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ASYMPTOTIC PROPERTIES OF ROOTED 3-CONNECTED MAPS ON SURFACES

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Abstract

In this paper we obtain asymptotics for the number of rooted 3-connected maps on an arbitrary surface and use them to prove that almost all rooted 3-connected maps on any fixed surface have large edge-width and large face-width. It then follows from the result of Roberston and Vitray [10] that almost all rooted 3-connected maps on any fixed surface are minimum genus embeddings and their underlying graphs are uniquely embeddable on the surface.

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1. Introduction

We begin with some definitions:

- A *map* is a connected graph G embedded in a surface S (a closed 2-manifold) such that all components of $S - G$ are simply connected regions, which are called *faces*. G is called the underlying graph of M, and is denoted by $G(M)$. Loops and multiple edges are allowed in G.
- A map is *rooted* if an edge is distinguished together with a vertex on the edge and a side of the edge. All maps shall be rooted.
- We use Tutte's definition [11] of connectivity: a graph (or the corresponding map) is k*-connected* (abbreviated k-c) if the girth is at least k and it requires removing at least k vertices to separate the graph.

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- By a *cycle* in a map, we mean a simple closed curve consisting of edges of the map. A cycle is called *separating* if deleting it separates the underlying graph and is called *facial* if it bounds a face of the map.
- A cycle is called *contractible* if it is homotopic to a point; otherwise it is called *non-contractible* and denoted by *nc-cycle*.
- The *edge-width* of a map M, denoted by $\mathrm{ew}(M)$, is the length of a shortest nc-cycle of M. The *face-width* (also called *representativity* in [10]) of a map M, denoted by $\mathrm{fw}(M)$, is the minimum of $|G(M) \cap C|$ taken over all non-contractible simple closed curves C that lie in the surface and contain no vertices of $G(M)$. It is easily seen that $\mathrm{ew}(M) \geq \mathrm{fw}(M)$ for any map M.

We will prove the following theorem and its corollaries.

THEOREM 1. *Let t_g and p_g be the constants defined in Theorem 1 of [1]. The number of 3-connected maps on a surface of Euler characteristic $\chi = 2 - 2g$ is given asymptotically by*

$$t_g(9n)^{5\chi/4}4^n \quad \text{on orientable surfaces, and}$$
$$p_g(9n)^{5\chi/4}4^n \quad \text{on non-orientable surfaces.}$$

In the following corollaries, 'almost all' means that the fraction of maps having the property approaches 1 as $n \to \infty$.

COROLLARY 1. *Almost all n-edged 3-connected maps on a given surface have face-width greater than $\delta \log n$ for some constant $\delta > 0$.*

COROLLARY 2. *Almost all n-edged 3-connected maps on a given surface are not hamiltonian.*

While interesting in itself, the theorem is also important because it shows that the number of 3-c maps on a surface 'grows normally.' This concept is defined as follows:

- Let $\mathscr{F}$ be some family of maps and let $\mathscr{F}_n(S)$ be the set of n-edged maps in $\mathscr{F}$ that lie on a surface S. We say that $\mathscr{F}$ *grows normally* if

$$|\mathscr{F}_n(S)| \sim A(S, \mathscr{F})n^{-5\chi/4}\rho(\mathscr{F})^n$$

for some $A(S, \mathscr{F})$ and $\rho(\mathscr{F})$, where χ is the Euler characteristic of S and the limit is taken through those n for which $\mathscr{F}_n(S) \neq \emptyset$.

A variety of families of maps exhibiting normal growth are listed in [7]. Properties of such families are discussed in [3, 4] and imply the corollaries.

The motivation for the first corollary may not immediately be apparent. Robertson and Vitray [10] have studied graph embeddings with large face-width. They have shown that they share many properties with planar embeddings. For example, if M is a map of genus g which has face-width exceeding $2g+2$, then it is the minimum genus embedding of $G(M)$, and if $G(M)$ is also 2-connected, then any other embedding of $G(M)$ of genus g is obtained from M by a sequence of '2-switchings' (defined by Whitney [13] who proved the planar case). It follows from this that, if a 3-connected map of genus g has face-width exceeding $2g+2$, then $G(M)$ has a unique embedding of genus g.

After establishing the connection between certain types of quadrangulations and 3-connected maps, we will focus on quadrangulations for two sections. In the last section, we use these results to prove Theorem 1 and its corollaries. To avoid too many technical details, we shall only prove our results for maps on orientable surfaces. Similar arguments work for maps on non-orientable surfaces.

2. Quadrangulations and 3-connected maps

- A *bipartite quadrangulation* is a map whose underlying graph is bipartite and whose faces are all quadrangles. All quadrangulations shall be bipartite.
- A quadrangulation is called *near-simple* if it has no contractible 2-cycles and no contractible non-facial 4-cycles, and is called *simple* if it has no 2-cycles and all 4-cycles are facial.

The following lemma connects these concepts with 3-connected maps.

LEMMA 1. *There is a bijection ϕ between n-edged maps and n-faced quadrangulations, such that* $\mathrm{fw}(M) = \mathrm{ew}(\phi(M))/2$. *Furthermore, $\phi(M)$ simple implies M 3-connected, which implies that $\phi(M)$ is near-simple.*

PROOF. The proof of the first statement is a straightforward extension of the bijection on the sphere given by Brown [6]: For any map M, place a vertex in each face and join it to the vertices on the boundary of the face through every corner and remove all the original edges of M. This gives a bipartite quadrangulation Q, whose root corner can be chosen the same as the root corner of M. This is clearly a bijection, and any nc-cycle of length $2k$ in Q intersects $G(M)$ in exactly k vertices.

Suppose now that $\phi(M)$ is not near-simple, that is, $\phi(M)$ has either a contractible 2-cycle or a contractible non-facial 4-cycle. Then in the former case, one vertex in the 2-cycle is a cut vertex of M, and in the latter case, two non-adjacent vertices in the 4-cycle form a cut-pair of M. So M 3-connected implies $\phi(M)$ near-simple.

Now suppose that M is not 3-connected; then the edges of M can be partitioned into two classes, say blue and red, such that only two vertices, say v_1 and v_2, are incident to both blue and red edges. Pick a face f_1 incident to both blue and red edges at v_1. Tracing around f_1 from the blue edge, we must eventually reach a red edge. This must occur at v_1 or v_2. If it is at v_1, then v_1 and f_1 have multiple incidence, and this gives a 2-cycle in $\phi(M)$. If it is at v_2, we can then pick another such face f_2 and repeat the process, thereby finding two faces f_1 and f_2 both incident to v_1 and v_2. They form a non-facial 4-cycle in $\phi(M)$. So $\phi(M)$ being simple implies that M is 3-connected.

Our approach is similar to that used in [5]. We obtain asymptotics for the number of near-simple quadrangulations and then show that almost all near-simple quadrangulations are simple. It then follows from Lemma 1 that the numbers of near-simple quadrangulations, simple quadrangulations and 3-connected maps are all asymptotically the same.

3. Enumerating near-simple quadrangulations

On the orientable surface of genus $g = 1 - \chi/2$, with x marking the number of faces, define the following generating functions:

$Q_g(x)$: quadrangulations,
$\hat{Q}_g(x)$: quadrangulations without contractible 2-cycles,
$Q_g^*(x)$: near-simple quadrangulations.

Let $R_1 = \sqrt{1-12x}$, $R_2 = \sqrt{1-27x/4}$, and $R_3 = \sqrt{1-4x}$.

It follows from Lemma 1 above, [2] and [1, Theorem 1 and Lemma 3] that $Q_g(x)$ is algebraic and has a Laurent series expansion in R_1:

$$(3.1) \qquad Q_g(x) = \begin{cases} \frac{1}{3} - \frac{4}{3}R_1^2 + (8/3)R_1^3(1+O(R_1)), & \text{if } g = 0, \\ A_g R_1^{3-5g}(1+O(R_1)), & \text{if } g > 0, \end{cases}$$

where the A_g's are constants. We will prove similar results for $\hat{Q}_g$ and Q_g^*:

THEOREM 2. *$\hat{Q}_g(x)$ and $Q_g^*(x)$ are algebraic and have the following Laurent series expansions:*

$$(3.2) \qquad \hat{Q}_g(x) = \begin{cases} \frac{1}{3} - \frac{4}{9}R_2^2 + \frac{8}{27\sqrt{3}}R_2^3(1+O(R_2)), & \text{if } g = 0, \\ (A_g/3)(R_2/\sqrt{3})^{3-5g}(1+O(R_2)), & \text{if } g > 0, \end{cases}$$

$$(3.3) \qquad Q_g^*(x) = \begin{cases} \frac{407}{4320} - \frac{14539}{64800}R_3^2 + \frac{8}{729}R_3^3(1+O(R_3)), & \text{if } g = 0, \\ (A_g/9)(R_3/3)^{3-5g}(1+O(R_3)), & \text{if } g > 0, \end{cases}$$

where A_g is given by (3.1). *Moreover, the only possible finite singularities of $\hat{Q}_g(x)$ are at $x = 4/27$, $-16/27$ and -4 and the only singularity of $Q_g^*(x)$ on its circle of convergence is at $x = 1/4$.*

PROOF. Lemma 1 allows us to convert statements about maps to statements about quadrangulations. Tutte's formula for Q_0 [12, (5.2)], can be written as

$$1 + Q_0(x) = \frac{4(1+2R_1)}{3(1+R_1)^2}. \tag{3.4}$$

Tutte also proved [12, p.257]

$$\hat{Q}_0(x) = t(2-3t), \quad \text{where} \quad x = t(1-t)^2 \quad \text{and} \quad t(0) = 0, \tag{3.5}$$

from which the $g = 0$ case of (3.2) follows. The formula for $Q_0^*(x)$ can be obtained from $Q_N^*(x, x)$ given by [8, (6.24–25)]. Thus, we need only consider the cases $g > 0$.

We require some definitions.

- A contractible cycle in a non-planar map separates the map into a planar piece and a non-planar piece. The planar piece is called the *interior* of the cycle and we also say that the cycle *contains* anything in its interior. Since we usually draw a planar map such that the root face is the unbounded face, we define the interior of a cycle in a planar map to be the piece which does not contain the root face.
- A 2-cycle or 4-cycle is called *maximal* (*minimal*) if it is contractible and its interior is maximal (minimal).

It is important to note that, in any quadrangulation, all maximal 2-cycles have disjoint interiors, and that, in any non-planar quadrangulation without contractible 2-cycles, all maximal 4-cycles have disjoint interiors. (This is simpler than the planar case [8, p. 260].) Therefore, we can close all maximal 2-cycles in quadrangulations to obtain quadrangulations without contractible 2-cycles, and remove the interior of each maximal contractible 4-cycle to obtain near-simple quadrangulations. The process can be reversed and used to construct quadrangulations from near-simple quadrangulations.

To study $\hat{Q}_g$, we use an approach similar to that in [5]. All quadrangulations of genus $g > 0$ can be divided into two classes according to whether or not the root face lies in the interior of some contractible 2-cycle.

For any quadrangulation in the first class, let C be the minimal 2-cycle containing the root face. Cutting along C, filling holes with disks and closing those two digons, we obtain a general quadrangulation of genus g and a planar quadrangulation with a distinguished edge. Taking the latter quadrangulation, cutting along all of its maximal 2-cycles and closing as before gives a quadrangulation without contractible 2-cycles,

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together with a set of planar quadrangulations extracted from within the maximal 2-cycles. Thus the generating function for the first class is $2u\hat{Q}_0'(u)Q_g(x)/(1+Q_0(x))$, where

$$u = x(1 + Q_0(x))^2. \tag{3.6}$$

For any quadrangulation in the second class, closing all maximal contractible 2-cycles gives quadrangulations without contractible 2-cycles. Thus the generating function for this class is $\hat{Q}_g(u)$. Combining the two classes, we have $Q_g(x) = \hat{Q}_g(u) + 2u\hat{Q}_0'(u)Q_g(x)/(1 + Q_0(x))$ and so

$$\hat{Q}_g(u) = \left(1 - \frac{2u\hat{Q}_0'(u)}{1 + Q_0(x)}\right) Q_g(x), \tag{3.7}$$

which is the same as the first line of [5, (4.1)]. Using (3.1), (3.2) with $g = 0$, and (3.7), we obtain (3.2) for $g > 0$.

We now use a similar argument to derive $Q_g^*(x)$ from $\hat{Q}_g(x)$. For any quadrangulation without contractible 2-cycles, let C be the maximal contractible 4-cycle containing the root face. Cutting along C and filling holes with disks, we obtain

(a) a planar quadrangulation which has no 2-cycles and has a distinguished face other than the root face, and
(b) a quadrangulation of genus g which, after the removal of the interiors of all maximal 4-cycles, gives a near-simple quadrangulation.

Note that

$$v = v(x) = \frac{\hat{Q}_0(x) - 2x}{x} \tag{3.8}$$

enumerates by the number of interior faces the planar quadrangulations having at least one interior face and having no 2-cycles. It follows from the construction that

$$x^2\hat{Q}_g(x) = \left(x^2 v'(x)\right) \times \left((x/v)Q_g^*(v)\right), \tag{3.8}$$

which gives

$$Q_g^*(v) = \frac{v}{xv'}\hat{Q}_g(x). \tag{3.9}$$

Using (3.2), (3.8) and (3.9), we obtain (3.3) for $g > 0$.

The sources of the singularities of $\hat{Q}_g(u)$ fall into three classes:

(1) singularities of $Q_g(x)$ that are carried over to $\hat{Q}_g(u)$ by (3.7),
(2) singularities that arise when (3.6) is solved for $x(u)$ to use in (3.7), and
(3) singularities of $\hat{Q}_0(u)$.

By [1, Lemma 3], the only possible singularities of $Q_g(x)$ are at $1/12$ and $-1/4$, the latter requiring $R_1(-1/4) = -2$. These lead to $u = 4/27$ and $u = -4$, respectively.

We now turn to the second source of singularities. Following [5], we rewrite (3.4) as

$$27u(1+R_1)^3 + 4(R_1-1)(2R_1+1)^2 = 0$$

and observe that this can be further rewritten as a polynomial equation in x and u. After some algebra, one finds that the leading coefficient in x vanishes at $u = -16/27$, and multiple roots can occur when $R_1 = 0$. This determines all possible singularities for $\hat{Q}_g$ in the second class. By (3.5), the last class corresponds to multiple roots $t(u)$ of $t(1-t)^2 = u$, which leads to $u = 4/27$.

The same sort of argument is used with (3.9) to study the singularities of $Q_g^*(v)$. Since Q_0^* does not appear in that formula, the third source of singularities does not arise. The three possible singularities $x = 4/27$, $-16/27$ and -4 of $\hat{Q}_g$ give only $v(x) = 1/4$, $-3/8$ and $-3/4$ as possible singularities of magnitude less than 1. We now turn to the second source of singularities. From (3.8) and (3.5) we have $v = t(1-2t)/(1-t)^2$, where $t(0) = 0$. A little algebra leads to

$$t = \frac{2v + 1 \pm \sqrt{1-4v}}{v+2},$$

which has singularities at $v = 1/4$ and $v + 2 = 0$.

4. Almost all near-simple quadrangulations are simple

In this section, we prove

THEOREM 3. *Almost all near-simple quadrangulations on any fixed surface are simple; that is, the ratio of the number of n-faced near-simple quadrangulations to n-faced simple quadrangulations approaches* 1 *as $n \to \infty$.*

PROOF. We must prove that almost all n-faced near-simple quadrangulations have no nc-cycles of length two or four. When $g = 0$, there is nothing to prove. For simplicity, we shall only consider non-separating nc-cycles. Separating nc-cycles can be handled more easily as in the proof of [4, Theorem 1]. Let Q be a near-simple quadrangulation of genus $g > 0$.

We first consider the case where Q has no non-contractible 2-cycles but has a non-separating non-contractible 4-cycle C. Cutting along C and filling holes with disks gives a quadrangulation Q' of genus $g-1$ with two distinguished facial 4-cycles C_1 and C_2. Q' may have some contractible non-facial 4-cycles which correspond to the 4-cycles in Q homotopic to C. Since Q has no contractible non-facial 4-cycles, all contractible non-facial 4-cycles in Q' must contain either C_1 or C_2.

To visualise the next step, it may be helpful to imagine that C_1 is drawn in the plane. Suppose that there are some non-facial 4-cycles containing C_1. Cutting along all these non-facial 4-cycles and filling the resulting holes with disks gives a sequence S_1 of quadrangulations. We can regard each of these as having a distinguished face as well as a root face, in order to mark the two distinct faces coming from the cuts along 4-cycles.

We must characterize the possibilities for the sequence S_1. We firstly describe this in terms of the map M corresponding to the quadrangulation Q. The operation of cutting along C in Q corresponds to cutting M along a closed curve through two vertices and two faces, and adding two edges e_1 and e_2 between the vertices sliced by the curve, to produce a map M'. The edges e_1 and e_2 correspond to C_1 and C_2 in Q'. Since Q has no 2-cycles, neither does Q', and hence M' is 2-connected. Thus, by the results of Tutte [11], M' decomposes into 'cleavage units' each of which is either a 'polygon' (a cycle), a 'bond' (two vertices joined by a multiple edge), or is 3-connected. These cleavage units are joined together in a tree T, where two components adjacent in T contain the same 'hinge' of M'. Each hinge is a pair of vertices whose removal disconnects M', and two faces each incident with each of these vertices determine to a 4-cycle of Q'. No two polygons are adjacent and no two bonds are adjacent. Since every contractible non-facial 4-cycle in Q' contains C_1 or C_2, no pair of vertices whose removal disconnects M' can leave e_1 and e_2 in the same planar component. It follows that in the subforest of T induced by the planar cleavage units, the components containing cleavage units containing e_1 and e_2 are paths, and that all bonds and polygons have at most four edges. (This situation is similar to the decomposition of 3-connected graphs on removal of an edge, as encountered in [14]; but the situation there resembles the situation here when e_1 and e_2 share a face, in which case there can be no bonds with four edges.)

We introduce some shorthand terminology. Let $Q3A$ and $Q4A$ denote the quadrangulations which correspond to the maps which are just a 3-cycle and a 4-cycle respectively. Each of these quadrangulations has precisely one other rooting, giving the quadrangulations $Q3B$ and $Q4B$ respectively. These correspond to the maps dual to the 3-cycle and 4-cycle, which contain just a triple edge and just a quadruple edge respectively. We refer to both $Q3A$ and $Q3B$ as $Q3$, and the same for $Q4$.

Consideration of the fact that Q is near-simple shows that the sequence of cleavage units of M' corresponding to S_1 can contain no 'polygon' or 'bond' of order greater than 4, and that in each $Q4$ in S_1, the distinguished face and the root face are non-adjacent.

We can now state the conditions on the sequence of quadrangulations in S_1 which follow from the properties of the decomposition of M' into cleavage units:

(1) Each of the quadrangulations in S_1 either is simple with more than four faces or is one of the four quadrangulations $Q3A$, $Q3B$, $Q4A$ and $Q4B$.

(2) Two consecutive elements of S_1 cannot both be in $\{Q3A, Q4A\}$ nor both in $\{Q3B, Q4B\}$.
(3) Given the root face of either $Q4A$ or $Q4B$, there is only one valid choice for the distinguished face.

The 4-cycles containing C_2 give a similar sequence S_1. Therefore Q' is decomposed into two sequences of the type of S_1 and a near-simple quadrangulation of genus $g-1$ which has a distinguished face. In addition, the original rooting of Q lies in one of these components. To give an upper bound for the number of maps Q', let $F_1(x)$ be the generating function for the sequence of consecutive $Q3$'s and $Q4$'s and let $F_2(x)$ be the generating function for planar simple quadrangulations with more than 4 faces. Then we have

$$F_1(x) = (4x^3 + 2x^4)/(1 - 2x - x^2), \tag{4.1}$$

since the first element of the sequence can be of type A or B, and from then on there is only one way to attach a $Q4$ but two ways to attach a $Q3$.

Since the sequence S_1 can begin with either F_1 or F_2 and end with either F_1 or F_2, we obtain the following upper bound for the counting series for a non-empty S_1:

$$G_1(x) = (F_1 + xF_2' + 2F_1F_2'/x)/(1 - F_1F_2'/x^3). \tag{4.2}$$

Therefore the counting series for Q' is bounded by

$$H_1(x) = 2(x^{-2}(Q_{g-1}^*)'(1 + G_1(x))^2)'. \tag{4.3}$$

(The factor of 2 appears because there are two ways to identify the distinguished faces C_1 and C_2 of Q' to retrieve Q. The outer derivative chooses the original rooting of Q.)

We now consider the case where the quadrangulation Q has some non-contractible 2-cycles. By cutting along 2-cycles analogously to the 4-cycle case, but closing up the boundary of each resulting digon to form an edge, Q can be decomposed into two sequences of planar quadrangulations without 2-cycles, each of which having a distinguished edge as well as a root edge, and a quad Q' of genus $g-1$ as described in case 1 (with the exception that Q' here has two distinguished edges instead of two distinguished faces). Let S_2 denote one of these planar sequences. In each element of S_2, all non-facial 4-cycles must contain either the distinguished edge or the root edge, so each element can be further decomposed into a sequence of the type of S_1 as described in the first case above. Therefore the counting series for S_2 is bounded by

$$1/(1 - 2G_1(x)) \tag{4.4}$$

where the factor of 2 is introduced in order to convert the distinguished face of one of the objects counted by G_1 into a distinguished edge. Thus the counting series for Q'

in this case is bounded by

$$H_2(x) = (x^{-2}(Q^*_{g-1})'(1/(1-2G_1(x)))^2)'. \tag{4.5}$$

Using (4.1) and $F_2(x) = Q^*_0 - x^2 - 2x^3$ we can show (with the aid of Maple) that the denominators of (4.2) and (4.4) are not zero at $x = 1/4$. Therefore by (3.3) the lowest terms in R_3 of (4.3) and (4.5) are R_3^{4-5g}, which is negligible compared with $Q^*_g(x)$.

5. 3-connected maps

The proof of Theorem 1 follows easily from Theorem 2 and the definitions of t_g and p_g in [1].

We now prove Corollary 1. It was shown in [4, Theorem 1] that, if $\mathscr{F}$ is a class of maps with normal growth, then the edge-width of most maps in $\mathscr{F}_n(S)$ is about $\log n$. In fact, the argument used there can also be used to show that the face-width of most such maps is also about $\log n$, thus proving Corollary 1.

Since the class of 3-connected maps on a surface has normal growth, it follows from [3, Corollary 1] that almost all of them contain any given 3-connected planar map M. One particular map M was used in [9] to show that almost all 3-connected triangulations of the plane are not hamiltonian. The same map now suffices to complete the proof of Corollary 2.

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