

INTERSECTIONS OF MULTICURVES FROM DYNNIKOV COORDINATES

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Abstract

We present an algorithm for calculating the geometric intersection number of two multicurves on the n -punctured disk, taking as input their Dynnikov coordinates. The algorithm has complexity $O(m^2n^4)$, where m is the sum of the absolute values of the Dynnikov coordinates of the two multicurves. The main ingredient is an algorithm due to Cumplido for relaxing a multicurve.

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1. Introduction

Determining the geometric intersection number of two simple closed curves, or of two multicurves (also known as *integral laminations*), on a surface S is a fundamental problem in computational topology. Algorithms such as those of Bell and Webb [2] and Schaefer *et al.* [9] take as input the *normal coordinates* of the multicurves: vectors of minimal intersection numbers with the edges of an ideal triangulation of S . They compute the geometric intersection number of two multicurves with polynomial complexity in the Euler characteristic of S and in $\log M$, where M is the sum of the normal coordinates.

In this paper we restrict to the case where $S = D_n$ is an n -punctured disk. In this setting, multicurves are beautifully described by their *Dynnikov coordinates* [6]: a collection of $2n - 4$ linear combinations of intersection numbers with the $3n - 5$ edges of a near-triangulation, which provide a bijection between the set of multicurves on D_n and \mathbb{Z}^{2n-4} . We describe an algorithm for calculating the geometric intersection number of two multicurves on D_n whose complexity is polynomial in n and in m , the sum of the absolute values of the coordinates. The advantages of this algorithm are that it works directly with Dynnikov coordinates and that it is straightforward to express and to code.

The main ingredient is an algorithm due to Cumplido [4] which *relaxes* a multicurve \mathcal{L} : that is, it finds a mapping class on D_n (expressed as a positive

raid) which sends \mathcal{L} to a multicurve each of whose components only intersects the horizontal diameter of the disk twice. Since the geometric intersection number is invariant under the action of the mapping class group, it only remains to provide an algorithm to calculate the geometric intersection number of an arbitrary multicurve with a relaxed one.

Section 2 is a brief introduction to multicurves, Dynnikov coordinates and the *update rules* which describe the action of the braid group B_n on Dynnikov coordinates. In Section 3, we derive a formula for calculating the geometric intersection number with a relaxed multicurve (this is a corrected version of a formula described in [10, Theorem 11]). Cumplido’s algorithm is stated in Section 4 and in Section 5 we state our algorithm and analyse its complexity.

We work throughout with *extended* Dynnikov coordinates in \mathbb{Z}^{2n} , obtained by adjoining four redundant coordinates to the standard Dynnikov ones, which brings computational and notational advantages. For the sake of brevity, we refer to these extended coordinates simply as Dynnikov coordinates, and to the usual Dynnikov coordinates as *reduced*. Remark 2.2 provides formulae for translating between the two types of coordinates.

2. Multicurves and Dynnikov coordinates

2.1. Multicurves on the punctured disk. Let $n \geq 3$ and let D_n be a standard model of the n -punctured disk in the plane, with the punctures arranged along the horizontal diameter (henceforth referred to simply as *the diameter*). A simple closed curve in D_n is *inessential* if it bounds an unpunctured disk, a once-punctured disk, or an n -punctured disk, and is *essential* otherwise.

A *multicurve* \mathcal{L} in D_n is a finite union of pairwise disjoint unoriented essential simple closed curves in D_n up to isotopy (that is, \mathcal{L} is the isotopy class of such a union of simple closed curves). We write \mathcal{L}_n for the set of multicurves on D_n (including the empty multicurve).

Given two multicurves $\mathcal{L}^{(1)}, \mathcal{L}^{(2)} \in \mathcal{L}_n$ we write $\iota(\mathcal{L}^{(1)}, \mathcal{L}^{(2)})$ for the *geometric intersection number* of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$, namely

$$\iota(\mathcal{L}^{(1)}, \mathcal{L}^{(2)}) = \min\{\#L^{(1)} \cap L^{(2)} : L^{(1)} \in \mathcal{L}^{(1)} \text{ and } L^{(2)} \in \mathcal{L}^{(2)}\}.$$

The aim of this paper is to describe an algorithm for calculating $\iota(\mathcal{L}^{(1)}, \mathcal{L}^{(2)})$ from the *Dynnikov coordinates* of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$.

We will regard n as being fixed throughout and suppress the dependence of some objects upon it.

2.2. The Dynnikov coordinate system. The Dynnikov coordinate system [6] provides, for each $n \geq 3$, a bijection $\rho_r: \mathcal{L}_n \rightarrow \mathbb{Z}^{2n-4}$, which we now define (see also Remark 2.2 below).

Construct *Dynnikov arcs* α_i ($-1 \leq i \leq 2n - 2$) and β_i ($0 \leq i \leq n$) in D_n as depicted in Figure 1. (The unconventional indexing starting with $i = -1$ is to maintain consistency with reduced Dynnikov coordinates, where the arcs $\alpha_{-1}, \alpha_0, \alpha_{2n-3}, \alpha_{2n-2}, \beta_0$ and β_n

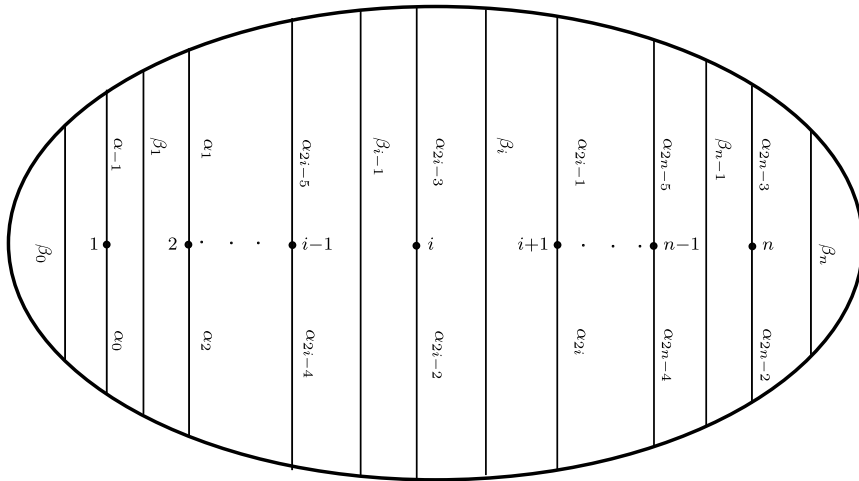


FIGURE 1. Arcs α_i and β_i .

are not used.) Given $\mathcal{L} \in \mathcal{L}_n$, let L be a representative of \mathcal{L} which intersects each of these arcs minimally (such an L is called a *minimal representative* of \mathcal{L}). Write α_i (respectively β_i) for the number of intersections of L with the arc α_i (respectively the arc β_i). This overload of notation will not give rise to any ambiguity, since it will always be stated explicitly when the symbols α_i and β_i refer to arcs rather than to integers. We write $(\alpha; \beta) = (\alpha_{-1}, \dots, \alpha_{2n-2}; \beta_0, \dots, \beta_n)$ for the collection of intersection numbers associated to \mathcal{L} .

The (extended) Dynnikov coordinate function $\rho: \mathcal{L}_n \rightarrow \mathbb{Z}^{2n}$ is defined by

$$\rho(\mathcal{L}) = (a; b) = (a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1}),$$

where

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2} \quad \text{and} \quad b_i = \frac{\beta_i - \beta_{i+1}}{2} \tag{2.1}$$

for $0 \leq i \leq n - 1$.

Given $1 \leq i \leq n$, let Δ_i denote the subset of D_n bounded by the arcs β_{i-1} and β_i (which contains puncture i). Let L be a minimal representative of \mathcal{L} and consider the connected components of $L \cap \Delta_i$. By minimality, each such component can be classed as one of the following four types:

- a *right loop* component, which has both endpoints on the arc β_{i-1} and intersects both of the arcs α_{2i-3} and α_{2i-2} ;
- a *left loop* component, which has both endpoints on the arc β_i and intersects both of the arcs α_{2i-3} and α_{2i-2} ;
- an *above* component, which has one endpoint on each of the arcs β_{i-1} and β_i , and intersects the arc α_{2i-3} but not the arc α_{2i-2} ; or
- a *below* component, which has one endpoint on each of the arcs β_{i-1} and β_i , and intersects the arc α_{2i-2} but not the arc α_{2i-3} .

REMARK 2.1. Clearly there cannot be both left loop and right loop components. It follows immediately from (2.1) that there are $|b_{i-1}|$ loop components, which are left loops if $b_{i-1} < 0$ and right loops if $b_{i-1} > 0$; and hence that there are $\alpha_{2i-3} - |b_{i-1}|$ above components and $\alpha_{2i-2} - |b_{i-1}|$ below components.

The intersection numbers $(\alpha; \beta)$ (and hence the multicurve \mathcal{L}) can be recovered from the Dynnikov coordinates $(a; b) \in \mathbb{Z}^{2n}$ using the formulae

$$\beta_i = -2 \sum_{k=0}^{i-1} b_k \quad \text{and} \tag{2.2}$$

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{1}{2} \beta_{\lceil i/2 \rceil} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{1}{2} \beta_{1+\lceil i/2 \rceil} & \text{if } b_{\lceil i/2 \rceil} \leq 0, \end{cases} \tag{2.3}$$

where $\lceil x \rceil$ denotes the smallest integer which is not less than x . Equation (2.2) is immediate from (2.1) and the observation that $\beta_0 = 0$; while (2.3) follows from (2.1) and the equation $\alpha_{2i} + \alpha_{2i-1} = \max(\beta_i, \beta_{i+1})$.

REMARK 2.2. We have $a_0 = a_{n-1} = 0$ (since $\alpha_{-1} = \alpha_0$ and $\alpha_{2n-3} = \alpha_{2n-2}$) and $\sum_{i=0}^{n-1} b_i = 0$ (since $\beta_0 = \beta_n = 0$). In fact there is one further relation

$$b_0 = - \max_{1 \leq k \leq n-2} \left(|a_k| + b_k^+ + \sum_{j=1}^{k-1} b_j \right) \tag{2.4}$$

(where $x^+ := \max(x, 0)$), which arises from the fact that no component of a multicurve can enclose all n punctures (see for example [7, Lemma 1]). It follows that \mathcal{L} can be described by its *reduced* Dynnikov coordinates $(a_1, \dots, a_{n-2}, b_1, \dots, b_{n-2}) \in \mathbb{Z}^{2n-4}$. We can recover the (extended) coordinates by setting $a_0 = a_{n-1} = 0$, defining b_0 using (2.4), and finally setting $b_{n-1} = -\sum_{j=0}^{n-2} b_j$. The reduced Dynnikov coordinate system gives a bijection $\rho_r: \mathcal{L}_n \rightarrow \mathbb{Z}^{2n-4}$.

REMARK 2.3. If $\mathcal{L}^{(1)}, \dots, \mathcal{L}^{(N)} \in \mathcal{L}_n$ have $\iota(\mathcal{L}^{(k)}, \mathcal{L}^{(\ell)}) = 0$ for all k and ℓ , then there are pairwise mutually disjoint representatives $L^{(1)}, \dots, L^{(N)}$ of the multicurves. We write $\mathcal{L} = \bigsqcup_{k=1}^N \mathcal{L}^{(k)}$ for the multicurve represented by the disjoint union $\bigsqcup_{k=1}^N L^{(k)}$, and observe that $\rho(\mathcal{L}) = \sum_{k=1}^N \rho(\mathcal{L}^{(k)})$.

The following notation will be useful when we discuss the complexity of algorithms involving Dynnikov coordinates.

NOTATION 2.4 ($|\mathcal{L}|$). Let $\mathcal{L} \in \mathcal{L}_n$ with $\rho(\mathcal{L}) = (a; b)$. We write $|\mathcal{L}| = \sum_{i=0}^{n-1} (|a_i| + |b_i|)$.

2.3. The action of the braid group. The mapping class group $\text{MCG}(D_n)$ of D_n is isomorphic to the n -braid group B_n modulo its centre [1], so that elements of $\text{MCG}(D_n)$ can be represented in terms of the Artin braid generators σ_i ($1 \leq i \leq n-1$). In this paper we adopt the convention of Birman’s book [3], that σ_i exchanges punctures i and $i+1$ in the counter-clockwise direction.

The action of $MCG(D_n)$ on \mathcal{L}_n can be calculated using *update rules* (see for example [5–8, 10]), which describe how Dynnikov coordinates transform under the action of the Artin generators and their inverses. In this paper we only need the transformation under the positive generators σ_i , which is given by Theorem 2.5 below. In the statement of the theorem, we again use the notation x^+ to denote $\max(x, 0)$.

THEOREM 2.5 (Update rules for positive generators). *Let $\mathcal{L} \in \mathcal{L}_n$ have Dynnikov coordinates $(a; b)$, and let $1 \leq i \leq n - 1$. Denote by $(a'; b')$ the Dynnikov coordinates of the multicurve $\sigma_i(\mathcal{L})$. Then $a'_j = a_j$ and $b'_j = b_j$ for all $j \notin \{i - 1, i\}$, and*

$$\begin{aligned} a'_{i-1} &= \max(a_{i-1} + b_{i-1}^+, a_i + b_{i-1}), \\ a'_i &= b_i - \max(-a_{i-1}, b_i^+ - a_i), \\ b'_{i-1} &= a_i + b_{i-1} + b_i - \max(a_{i-1} + b_{i-1}^+ + b_i^+, a_i + b_{i-1}), \\ b'_i &= \max(a_{i-1} + b_{i-1}^+ + b_i^+, a_i + b_{i-1}) - a_i. \end{aligned} \tag{2.5}$$

3. Geometric intersection number with an elementary multicurve

DEFINITION 3.1 (Elementary multicurve $\mathcal{L}_{i,j}$). Let $1 \leq i < j \leq n$, with $(i, j) \neq (1, n)$. The *elementary multicurve $\mathcal{L}_{i,j} \in \mathcal{L}_n$ about punctures i through j* is the multicurve with Dynnikov coordinates $(a; b) \in \mathbb{Z}^{2n}$ which are all zero except for $b_{i-1} = -1$ and $b_{j-1} = 1$. (This is equivalent to saying that $\mathcal{L}_{i,j}$ is represented by a simple closed curve which bounds a disk containing punctures i through j , and intersects the diameter of D_n exactly twice.)

In this section we obtain a formula for $\iota(\mathcal{L}, \mathcal{L}_{i,j})$, given a multicurve $\mathcal{L} \in \mathcal{L}_n$. We start by introducing some notation.

NOTATION 3.2 ($A_i, B_i, A_{\ell,m}$ and $B_{\ell,m}$). Let $\mathcal{L} \in \mathcal{L}_n$ be a multicurve with Dynnikov coordinates $(a; b)$ and intersection numbers $(\alpha; \beta)$ with the Dynnikov arcs. For each i with $1 \leq i \leq n$, write

$$A_i = \alpha_{2i-3} - |b_{i-1}| \quad \text{and} \quad B_i = \alpha_{2i-2} - |b_{i-1}|.$$

For each ℓ and m with $1 \leq \ell \leq m \leq n$, write

$$A_{\ell,m} = \min_{\ell \leq k \leq m} A_k \quad \text{and} \quad B_{\ell,m} = \min_{\ell \leq k \leq m} B_k. \tag{3.1}$$

REMARK 3.3. By Remark 2.1, A_i and B_i are, respectively, the number of above and below components in Δ_i .

Given $1 \leq \ell \leq m \leq n$, let $\Delta_{\ell,m} = \bigcup_{i=\ell}^m \Delta_i$ be the subset of D_n bounded by $\beta_{\ell-1}$ and β_m . If L is a minimal representative of a multicurve \mathcal{L} , then a component of $L \cap \Delta_{\ell,m}$ is called a *large over* (respectively *large under*) component if it lies entirely above (respectively below) the diameter of D_n . Since large over components are the highest components in each of the Δ_i , it follows that $A_{\ell,m}$ is the number of large over components of $L \cap \Delta_{\ell,m}$ and, analogously, $B_{\ell,m}$ is the number of large under components.

LEMMA 3.4 (Intersections with an elementary multicurve). *Let $1 \leq i < j \leq n$ with $(i, j) \neq (1, n)$ and let $\mathcal{L} \in \mathcal{L}_n$ be a multicurve. Write*

$$R = \min(A_{i,j-1} - A_{i,j}, B_{i,j-1} - B_{i,j}, b_{j-1}^+), \text{ and}$$

$$L = \min(A_{i+1,j} - A_{i,j}, B_{i+1,j} - B_{i,j}, (-b_{i-1})^+),$$

where $A_{i,j}$ and $B_{i,j}$ are defined by (3.1). Then

$$i(\mathcal{L}, \mathcal{L}_{i,j}) = \beta_{i-1} + \beta_j - 2(R + L + A_{i,j} + B_{i,j}).$$

PROOF. Let $C_{i,j}$ be a minimal representative of $\mathcal{L}_{i,j}$, and let L be a representative of \mathcal{L} which is minimal with respect both to the Dynnikov arcs and to $C_{i,j}$. Every component of $L \cap \Delta_{i,j}$ is therefore either disjoint from $C_{i,j}$ or intersects it exactly twice.

The components of $L \cap \Delta_{i,j}$ which are disjoint from $C_{i,j}$ are precisely:

- components which are contained in the interior of $\Delta_{i,j}$;
- large over and large under components, which have one endpoint on β_{i-1} and one on β_j ;
- large right loop components, which have both endpoints on the arc β_{i-1} and intersect the diameter of D_n only between β_j and puncture j , and
- large left loop components, which have both endpoints on the arc β_j and intersect the diameter of D_n only between β_{i-1} and puncture i .

The total number of large over and under components is $A_{i,j} + B_{i,j}$, by Remark 3.3. The proof can therefore be completed by showing that the number of large right (respectively left) loop components is R (respectively L).

Since $A_{i,j-1} - A_{i,j}$ (respectively $B_{i,j-1} - B_{i,j}$) is the number of large over (respectively under) components of $L \cap \Delta_{i,j-1}$ which are not contained in large over (respectively under) components of $L \cap \Delta_{i,j}$, and b_{j-1}^+ is the number of right loop components of Δ_j (see Remark 2.1), the number of large right loop components is the minimum of these three numbers, namely R (see Figure 2). The argument that there are L large left loop components is analogous. □

4. Cumplido’s relaxation algorithm

DEFINITION 4.1 (Relaxed multicurve). A multicurve $\mathcal{L} \in \mathcal{L}_n$, with Dynnikov coordinates $(a; b) \in \mathbb{Z}^{2n}$, is said to be *relaxed* if $a_i = 0$ for all i .

We observe that a multicurve is relaxed if and only if it is a disjoint union $\mathcal{L} = \bigsqcup_{k=1}^N \mathcal{L}_{i_k, j_k}$ of elementary multicurves. On the one hand, it is immediate that such a multicurve has $a_i = 0$ for all i . Conversely, the following algorithm—essentially a bracket matching algorithm—parses a relaxed multicurve as such a disjoint union.

ALGORITHM 4.2 (Parsing a relaxed multicurve). Let $\mathcal{L} \in \mathcal{L}_n$ be a relaxed multicurve with Dynnikov coordinates $(a; b) \in \mathbb{Z}^{2n}$. The following algorithm returns a list $C = ((i_1, j_1), \dots, (i_N, j_N))$ with the property that $\mathcal{L} = \bigsqcup_{k=1}^N \mathcal{L}_{i_k, j_k}$.

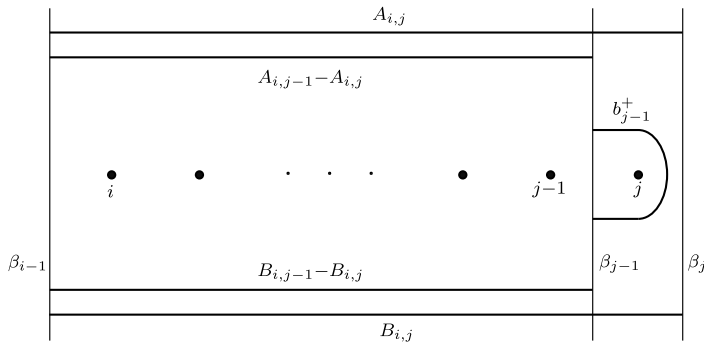


FIGURE 2. The number of large right loop components in $\Delta_{i,j}$.

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1:  $C \leftarrow$  empty list
2:  $s \leftarrow$  empty stack
3: for  $i$  from 1 to  $n$  do
4:   if  $b_{i-1} < 0$  then
5:     push  $-b_{i-1}$  copies of  $i$  onto  $s$ 
6:   else
7:     pop  $b_{i-1}$  top entries  $\ell_1, \dots, \ell_{b_{i-1}}$  from  $s$ 
8:     add  $(\ell_1, i), \dots, (\ell_{b_{i-1}}, i)$  to  $C$ 
9: return  $C$ 

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Note that since $\sum_{k=0}^{i-1} b_k = -\beta_i/2 \leq 0$ for each i by (2.2), the stack s is never empty at line 7. Moreover, if (i, j) and (i', j') are in C with $i < i'$, then either $j < j'$ or $j \geq j'$, so that $t(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'}) = 0$. The multicurves \mathcal{L}_{i_k, j_k} can therefore be realised disjointly by Remark 2.3. That $\mathcal{L} = \bigsqcup_{k=1}^N \mathcal{L}_{i_k, j_k}$ has Dynnikov coordinates $(a; b)$ then follows from Remark 2.1, since if L is a minimal representative of \mathcal{L} , then $L \cap \Delta_i$ has $-b_{i-1}$ left loop components if $b_{i-1} \leq 0$, and b_{i-1} right loop components if $b_{i-1} \geq 0$.

Cumplido [4] gives an algorithm which takes as input the Dynnikov coordinates $(a; b) \in \mathbb{Z}^{2n}$ of $\mathcal{L} \in \mathcal{L}_n$, and produces as output a braid $\beta \in B_n^+$ (the positive braid monoid) and a relaxed multicurve $\mathcal{L}' = \beta(\mathcal{L})$ (in fact, β is the unique prefix-minimal positive braid which relaxes \mathcal{L} in this way).

ALGORITHM 4.3 (Cumplido’s relaxation algorithm). Let $(a; b) \in \mathbb{Z}^{2n}$ be the Dynnikov coordinates of $\mathcal{L} \in \mathcal{L}_n$. The following algorithm returns $\beta \in B_n^+$ and the Dynnikov coordinates of a relaxed multicurve \mathcal{L}' with $\mathcal{L}' = \beta(\mathcal{L})$.

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1:  $\beta \leftarrow$  id
2:  $j \leftarrow 1$ 
3: while  $j < n$  do
4:   if  $a_j > a_{j-1}$  then
5:      $(a; b) \leftarrow \sigma_j(a; b)$  ▷ Use (2.5)
6:      $\beta \leftarrow \beta \cdot \sigma_j$ 
7:      $j \leftarrow 1$ 

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8:   |   else
9:   |   |   j ← j + 1
10: return (β, (a; b))
    
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The following result is contained in [4, Corollaries 44 and 46].

THEOREM 4.4 (Cumplido). *Let $\mathcal{L} \in \mathcal{L}_n$ and write $m = |\mathcal{L}|$. Then Algorithm 4.3 requires $O(n^2m)$ arithmetic operations; and the word length of the relaxing braid β which it returns is $O(n^2m)$.*

5. The geometric intersection number algorithm

It is now straightforward to state the geometric intersection number algorithm, which relies on the fact that if $\mathcal{L}^{(1)}, \mathcal{L}^{(2)} \in \mathcal{L}_n$ and $\beta \in B_n$, then $\iota(\beta(\mathcal{L}^{(1)}), \beta(\mathcal{L}^{(2)})) = \iota(\mathcal{L}^{(1)}, \mathcal{L}^{(2)})$.

ALGORITHM 5.1 (Geometric intersection number algorithm).

Let $(a^{(1)}; b^{(1)})$ and $(a^{(2)}; b^{(2)})$ be the Dynnikov coordinates of $\mathcal{L}^{(1)}, \mathcal{L}^{(2)} \in \mathcal{L}_n$. The following algorithm returns $\iota(\mathcal{L}^{(1)}, \mathcal{L}^{(2)})$.

- 1: If $(a^{(1)}; b^{(1)}), (a^{(2)}; b^{(2)})$ are reduced coordinates, extend them. ▶ Use Remark 2.2
- 2: Find $\beta \in B_n^+$ such that $\mathcal{L}^{(1)'} = \beta(\mathcal{L}^{(1)})$ is relaxed. ▶ Use Algorithm 4.3
- 3: Parse $\mathcal{L}^{(1)'} = \bigsqcup_{k=1}^N \mathcal{L}_{i_k, j_k}$. ▶ Use Algorithm 4.2
- 4: Calculate the Dynnikov coordinates of $\mathcal{L}^{(2)'} = \beta(\mathcal{L}^{(2)})$. ▶ Use (2.5)
- 5: Determine the intersection numbers $(\alpha; \beta)$ of $\mathcal{L}^{(2)'}$. ▶ Use (2.2) and (2.3)
- 6: **Return** $\sum_{i=1}^k \iota(\mathcal{L}^{(2)'}, \mathcal{L}_{i_k, j_k})$. ▶ Use Lemma 3.4

REMARK 5.2. The same algorithm may be used to compute the measure of a multicurve \mathcal{L} with respect to a measured foliation (\mathcal{F}, μ) on D_n , described by its Dynnikov coordinates $\rho(\mathcal{F}, \mu) \in \mathbb{R}^{2n}$ (or its reduced Dynnikov coordinates $\rho_r(\mathcal{F}, \mu) \in \mathbb{R}^{2n-4}$).

THEOREM 5.3 (Complexity of the geometric intersection number algorithm). *Let $\mathcal{L}^{(1)}, \mathcal{L}^{(2)} \in \mathcal{L}_n$, and write $m = |\mathcal{L}^{(1)}| + |\mathcal{L}^{(2)}|$. Then Algorithm 5.1 has complexity $O(m^2n^4)$.*

PROOF. We first observe that the number of arithmetic operations (addition, subtraction, comparing, taking the maximum, or taking the minimum of two integers) required for each of the six steps of the algorithm is $O(n^2m)$.

- (1) Extending reduced coordinates using Remark 2.2 requires $O(n)$ arithmetic operations.
- (2) Algorithm 4.3 requires $O(n^2m)$ arithmetic operations by Theorem 4.4. We observe that $\mathcal{L}^{(1)}$, and hence $\mathcal{L}^{(1)'}$, has $O(m)$ components (each component must form left/right loops around at least two punctures and hence contributes at least two to $|\mathcal{L}|$ by Remark 2.1).

- (3) Parsing $\mathcal{L}^{(1)'}$ into its $O(m)$ components using Algorithm 4.2 requires $O(m + n)$ arithmetic operations.
- (4) Each application of a braid generator σ_j to $\mathcal{L}^{(2)}$ using (2.5) requires $O(1)$ arithmetic operations. Since β has $O(n^2m)$ generators by Theorem 4.4, calculating the Dynnikov coordinates of $\mathcal{L}^{(2)'}$ requires $O(n^2m)$ arithmetic operations.
- (5) Calculating the intersection numbers $(\alpha; \beta)$ using (2.2) and (2.3) requires $O(n)$ arithmetic operations.
- (6) For each k , determining the geometric intersection number $\iota(\mathcal{L}_{i_k, j_k}, \mathcal{L}^{(2)'})$ using Lemma 3.4 requires $O(n)$ operations. Calculating the sum of $O(m)$ such numbers therefore requires $O(mn)$ arithmetic operations.

By (2.5), there is a constant factor K such that $|\sigma_j(\mathcal{L})| \leq K|\mathcal{L}|$ for every multicurve \mathcal{L} and every j . Therefore, the integers involved in each arithmetic operation are $O(K^{mn^2}m)$. Since each arithmetic operation on such integers has complexity $O(mn^2)$, the complexity of Algorithm 5.1 is $O(m^2n^4)$ as required. \square

The algorithm has been implemented as part of the second author's program Dynn, available at <http://pcwww.liv.ac.uk/math/toobyhall/software>. Experimentally, it appears to scale much better than m^2n^4 for small values of n (up to 100) and m (up to 10^6).

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