REMARKS ON THE UNIQUENESS THEOREM OF SOLUTIONS OF THE DARBOUX PROBLEM

James S. W. Wong

(received January 7, 1965)

Recently B. Palczewski and W. Pawelski [1] have given some sufficient conditions of uniqueness for the solutions of the Darboux problem for equations of the form:

$$(1) \qquad \frac{\partial^2 u}{\partial x \partial y} = f(x, y, u) .$$

The criteria given there for equations of the form (1) are natural generalizations of the criteria given by Krasnosielski and Krein [2] in the corresponding ordinary differential case. The purpose of the present note is to give a further generalization of the above result and two other uniqueness conditions for the solutions of the Darboux problem of the same form.

Let D denote the rectangle, $0 \leq x \leq a$, $0 \leq y \leq b$ $(a, b > 0)$ and let $f(x, y, u)$ be a function defined and continuous on the set $E = D \times \{-\infty < u < \infty\}$. Then every solution $u(x, y)$ of the Darboux problem of the form (1), satisfying the conditions:

$$(2) \qquad u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y), \quad \sigma(0) = \tau(0) = u_o$$

is a solution of the following integral equation:

$$(3) \qquad u(x, y) = \sigma(x) + \tau(y) - u_o + \int_o^x \int_o^y f(s, t, u(s, t)) ds\, dt$$

and conversely. Here the functions $\sigma(x)$, $\tau(y)$ may be taken as functions of the class C' defined respectively on $[0, a]$ and $[0, b]$. (Note: Some regularity restrictions on $\sigma(x)$ and $\tau(y)$

Canad. Math. Bull. vol. 8, no. 6, 1965

https://doi.org/10.4153/CMB-1965-059-1 Published online by Cambridge University Press

are necessary to ensure the existence of solutions to (1) under conditions (2). We are not concerned with these conditions here.)

THEOREM 1. If $f(x,y,u)$ is defined, continuous and bounded on E, and it satisfies in addition the following:

$$(4) \quad \begin{aligned} |f(x,y,u_1) - f(x,y,u_2)| &\le \frac{k}{xy} |u_1 - u_2| , \quad k > 0 \\ |f(x,y,u_1) - f(x,y,u_2)| &\le \frac{C}{x^\beta y^\beta} |u_1 - u_2|^\alpha , \quad C > 0 \end{aligned}$$

with $0 < \alpha < 1$, $\beta < \alpha$, and $k(1-\alpha)^2 < (1-\beta)^2$ for all $(x,y,u) \in E$, then there exists at most one solution $u(x,y)$ of the Darboux problem for equation (1) satisfying conditions (2).

Proof: Let $M = \sup_E |f(x,y,u)|$, and assume $u(x,y)$ and $v(x,y)$ are two solutions to the Darboux problem satisfying condition (2). We obtain from (3) that

$$|u(x,y) - v(x,y)| \le 2Mxy$$

for all $(x,y) \in D$. From (4) it follows:

$$|u(x,y) - v(x,y)| \le \int_0^x \int_0^y |f(s,t, u(s,t)) - f(s,t, v(s,t))| ds\, dt$$

$$\le C \int_0^x \int_0^y (2M)^\alpha (st)^{\alpha-\beta} ds\, dt$$

$$\le C (2M)^\alpha \frac{(xy)^{(1-\beta)+\alpha}}{((1-\beta)+\alpha)^2}$$

$$\le C (2M)^\alpha (xy)^{(1-\beta)+\alpha} ,$$

and in general:

https://doi.org/10.4153/CMB-1965-059-1 Published online by Cambridge University Press

$$|u(x,y) - v(x,y)| \leq C^{1+\alpha+\ldots+\alpha^m} (2M)^{\alpha^{m+1}} (xy)^{(1-\beta)(1+\alpha+\ldots+\alpha^m) + \alpha^{m+1}}$$

for $m = 1, 2, 3, \ldots$. Therefore, we have the following estimate

$$(5) \qquad |u(x,y) - v(x,y)| \leq C^{\frac{1}{1-\alpha}} (xy)^{\frac{1-\beta}{1-\alpha}} .$$

Define $Q(x,y) = (xy)^{-\sqrt{k}} |u(x,y) - v(x,y)|$ for $xy > 0$. Then, it follows from (5) that

$$0 \leq Q(x,y) = Q(s) \leq C^{\frac{1}{1-\alpha}} (xy)^{\frac{(1-\beta)-\sqrt{k}(1-\alpha)}{1-\alpha}}$$

Hence, we have $\lim_{s \to s_o} Q(s) = 0$, where $s \in D$ and $s_o \in \Gamma = \{ s : s = (x,y) \in D$ and $x = 0$ or $y = 0\}$. Clearly Q is continuous on D if we define $Q(s_o) = 0$ for $s_o \in \Gamma$. We wish to show that $Q(x,y) \equiv 0$ on D. Assume the contrary. Then there exists a point $(\bar{x}, \bar{y})$ such that:

$$0 < r = Q(\bar{x}, \bar{y}) = \sup_D Q(x,y) .$$

On the other hand, if we use (3) and (4), we obtain:

$$r = Q(\bar{x},\bar{y}) \leq (\bar{x}\,\bar{y})^{-\sqrt{k}} \int_0^{\bar{x}} \int_0^{\bar{y}} |f(s,t,u(s,t)) - f(s,t,v(s,t))| ds\, dt$$

$$\leq (\bar{x}\,\bar{y})^{-\sqrt{k}} \int_0^{\bar{x}} \int_0^{\bar{y}} k(st)^{\sqrt{k}-1} Q(s,t) ds\, dt$$

$$< r\, (\bar{x}\,\bar{y})^{-\sqrt{k}} \int_0^{\bar{x}} \sqrt{k}\, s^{\sqrt{k}-1} ds \int_0^{\bar{y}} \sqrt{k}\, t^{\sqrt{k}-1} dt$$

$$= r ,$$

which is the desired contradiction.

THEOREM 2. If $f(x, y, u)$ is defined and continuous on E, and it satisfies in addition the following:

$$(6) \qquad |f(x,y,u)| \leq A(xy)^p \quad p > -1, \quad A > 0,$$

$$|f(x,y,u_1) - f(x,y,u_2)| \leq \frac{C}{(xy)^r} |u_1 - u_2|^q \quad q \geq 1, \quad C > 0,$$

with

$$q(1+p) - r = p,$$

$$\zeta = \frac{C(2A)^{q-1}}{(p+1)^q} < 1$$

for all $(x, y, u) \in E$, then there exists at most one solution $u(x, y)$ of the Darboux problem for equation (1) satisfying conditions (2).

Proof: Let $u(x, y)$ and $v(x, y)$ be two solutions to the Darboux problem satisfying condition (2). We obtain from (3) and (6) that

$$(7) \qquad |u(x,y) - v(x,y)| \leq \frac{2A}{p+1} (xy)^{p+1}$$

for all $(x, y) \in D$. Using (6) and the estimate of (7), we obtain:

$$|u(x,y) - v(x,y)| \leq C\left(\frac{2A}{p+1}\right)^q \int_0^x \int_0^y (st)^{(p+1)q-r} \, ds \, dt$$

$$= \zeta \left(\frac{2A}{p+1}\right) (xy)^{p+1}$$

and hence successively,

$$|u(x,y) - v(x,y)| \leq \zeta^{1+q+\ldots+q^m} \left(\frac{2A}{p+1}\right) (xy)^{p+1}$$

https://doi.org/10.4153/CMB-1965-059-1 Published online by Cambridge University Press

for $m = 1, 2, \ldots$. Since $q \geq 1$, we conclude that $v(x,y) = u(x,y)$.

THEOREM 3. If $f(x,y,u)$ is defined and continuous on E, and it satisfies in addition the following:

$$(8) \qquad |f(x,y,u_1) - f(x,y,u_2)| \leq \frac{k}{xy} |u_1 - u_2|$$

with $k \leq 1$ for all $(x,y,u) \in E$, then there exists at most one solution of the Darboux problem for equation (1) satisfying conditions (2).

Proof. Assume $u(x,y)$ and $v(x,y)$ are two solutions to the Darboux problem satisfying conditions (2). We obtain from (8) the following estimate:

$$(9) \qquad |u(x,y) - v(x,y)| \leq k \int_0^x \int_0^y \frac{|u(s,t) - v(s,t)|}{st} \, ds \, dt .$$

Define $B(x,y) = \dfrac{|u(x,y) - v(x,y)|}{xy}$ for $xy > 0$. Since $f(x,y,u)$ is continuous, we note that $|f(x,y,u(x,y)) - f(x,y,v(x,y))| \leq M_{xy}$ where M_{xy} tends to zero as x or y tends to zero, or both.

Therefore it follows that $B(x,y) \geq 0$ for all $(x,y) \in D$ and $\lim_{s \to s_o} B(s) = 0$ where $s = (x,y) \in D$ and $s_o \in \Gamma = \{ s : x=0 \text{ or } y=0 \}$.

Since $B(x,y)$ is now continuous over the compact region D, it attains its maximum at some point (x_o, y_o). Using (9), we observe:

$$|u(x_o,y_o) - v(x_o,y_o)| \leq k \int_o^{x_o} \int_o^{y_o} B(s,t) ds \, dt$$

$$< k \, B(x_o,y_o) \, x_o \cdot y_o$$

$$= k \, |u(x_o,y_o) - v(x_o,y_o)|$$

which is the desired contradiction.

https://doi.org/10.4153/CMB-1965-059-1 Published online by Cambridge University Press

REMARK 1. Theorem 1 reduces to the result of Palczewski and Pawelski [1], by taking $\beta = 0$.

REMARK 2. The criteria given in Theorems 1 and 2 are strictly connected with the results of [3] and [4] for the case of ordinary differential equations.

REMARK 3. Theorem 3 gives the uniqueness of solutions to the Darboux problem under conditions analogous to the Nagumo-Perron criteria in the case of ordinary differential equations (Cf. [5]).

REMARK 4. All these results may easily be generalized to respective theorems for higher orders and to theorems concerning systems of equations of the type (1) (Cf. Theorem 2 [1]).

REFERENCES

1. B. Palczewski and W. Pawelski, Some remarks on the uniqueness of solutions to the Darboux problem with conditions of the Krasnosielski-Krein type, Ann. Polon. Math. XIV (1964), pp. 97-100.

2. M.A. Krasnosielski and S.G. Krein, On a class of uniqueness theorems for the equation $y' = f(x, y)$, Uspehi Mat. Nauk. (N.S.) 11 (1956), No. 1 (67), 209-213.

3. O. Kooi, The method of successive approximations and a uniqueness theorem of Krasnosielski and Krein in the theory of differential equations, Indag. Math. 20 (1958), pp. 322-327.

4. O. Kooi, Existentie, eenduidigheids- en convergentie stellingen in de theorie der gewone differentiaal vergelijkingen, Thesis V.U., Amsterdem (1956).

5. M. Nagumo, Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung, Japan J. Math. 3 (1926) pp. 107-112.

University of Alberta, Edmonton

https://doi.org/10.4153/CMB-1965-059-1 Published online by Cambridge University Press