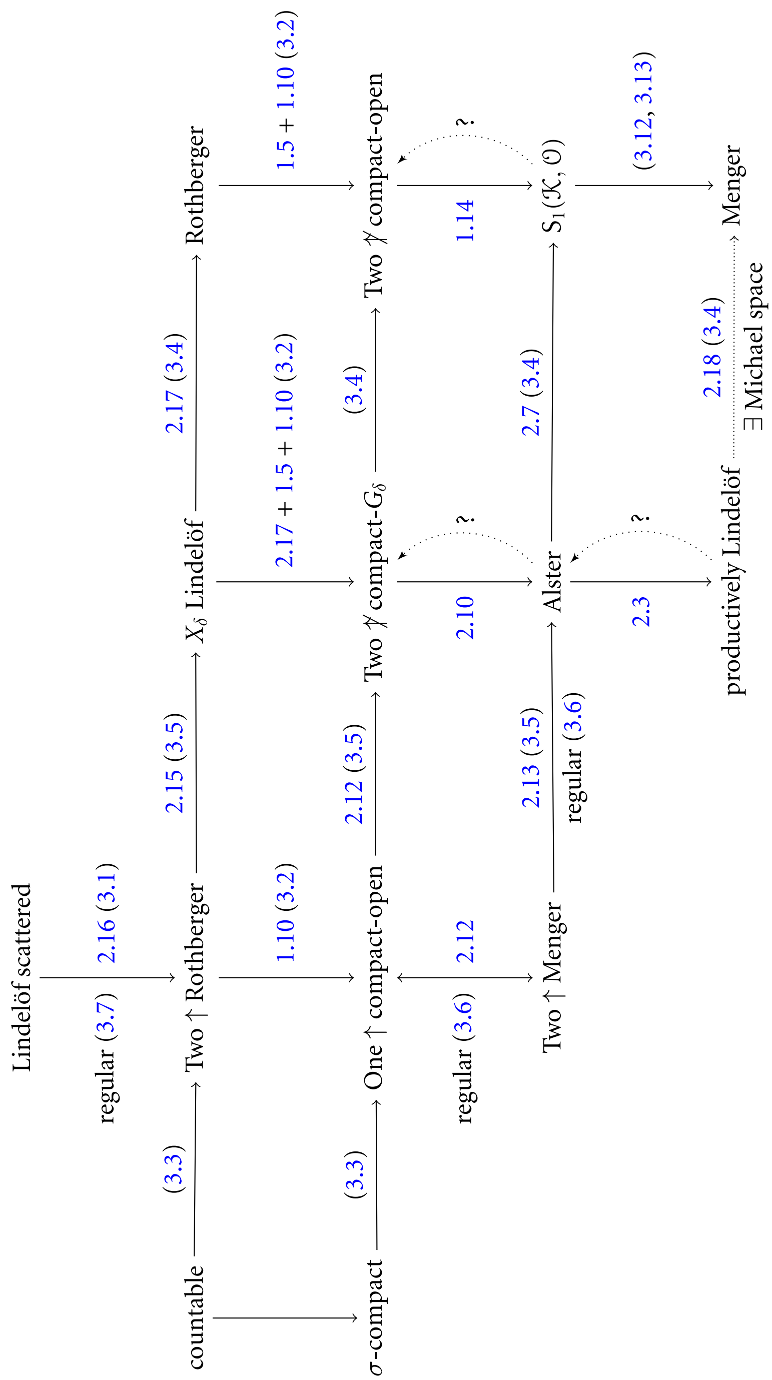
Lindelöf scattered
regular (3.7)
2.16 (3.1)
countable
(3.3)
Two ↑ Rothberger
2.15 (3.5)
X_δ Lindelöf
2.17 (3.4)
Rothberger
1.10 (3.2)
2.17 + 1.5 + 1.10 (3.2)
1.5 + 1.10 (3.2)
σ-compact
(3.3)
One ↑ compact-open
2.12 (3.5)
Two ↑̸ compact-G_δ
(3.4)
Two ↑̸ compact-open
regular (3.6)
2.12
2.10
?
1.14
?
Two ↑ Menger
2.13 (3.5)
regular (3.6)
Alster
2.7 (3.4)
$S_1(\mathcal{K}, \mathcal{O})$
2.3
?
(3.12, 3.13)
productively Lindelöf
2.18 (3.4)
∃ Michael space
Menger

Figure 1

Example 3.2 A compact Hausdorff space that is not Rothberger.

The Cantor set 2^ω satisfies these conditions: it is folklore that the sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of 2^ω defined by $\mathcal{U}_n = \{\pi_n^{-1}[\{0\}], \pi_n^{-1}[\{1\}]\}$, where $\pi_n\colon 2^\omega \to 2$ is the projection onto the n-th coordinate, witnesses the failure of the Rothberger property.

Example 3.3 A Lindelöf scattered regular space that is not σ-compact.

This is the one-point Lindelöfication of an uncountable discrete space, *i.e.,* the space $X = A \cup \{p\}$, where A is uncountable and $p \notin A$, in which every point of A is isolated and cocountable subsets of X are open.

Example 3.4 A Rothberger regular space that is not productively Lindelöf.

J. Moore's L space [22] is Rothberger (see [32, section 4]) but has a non-Lindelöf finite power (see [38, Theorem 3.4(2)] and [2, Theorem 2]).[5]

Example 3.5 There is a Lindelöf regular nonscattered space Y such that Y_δ is Lindelöf (hence, in particular, Y is Alster) and TWO does not have a winning strategy in the Menger game on Y.

Let Y be the space considered by Telgársky in [36, Section 7]. For each $\lambda \in \lim(\omega_1) = \{\gamma \in \omega_1 : \gamma \text{ is a limit ordinal}\}$, fix a cofinal subset $C_\lambda \subseteq \lambda$ such that $|C_\lambda \cap \alpha| < \aleph_0$ whenever $\alpha < \lambda$. The set

$$Y = \{\chi_{C_\lambda} : \lambda \in \lim(\omega_1)\} \cup \{\chi_F : F \in [\omega_1]^{<\aleph_0}\}$$

is then regarded as a subspace of 2^{ω_1} with the countable box product topology τ.[6]

Note that $Y = Y_\delta$ and that $\{\chi_F : F \in [\omega_1]^{<\aleph_0}\}$ is a closed subset of Y without isolated points. In [36], it was proved that the compact-open game on Y is undetermined; thus, in view of Theorem 2.12, there is no winning strategy for TWO in the Menger game on Y.

As in Corollary 2.13, we can give a proof for this last fact that does not rely on the equivalence between (ii) and Theorem 2.12(iii).

A direct proof for Example 3.5 For each $p \in Y$ and each $\alpha \in \omega_1$, we shall write $V(p, \alpha) = \{y \in Y : y \restriction \alpha = p \restriction \alpha\} \in \tau$.

Let σ be a strategy for TWO in the Menger game on Y. By expanding the answers of TWO if necessary, we can regard σ as a function

$$\sigma\colon ({}^{<\omega}\lim(\omega_1)) \setminus \{\varnothing\} \to [\omega_1]^{<\aleph_0},$$

meaning that, if ONE gives an open cover $\{V(y, \alpha) : y \in Y\}$ of Y with $\alpha \in \lim(\omega_1)$ (note that, as Y is Lindelöf and ω_1 is regular, any open cover of Y has an open refinement of this form), TWO responds by choosing, for some $F \in [\alpha]^{<\aleph_0}$, the open sets $V(y, \alpha)$ with $y \in \{\chi_G : G \subseteq F\} \cup \{\chi_{C_\gamma} : \gamma \in F \cap \lim(\omega_1)\} \cup \{\chi_{C_\alpha}\}$.

[5]We thank Marion Scheepers and Boaz Tsaban for pointing this out to us.

[6]Here, χ_A denotes the function in ${}^{\omega_1}2$ satisfying $\{\alpha \in \omega_1 : \chi_A(\alpha) = 1\} = A$.

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For each $t \in {}^{<\omega}\lim(\omega_1)$, we have $\max(\sigma(t^\frown\alpha)) < \alpha$ for all $\alpha \in \lim(\omega_1)$; thus, it follows from Fodor's Lemma ([10, Theorem 2]; see *e.g.*, [17, Theorem 21.12]) that there exist $\beta_t \in \omega_1$ and a stationary set $S_t \subseteq \lim(\omega_1)$ such that for all $\alpha \in S_t$, we have $\max(\sigma(t^\frown\alpha)) = \beta_t$. Let M be a countable elementary submodel of H_θ for a convenient choice of θ (see *e.g.*, [9] or [17, Chapter 24]) such that $Y, \tau, \sigma \in M$, and consider $\lambda = M \cap \omega_1 \in \lim(\omega_1)$. By elementarity, it follows that for each $t \in {}^{<\omega}\lim(\lambda)$ there exist $\beta_t \in \lambda$ and an unbounded subset S_t of λ with $S_t \subseteq \lim(\lambda)$ such that $\max(\sigma(t^\frown\alpha)) = \beta_t$ for all $\alpha \in S_t$.

We shall now prove that σ is a not a winning strategy by showing that ONE can prevent the point $\chi_{C_\lambda} \in Y$ from being covered if TWO plays according to σ.

In order to accomplish this, ONE starts by picking $\xi_0 \in C_\lambda$ with $\beta_\varnothing < \xi_0$ and then plays $\alpha_0 \in S_\varnothing$ such that $\xi_0 < \alpha_0$ — which, we recall, is short for saying that she plays the open cover $\{V(y, \alpha) : y \in Y\}$. Since TWO follows the strategy σ, he responds with $\sigma((\alpha_0)) \in [\lambda]^{<\aleph_0}$; now ONE picks $\xi_1 \in C_\lambda$ satisfying $\beta_{(\alpha_0)} < \xi_1$ and then plays $\alpha_1 \in S_{(\alpha_0)}$ with $\xi_1 < \alpha_1$. In general, in the n-th inning, if $t_n = (\alpha_k)_{k<n}$ is the sequence of ONE's moves so far, she picks $\xi_n \in C_\lambda$ such that $\beta_{t_n} < \xi_n$ and then plays $\alpha_n \in S_{t_n}$ with $\xi_n < \alpha_n$. It is clear that the point $\chi_{C_\lambda} \in Y$ is not covered in any of the innings, since for all $n \in \omega$ we have $\max\sigma((\alpha_0, \dots, \alpha_n)) < \xi_n < \alpha_n$ and $\chi_{C_\lambda}(\xi_n) = 1$. ∎

Example 3.6 There is a Hausdorff non-regular space X such that $\mathsf{S}_1(\mathcal{K}_X, \mathcal{O}_X)$ fails and yet TWO has a winning strategy in the Menger game on X; in particular, X is a Menger space.

This is the space X obtained by taking the real line $\mathbb{R}$ (with the usual topology) and then declaring every countable subset closed. Since every compact subset of X is finite, it follows from [29, Theorem 17] that $\mathsf{S}_1(\mathcal{K}_X, \mathcal{O}_X)$ is equivalent to $S_1(\mathcal{O}_X, \mathcal{O}_X)$, which does not hold since $\mathbb{R}$ is not a Rothberger space.

Now write

$$\{2k+1 : k \in \omega\} = \dot{\bigcup}_{j\in\omega} A_j$$

with $|A_j| = \aleph_0$ for each $j \in \omega$ and let ϱ be a winning strategy for TWO in the Menger game played on the real line with the usual topology; such a strategy exists, since $\mathbb{R}$ is σ-compact. We may assume that in the Menger game on X ONE only plays covers constituted by basic open sets of the form $U \setminus C$, where U is open in $\mathbb{R}$ and $C \subseteq \mathbb{R}$ is countable. For each such basic open set W, fix $U(W)$ open in $\mathbb{R}$ and $C(W) \in [\mathbb{R}]^{\leq\aleph_0}$ with $W = U(W) \setminus C(W)$. Then, for each basic open cover $\mathcal{W}$ of X, define $\mathcal{U}(\mathcal{W}) = \{U(W) : W \in \mathcal{W}\} \in \mathcal{O}_\mathbb{R}$.

We shall now describe a winning strategy for TWO in the Menger game on X. In each even inning $2k \in \omega$ if $(\mathcal{W}_i)_{i\leq 2k}$ is the sequence of open covers played by ONE so far, TWO responds with $\mathcal{F}_{2k} \in [\mathcal{W}_{2k}]^{<\aleph_0}$ such that

$$\varrho\big((\mathcal{U}(\mathcal{W}_{2i}))_{i\leq k}\big) = \{U(W) : W \in \mathcal{F}_{2k}\};$$

i.e., $\{U(W) : W \in \mathcal{F}_{2k}\}$ is TWO's answer to the sequence $(\mathcal{U}(\mathcal{W}_{2i}))_{i\leq k}$ in the Menger game on $\mathbb{R}$ according to the strategy ϱ. Now TWO makes use of the innings in A_k to cover the countably many points in $\bigcup_{W\in\mathcal{F}_{2k}} C(W)$. The fact that ϱ is a winning

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strategy for TWO in the Menger game on $\mathbb{R}$ guarantees that X will be covered by TWO through this procedure.

Example 3.7 If there is a Luzin subset of the real line,[7] then there is a Hausdorff nonregular space X that is Lindelöf scattered and such that X_δ is not Lindelöf.

Let $L \subseteq \mathbb{R}$ be a Luzin set, which we may assume to consist only of irrational numbers. On the set $X = L \cup \mathbb{Q}$, consider the topology in which every point of L is isolated and basic neighbourhoods of $q \in \mathbb{Q}$ are of the form

$$\{q\} \cup \left\{ x \in L : |x - q| < \frac{1}{n+1} \right\}$$

for $n \in \omega$. It is clear that X is scattered and that X_δ, being discrete and uncountable, is not Lindelöf. Yet X is Lindelöf. From any open cover $\mathcal{U}$ of X, we can extract a countable subset $\mathcal{U}_0$ that covers $\mathbb{Q}$; as L is a Luzin set, $\mathcal{U}_0$ leaves only countably many points of L uncovered.

Definition 3.8 An open cover $\mathcal{U}$ of a topological space X is an *R-cover* if every Rothberger subspace of X is included in some element of $\mathcal{U}$. The set of all R-covers of X will be denoted by $\mathcal{R}_X$.

The following is a straightforward generalization of the implication (3) $\Rightarrow$ (1) of [29, Theorem 17]; its proof is essentially the same.

Proposition 3.9 *A topological space X satisfies $\mathsf{S}_1(\mathcal{R}, \mathcal{O})$ if and only if X is a Rothberger space.*

Corollary 3.10 *Let X be a topological space such that every compact subspace of X is Rothberger. Then X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$ if and only if X is a Rothberger space.*[8]

Proof Since in this case we have $\mathcal{R}_X \subseteq \mathcal{K}_X$, it follows that $\mathsf{S}_1(\mathcal{K}_X, \mathcal{O}_X)$ implies $\mathsf{S}_1(\mathcal{R}_X, \mathcal{O}_X)$, which, by Proposition 3.9, is equivalent to X being Rothberger. ■

Corollary 3.11 *Let X be a topological space every compact subspace of which has an isolated point. Then X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$ if and only if X is a Rothberger space.*

Proof This follows directly from Corollary 3.10, since every compact scattered space is Rothberger (folklore; see *e.g.,* [3, Proposition 5.5]). ■

Example 3.12 If $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$, then there is a Menger regular space that does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

It follows from [11, Theorem 5] that $\mathrm{cov}(\mathcal{M})$ is the least cardinality of a non-Rothberger subspace of the real line. Let $X \subseteq \mathbb{R}$ be such a subspace. As $|X| < \mathfrak{d}$, it follows from [16, Theorem 5] (see also [11, Theorem 3]) that X is a Menger space. By

[7] That is, an uncountable set $L \subseteq \mathbb{R}$ such that $L \cap A$ is countable for every nowhere dense subset A of $\mathbb{R}$. The Continuum Hypothesis implies the existence of a Luzin set [19, Theorem 1].

[8] This has also been observed independently in [4].

the Čech–Pospíšil Theorem ([8]; see *e.g.,* [14, Theorem 7.19]), any compact subspace of X without isolated points would have size at least $\mathfrak{c} \geq \mathfrak{d} > |X|$, which is impossible. Thus, by Corollary 3.11, X does not satisfy $S_1(\mathcal{K}, \mathcal{O})$.

Example 3.13 If there is a Sierpiński subset of the real line,[9] then there is a Menger regular space that does not satisfy $S_1(\mathcal{K}, \mathcal{O})$.

Pick a Sierpiński set $S \subseteq \mathbb{R}$ and endow it with the Sorgenfrey line topology. By [27, Corollary 3.6], S is Menger. Since S does not have measure zero, it cannot be Rothberger [26]. Thus, as every compact subset of the Sorgenfrey line is countable, it follows from Corollary 3.10 that S does not satisfy $S_1(\mathcal{K}, \mathcal{O})$.

In view of Examples 3.12 and 3.13, it is natural to ask the following question.

Problem 3.14 Is it consistent with ZFC that every Menger regular space satisfies $S_1(\mathcal{K}, \mathcal{O})$?

We conjecture that the answer is negative. Note that this could be proved by means of a dichotomic argument, *e.g.*, by showing that a counterexample exists under $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$. Should this be the case, one might still ask the following question.

Problem 3.15 Is there a ZFC example of a Menger regular space that does not satisfy $S_1(\mathcal{K}, \mathcal{O})$?

We point out that a set of reals satisfying these conditions would have to be, in particular, a ZFC example of a Menger non-σ-compact subspace of $\mathbb{R}$ — a kind of set that only recently has been constructed (see [7, Theorem 16]), even though the existence of a Menger non-σ-compact set of reals had been known to follow from ZFC since D. Fremlin and A. Miller's dichotomic proof from [11, Theorem 4]. Note also that if the regularity requirement is dropped, then Example 3.6 answers Problem 3.15 in the affirmative.

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[9] That is, an uncountable set $S \subseteq \mathbb{R}$ such that $S \cap A$ is countable for every (Lebesgue) measure zero subset A of $\mathbb{R}$. The Continuum Hypothesis implies the existence of a Sierpiński set [33, 4.6].

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