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Just non-finitely-based varieties of groups

M. F. Newman

A variety of groups is *just non-finitely-based* if it does not have a finite basis for its laws while all its proper subvarieties do have a finite basis. Recent work of Ol'šanskiĭ, Vaughan-Lee and Adjan guarantees the existence of at least one just non-finitely-based variety. In this note an infinite number of just non-finitely-based varieties are shown to exist by proving that for every prime p there is a non-finitely based variety of p-groups.

A variety of groups which does not have a finite basis for its laws contains, by a routine application of Zorn's Lemma, a variety which is minimal with respect to not having a finite basis for its laws. Call such a minimal variety *just non-finitely-based*. The previous sentence can then be rewritten: every non-finitely-based variety contains a just non-finitely-based variety.

It follows from Vaughan-Lee's work in [5] that the product variety $\underline{\underline{T}}_2\underline{\underline{T}}_2$ (where $\underline{\underline{T}}_2$ is the variety generated by the dihedral group of order 8), even the subvariety defined by the additional word $[[x_1, x_2, x_3], [x_4, x_5, x_6], [x_7, x_8]]$, contains a just non-finitely-based variety. Conceivably there is only one just non-finitely-based variety in $\underline{\underline{T}}_2\underline{\underline{T}}_2$. Although the non-finitely-based varieties described by Ol'šanskiĭ

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[4] and Adjan [1] are quite different from Vaughan-Lee's, they don't seem to guarantee the existence of even one more just non-finitely-based variety.

The purpose of this note is to prove:

There are an infinite number of just non-finitely-based varieties of groups.

This is done by proving for each prime p that there is a variety of p-groups which does not have a finite basis for its laws. Unexplained notation is used as described in Hanna Neumann's book [3].

In the word group W on the set $\{x, y, z, x_1, x_2, \ldots\}$ let $u(k) = [x_1, x_2] \ldots [x_{2k-1}, x_{2k}]$ and

$$v(k) = \left[[x, y, z], [x, y, z]^{u(k)}, \ldots, [x, y, z]^{u(k)^{p-1}}\right] .$$

It suffices to prove that there is a group in the variety $(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_p)\underline{\underline{T}}_p$ in which $v(1), \ldots, v(n-1)$ are laws but $v(n)$ is not a law (here for p odd $\underline{\underline{T}}_p$ is the variety generated by the non-abelian group of order p^3 and exponent p); for then a standard argument shows that the subvariety of $(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_p)\underline{\underline{T}}_p$ consisting of the groups in which $v(1), v(2), \ldots$ are all laws cannot have a finite basis for its laws.

Let A be a free group in $\underline{\underline{T}}_p$ of finite rank m. Let B be a free group of $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_p$ of rank the order of A freely generated by $\{b_a : a \in A\}$. There is a natural action of A on B described by $(b_a)^{a'} = b_{aa'}$. Let C be the splitting extension of B by A with this action. As usual A and B will be identified with the appropriate subgroups of C, and the elements of C will be taken to have the form ab with a in A and b in B (identity elements will be omitted). The required group will be exhibited as a factor group of C.

The order of the commutator subgroup A' of A is $p^{m(m-1)/2}$. The number of elements of A' which can be written as the product of $n - 1$ commutators is at most $p^{2m(n-1)}$ (since the identity e is a commutator this covers products of less than $n - 1$ commutators as well). Take $m \geq 4n$, then there is an element, d say, of A' which can be written as the product of n commutators but not of $n - 1$ (or fewer) commutators. Let D denote the subgroup generated by d.

The p-th term $\gamma_p(B)$ of the lower central series of B is an elementary abelian p-group generated (not freely) by the elements $\left[b_{a_1}, \ldots, b_{a_p}\right]$ which will be written $\left[[a_1, \ldots, a_p]\right]$ from now on. The result will be proved by exhibiting a subgroup R of $\gamma_p(B)$ which contains all the values in C of the words $v(1), \ldots, v(n-1)$ but not every value of $v(n)$; this is enough for then the verbal subgroup V of C corresponding to the set $\{v(1), \ldots, v(n-1)\}$ of words lies in R and consequently C/V will have the required properties.

Let M be the subgroup of $\gamma_p(B)$ generated by the $\left[[a_1, \ldots, a_p]\right]$ where $\{a_1, \ldots, a_p\}$ is a coset of D and L the subgroup generated by all the other $\left[[a_1, \ldots, a_p]\right]$. Every relation between the $\left[[a_1, \ldots, a_p]\right]$ is a consequence of relations of the types

$$(*)\quad \begin{cases} \left[[a_1, a_2, a_3, \ldots]\right]\left[[a_2, a_3, a_1, \ldots]\right]\left[[a_3, a_1, a_2, \ldots]\right] = e, \\ \left[[a_1, a_2, \ldots, a_i, a_{i+1}, \ldots]\right]\left[[a_1, a_2, \ldots, a_{i+1}, a_i, \ldots]\right]^{-1} = e, \\ \text{and } \left[[a_1, a_2, \ldots]\right]\left[[a_2, a_1, \ldots]\right] = e, \end{cases}$$

so $\gamma_p(B)$ is the direct product of M and L. Let U be a transversal of D in A' which contains e, and T a transversal of A' in A which contains e. The subgroup M is freely generated by the $g(t, u, i) = \left[[tud^i, tu, tud, \ldots, tud^{i-1}, tud^{i+1}, \ldots, tud^{p-1}]\right]$ where t, u run through T, U respectively and i through $\{1, \ldots, p-1\}$ because these commutators are basic in any order in which $tu < tud < \ldots < tud^{p-1}$ for all t, u (see for instance 4.05 of [2] for a direct proof). Let $g_t = g(t, e, 1)$. Let P be the subgroup of M

generated by the g_t with t running through T, and let N be the subgroup of M generated by all the products $g(t, u, i)g_t^{-i}$. Clearly M is the direct product of P and N. Take R to be the direct product of N and L.

Let θ be a homomorphism from the word group W to C such that $x\theta = b_e$, $y\theta = t$, $z\theta = t'$, where t, t' are distinct elements of $T\backslash\{e\}$, and $u(n)\theta = d$, then $v(n)\theta = g_e^{-1}g_tg_{t'}g_{tt'}^{-1}r$ with r in R. Thus $v(n)\theta$ does not belong to R, and one of the claims is established.

The following observation will be needed in proving the other claim. For all $a_1, \ldots, a_p$ in A and h in A',

$\left[[a_1, \ldots, a_p]\right]^h = \left[[a_1h, \ldots, a_ph]\right] = \left[[a_1, \ldots, a_p]\right]r$ for some r in R because if $\{a_1, \ldots, a_p\}$ is not a coset of D neither is $\{a_1h, \ldots, a_ph\}$, while if $\{a_1, \ldots, a_p\}$ is the coset tuD and $a_1 = tud^i$, $a_2 = tud^j$, then both $\left[[a_1, \ldots, a_p]\right]$ and $\left[[a_1h, \ldots, a_ph]\right]$ are congruent to g_t^{i-j} modulo N. Hence, by linearity, $[b_1, \ldots, b_p]^h$ is congruent to $[b_1, \ldots, b_p]$ modulo R for all $b_1, \ldots, b_p$ in B and h in A'.

Every value of $v(k)$ for k in $\{1, \ldots, n-1\}$ has the form $\left[b, b^c, \ldots, b^{c^{p-1}}\right]$ where b belongs to B and c is a product of at most $n - 1$ commutators from A'. All these elements will be shown to belong to R. It suffices to prove for b_1, b_2 in B and h in A' that $\left[b_1b_2, (b_1b_2)^h, \ldots, (b_1b_2)^{h^{p-1}}\right]$ is congruent to $\left[b_1, b_1^h, \ldots, b_1^{h^{p-1}}\right]\left[b_2, b_2^h, \ldots, b_2^{h^{p-1}}\right]$ modulo R, for then $\left[b, b^c, \ldots, b^{c^{p-1}}\right]$ is, by induction on the length of b as a product of generators b_a, congruent modulo R to a product of commutators

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$\left[b_a, b_a^c, \ldots, b_a^{c^{p-1}}\right]$ each of which lies in L because $\{a, ac, \ldots, ac^{p-1}\}$ is not a coset of D. Now $\left[b_1b_2, (b_1b_2)^h, \ldots, (b_1b_2)^{h^{p-1}}\right] = \prod\left[b_{f(0)}, b_{f(1)}^h, \ldots, b_{f(p-1)}^{h^{p-1}}\right]$ where f runs through the set F of functions from $\{0, \ldots, p-1\}$ to $\{1, 2\}$. Define a mapping τ on F by

$$f\tau(i) = f(i+1) \text{ for all } i \ \big(\text{taking } (p-1) + 1 = 0\big) .$$

The orbits of τ in F are all of length p except for two of length 1 corresponding to the two constant functions. The desired result comes by proving $b^* = \prod_{j=0}^{p-1}\left[b_{f\tau^j(0)}, b_{f\tau^j(1)}^h, \ldots, b_{f\tau^j(t-1)}^{h^{p-1}}\right]$ belongs to R for all f in F. Now by the definition of τ, $b^* = \prod_{j=0}^{p-1}\left[b_{f(j)}, b_{f(j+1)}^h, \ldots, b_{f(j-1)}^{h^{p-1}}\right]$. It follows from the observation in the last paragraph that $b^* = \prod_{j=0}^{p-1}\left[b_{f(j)}^{h^j}, \ldots, b_{f(j-1)}^{h^{j-1}}\right]^r$ for some r in R. Use of relations of the type (*) yields $b^* = r$ and the proof is complete.

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Institute of Advanced Studies,
Australian National University,
Canberra, ACT.