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TAUBERIAN THEOREMS FOR STRONG AND ABSOLUTE BOREL-TYPE METHODS OF SUMMABILITY(1)

BY

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1. **Introduction.** Suppose throughout that s, a_n $(n=0,1,2,\ldots)$ are arbitrary complex numbers, that $\alpha>0$ and β is real and that N is a non-negative integer such that $\alpha N+\beta\geq 1$. Let

$$s_n=\sum_{\nu=0}^{n} a_\nu \ (n\geq 0), \qquad s_{-1}=0,$$

$$s_{\alpha,\beta}(z)=\sum_{n=N}^{\infty} s_n\frac{z^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}, \qquad a_{\alpha,\beta}(z)=\sum_{n=N}^{\infty} a_n\frac{z^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)},$$

$$S_{\alpha,\beta}(z)=\alpha e^{-z}s_{\alpha,\beta}(z), \qquad A_{\alpha,\beta}(z)=\alpha e^{-z}a_{\alpha,\beta}(z)$$

where $z=x+iy$ is a complex variable and the power z^γ is assumed to have its principal value.

Borel-type methods are defined as follows:

(a) Summability: If $S_{\alpha,\beta}(x)$ exists for all $x\geq 0$ and tends to s as $x\to\infty$, we say that $s_n\to s(B,\alpha,\beta)$ or $\sum_0^\infty a_n=s(B,\alpha,\beta)$;

(b) Strong summability with index $p>0$: If $S_{\alpha,\beta-1}(x)$ exists for all $x\geq 0$ and

$$\int_0^x e^t\,|S_{\alpha,\beta-1}(t)-s|^p\,dt=o(e^x),$$

we say that $s_n\to s[B,\alpha,\beta]_p$;

(c) Absolute summability: If $s_n\to s(B,\alpha,\beta)$ and $S_{\alpha,\beta}(x)\in BV_x[0,\infty)$,(2) we say that $s_n\to s\,|B,\alpha,\beta|$;

(d) Boundedness: If $S_{\alpha,\beta}(x)$ exists and is bounded on $[0,\infty)$, we say that $s_n=0(1)(B,\alpha,\beta)$;

(e) Strong boundedness with index $p>0$: If $S_{\alpha,\beta-1}(x)$ exists for all $x\geq 0$ and

$$\int_0^x e^t\,|S_{\alpha,\beta-1}(t)|^p\,dt=0(e^x),$$

we say that $s_n=0(1)[B,\alpha,\beta]_p$.

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(2) $f(x)\in BV_x[0,\infty)$ means that $f(x)$ is of bounded variation with respect to x on $[0,\infty)$.

The summability method $(B, 1, 1)$ is the Borel exponential method B (see [7]). The (B, α, β) method is due to Borwein (see [2]) and the $[B, \alpha, \beta]_p$ and $|B, \alpha, \beta|$ methods are due to Borwein and Shawyer (see [4], [3] respectively). Strong Borel-type summability $[B, \alpha, \beta]$ (see [3]) is the $[B, \alpha, \beta]_1$ method.

The actual choice of the integer N in the above definitions is clearly immaterial. We shall therefore tacitly assume whenever a finite number of methods, with α fixed and $\beta = \beta_1, \beta_2, \ldots, \beta_k$, are under consideration that N is such that $\alpha N + \beta_r \geq 1$ $(r = 1, 2, \ldots, k)$.

The following known result establishes a natural scale for these summability methods. (Theorem A(i) is [1, (II)]. Theorem A(ii) is [3, Theorem 9] when $p = 1$ and part of [4, Theorem 9*(ii)] when $p \geq 1$. Theorem A(iii) is [8, Lemma].)

THEOREM A. *Let* $\beta > \mu$.

(i) *If* $s_n \to s(B, \alpha, \mu)$, *then* $s_n \to s(B, \alpha, \beta)$.
(ii) *If* $p \geq 1$ *and* $s_n \to s[B, \alpha, \mu]_p$, *then* $s_n \to s[B, \alpha, \beta]_p$.
(iii) *If* $s_n \to s\,|B, \alpha, \mu|$, *then* $s_n \to s\,|B, \alpha, \beta|$.

In [5] we established a number of tauberian theorems for the (B, α, β) method. In this paper we investigate all the corresponding results for the $[B, \alpha, \beta]_p$ method with $p \geq 1$ and either prove them or show, by means of counterexamples, that they are false. We also examine some of the corresponding results for the $|B, \alpha, \beta|$ method.

2. **Preliminary results.** We first state some known results.

LEMMA 1.

(i) *If* $p \geq 1$ *and* $s_n \to s[B, \alpha, \beta]_p$, *then* $a_n \to 0[B, \alpha, \beta]_p$.
(ii) *If* $s_n \to s\,|B, \alpha, \beta|$, *then* $a_n \to 0\,|B, \alpha, \beta|$.

LEMMA 2.

(i) *If* $p \geq 1$ *and* $s_n \to s[B, \alpha, \beta]_p$, *then* $s_n \to s(B, \alpha, \beta)$.
(ii) *If* $p > 0$ *and* $s_n \to s(B, \alpha, \beta)$, *then* $s_n \to s[B, \alpha, \beta + 1]_p$.

Lemma 1(i) is included in [3, Theorem 15] when $p = 1$ and in [4, Theorem 15*] when $p > 1$. Lemma 1(ii) is included in [3, Theorem 14]. Lemma 2(i) is [3, Theorem 3] when $p = 1$ while Lemma 2(i) follows from [4, Theorem 3*] and Theorem A(i) when $p > 1$. Lemma 2(ii) is [4, Theorem 5*].

Wherever it occurs in the following lemmas, we suppose that $f(x)$ is bounded and Lebesgue measurable on every finite interval $[0, X]$ and we let $f_\delta(x)$ be defined by

$$f_\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t)\, dt$$

where $\delta > 0$.

LEMMA 3. *If* $\delta>0$ *and* $\gamma>0$, *then*

$$f_{\delta+\gamma}(x)=\frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}f_\delta(t)\,dt.$$

LEMMA 4.

(i) *Let* $f(x)=s_{\alpha,\beta}(x)$ *and let* $\delta>0$. *Then* $s_{\alpha,\beta+\delta}(x)=f_\delta(x)$.
(ii) $A_{\alpha,\beta}(x)=S_{\alpha,\beta}(x)-S_{\alpha,\beta+\alpha}(x)-\alpha e^{-x}s_{N-1}(x^{\alpha N+\beta-1}/\Gamma(\alpha N+\beta))$.

Lemma 3 is a well-known result the proof of which is straightforward. Lemma 4(i) is [2, Lemma 2]. The proof of Lemma 4(ii) is also straightforward.

LEMMA 5. *If* $s_n=0(1)(B,\alpha,\beta)$, *then* $s_n=0(1)(B,\alpha,\beta+\delta)$ *for every* $\delta>0$.

Lemma 5 is [3, Theorem 8].

LEMMA 6. *Let* $p\geq 1$. *If* $s_n=0(1)[B,\alpha,\beta]_p$, *then*

(i) $s_n=0(1)(B,\alpha,\beta)$,
(ii) $s_n=0(1)[B,\alpha,\beta+\delta]_p$ *where* $0<\delta<1$, *and*
(iii) $s_n=0(1)[B,\alpha,\beta+\delta]_r$ *where* $r>0$ *and* $\delta\geq 1$.

Proof. (i) When $p=1$ the result is [3, Theorem 4]. Thus we suppose that $p>1$ and we let $1/p+1/q=1$. Using Hölder's inequality and Lemma 4(i), we have that

$$\begin{aligned}|S_{\alpha,\beta}(x)|&\leq e^{-x}\int_0^x e^t\,|S_{\alpha,\beta-1}(t)|\,dt\\ &\leq e^{-x}\left\{\int_0^x e^t\,|S_{\alpha,\beta-1}(t)|^p\,dt\right\}^{1/p}\left\{\int_0^x e^t\,dt\right\}^{1/q}\\ &\leq e^{-x}\{Ke^x\}^{1/p}\{e^x\}^{1/q}=K^{1/p}\end{aligned}$$

for some positive constant K since $s_n=0(1)[B,\alpha,\beta]_p$.

(ii) When $p=1$ the result is included in [3, Theorem 10]. Thus we again suppose that $p>1$ and we let $1/p+1/q=1$. Furthermore, we let $f(x)=\alpha s_{\alpha,\beta-1}(x)$, $L=2^p/\{\Gamma(\delta)\}^p$, and $M=[1/\{\Gamma(\delta)\}^p]\int_0^1 e^{(1-p)t}\,|f_\delta(t)|^p\,dt$. Then, using Lemma 4(i), Hölder's inequality, and part of the proof of (i), we have for $x\geq 1$ that

$$\begin{aligned}\int_0^x e^t\,|S_{\alpha,\beta+\delta-1}(t)|^p\,dt&=\frac{1}{\{\Gamma(\delta)\}^p}\int_0^x e^{(1-p)t}\left|\int_0^t (t-u)^{\delta-1}f(u)\,du\right|^p dt\\ &\leq L\int_1^x e^{(1-p)t}\left\{\int_0^{t-1}|f(u)|\,du\right\}^p dt\\ &\quad+L\int_1^x e^{(1-p)t}\left\{\int_{t-1}^t (t-u)^{\delta-1}\,|f(u)|^p\,du\right\}\\ &\quad\times\left\{\int_{t-1}^t (t-u)^{\delta-1}\,du\right\}^{p/q}dt+M\end{aligned}$$

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$$\leq L\int_0^x e^{(1-p)t}\{K^{1/p}e^t\}^p\,dt$$

$$+\frac{L}{\delta^{p/q}}\int_1^x e^{(1-p)t}\,dt\int_{t-1}^t (t-u)^{\delta-1}\,|f(u)|^p\,du+M$$

$$\leq LKe^x+\frac{L}{\delta^{p-1}}\int_0^x |f(u)|^p\,du\int_u^{u+1} e^{(1-p)t}(t-u)^{\delta-1}\,dt+M$$

$$\leq LKe^x+\frac{L}{\delta^p}\int_0^x e^{(1-p)u}\,|f(u)|^p\,du+M$$

$$=LKe^x+\frac{L}{\delta^p}\int_0^x e^u\,|S_{\alpha,\beta-1}(u)|^p\,du+M$$

$$=0(e^x)\quad\text{since}\quad s_n=0(1)[B,\alpha,\beta]_p.$$

This establishes the desired result.

(iii) If $\delta\geq 1$, then

$$\int_0^x e^t\,|S_{\alpha,\beta+\delta-1}(t)|^r\,dt\leq\int_0^x K^r e^t\,dt\leq K^r e^x$$

for some positive constant K by Lemma 6(i) and Lemma 5.

LEMMA 7. *If*

$$e^{-x}\int_0^x f(t)\,dt=o(1),$$

then

$$e^{-x}\int_0^x f_\delta(t)\,dt=o(1)$$

for every $\delta>0$.

The proof of Lemma 7 is essentially the same as the proof of [3, Lemma 5].

LEMMA 8. *Let* $p\geq 1$. *If*

$$e^{-x}\int_0^x e^{(1-p)t}\,|f(t)|^p\,dt=0(1)\quad\textit{and}\quad e^{-x}\int_0^x f(t)\,dt=o(1),$$

then

(i) $e^{-x}\int_0^x e^{(1-p)t}\,|f_\delta(t)|^p\,dt=o(1)$ *where* $0<\delta<1$ *and*

(ii) $e^{-x}\int_0^x e^{(1-r)t}\,|f_\delta(t)|^r\,dt=o(1)$ *where* $r>0$ *and* $\delta\geq 1$.

Proof. (i) Let $\varepsilon>0$. By hypothesis, there exists a number $Y\geq 0$ such that

$$\left|\int_0^x f(t)\,dt\right|\leq\varepsilon e^x$$

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for all $x \geq Y$. Let

$$N(\varepsilon) = \sup_{0 \leq x \leq Y} \left| \int_0^x f(t)\, dt \right| < \infty.$$

Now

$$\limsup_{x\to\infty} e^{-x} \int_0^x e^{(1-p)t} |f_\delta(t)|^p \, dt$$

$$= \limsup_{x\to\infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \frac{1}{\Gamma(\delta)} \int_0^t (t-u)^{\delta-1} f(u)\, du \right|^p dt$$

$$\leq \frac{2^p}{\{\Gamma(\delta)\}^p} \left\{ \limsup_{x\to\infty} I_1 + \limsup_{x\to\infty} I_2 \right\}\cdot$$

where

$$I_1 = e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \int_0^{t-\varepsilon} (t-u)^{\delta-1} f(u)\, du \right|^p dt$$

and

$$I_2 = e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \int_{t-\varepsilon}^t (t-u)^{\delta-1} f(u)\, du \right|^p dt$$

But, using the Second Mean Value Theorem,

$$\limsup_{x\to\infty} I_1 = \limsup_{x\to\infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \left| \varepsilon^{\delta-1} \int_{\mu(t)}^{t-\varepsilon} f(u)\, du \right|^p dt$$

$$\leq 2^p \varepsilon^{(\delta-1)p} \limsup_{x\to\infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \{N(\varepsilon) + \varepsilon e^t\}^p \, dt$$

$$\leq 2^{2p} \varepsilon^{(\delta-1)p} \limsup_{x\to\infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \{(N(\varepsilon))^p + \varepsilon^p e^{pt}\}\, dt$$

$$= 2^{2p} \varepsilon^{\delta p}$$

since

$$\left| \int_{\mu(t)}^{t-\varepsilon} f(u)\, du \right| \leq 2 \sup_{0 \leq y \leq t-\varepsilon} \left| \int_0^y f(u)\, du \right| \leq 2\{N(\varepsilon) + \varepsilon e^t\}$$

and

$$\lim_{x\to\infty} e^{-x} \int_\varepsilon^x e^{(1-p)t} \{N(\varepsilon)\}^p \, dt = 0.$$

Also, by hypothesis there is a number $K \geq 0$ such that

$$e^{-x} \int_0^x e^{(1-p)t} |f(t)|^p \, dt \leq K$$

for all $x \geq 0$, and therefore, when $p=1$,

$$\begin{aligned}\limsup_{x\to\infty} I_2 &\leq \limsup_{x\to\infty} e^{-x}\int_\varepsilon^x dt \int_{t-\varepsilon}^t (t-u)^{\delta-1}\,|f(u)|\,du\\ &\leq \limsup_{x\to\infty} e^{-x}\int_0^x |f(u)|\,du \int_u^{u+\varepsilon} (t-u)^{\delta-1}\,dt\\ &\leq K\frac{\varepsilon^\delta}{\delta},\end{aligned}$$

while, when $p>1$,

$$\begin{aligned}\limsup_{x\to\infty} I_2 \leq \limsup_{x\to\infty} e^{-x}\int_\varepsilon^x e^{(1-p)t}&\left\{\int_{t-\varepsilon}^t (t-u)^{\delta-1}\,|f(u)|^p\,du\right\}\\ &\times\left\{\int_{t-\varepsilon}^t (t-u)^{\delta-1}\,du\right\}^{p-1} dt\\ &=\left\{\frac{\varepsilon^\delta}{\delta}\right\}^{p-1} \limsup_{x\to\infty} e^{-x}\int_\varepsilon^x e^{(1-p)t}\,dt \int_{t-\varepsilon}^t (t-u)^{\delta-1}\,|f(u)|^p\,du\\ &\leq\left\{\frac{\varepsilon^\delta}{\delta}\right\}^{p-1} \limsup_{x\to\infty} e^{-x}\int_0^x |f(u)|^p\,du \int_u^{u+\varepsilon} (t-u)^{\delta-1}e^{(1-p)t}\,dt\\ &\leq\left\{\frac{\varepsilon^\delta}{\delta}\right\}^{p} \limsup_{x\to\infty} e^{-x}\int_0^x e^{(1-p)u}\,|f(u)|^p\,du \leq K\left\{\frac{\varepsilon^\delta}{\delta}\right\}^{p}.\end{aligned}$$

Thus for $p \geq 1$ we have that

$$\limsup_{x\to\infty} e^{-x}\int_0^x e^{(1-p)t}\,|f_\delta(t)|^p\,dt \leq \frac{2^p(2^{2p}+K\delta^{-p})}{\{\Gamma(\delta)\}^p}\,\varepsilon^{\delta p}$$

from which it follows that

$$\limsup_{x\to\infty} e^{-x}\int_0^x e^{(1-p)t}\,|f_\delta(t)|^p\,dt = 0$$

since ε is arbitrary. This establishes the desired result.

(ii) Since $e^{-x}f_1(x)=o(1)$ by hypothesis, we have, when $\delta=1+\mu$ where $\mu>0$, that

$$e^{-x}f_{1+\mu}(x)=e^{-x}\int_0^x f_\mu(t)\,dt=o(1),$$

using Lemma 3 and Lemma 7. Hence, for $\delta \geq 1$,

$$\begin{aligned}e^{-x}\int_0^x e^{(1-r)t}\,|f_\delta(t)|^r\,dt &= e^{-x}\int_0^x e^t\,|e^{-t}f_\delta(t)|^r\,dt\\ &= e^{-x}\int_0^x e^t o(1)\,dt\\ &= o(1).\end{aligned}$$

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If b is a real number, we let

$$H_b = \{z \mid \operatorname{Re} z \geq b\}.$$

A function $g(z)$ is said to be of exponential type in H_b if $g(z)$ is analytic in H_b and if there are positive numbers A, a such that $|g(z)| \leq Ae^{a|z|}$ for all z in H_b.

LEMMA 9. *If $g(z)$ is of exponential type in H_0 and if*

$$\int_0^\infty |g(x)|^p \, dx < \infty \qquad (p > 0),$$

then

$$\int_0^\infty |g'(x)|^p \, dx < \infty.$$

Lemma 9 is due to Gaier [6, Theorem 2].

LEMMA 10. *If $g(z)$ is of exponential type in H_b and $g(x) \in BV_x[b, \infty)$, then*

$$g^{(k)}(x) \in BV_x[b, \infty)$$

for every non-negative integer k.

Proof. Suppose that $g^{(k)}(x) \in BV_x[b, \infty)$ where k is a non-negative integer. Then

$$\int_0^\infty |g^{(k+1)}(x+b+1)| \, dx < \infty$$

and

$$\begin{aligned} |g^{(k+1)}(z+b+1)| &\leq \frac{(k+1)!}{2\pi} \int_0^{2\pi} |g(z+b+e^{i\theta})| \, d\theta \\ &\leq (k+1)! \, Ae^{a(|z|+|b|+1)} \end{aligned}$$

for all z in H_0 where A, a are positive constants. Hence, by Lemma 9,

$$\int_0^\infty |g^{(k+2)}(x+b+1)| \, dx = \int_{b+1}^\infty |g^{(k+2)}(x)| \, dx < \infty$$

i.e. $g^{(k+1)}(x) \in BV_x[b+1, \infty)$. Since $g^{(k+1)}(x) \in BV_x[b, b+1]$, therefore $g^{(k+1)}(x) \in BV_x[b, \infty)$. The desired result now follows by induction.

3. **Tauberian theorems for strong Borel-type summability with index $p \geq 1$.** We first show that the scale in Theorem A(ii) us proper. In [5] we showed that there is a sequence $\{s_n\}$ which tends to a limit (B, α, β) but does not tend to a limit $(B, \alpha, \beta-1)$. Hence, in view of Lemma 2, there is a sequence $\{s_n\}$ which tends to a limit $[B, \alpha, \beta+1]_p$ for every $p > 0$ but does not tend to a limit $[B, \alpha, \beta-1]_p$ for any $p \geq 1$.

THEOREM 1. *Let $p, r \geq 1$. If $s_n \to s[B, \alpha, \mu]_p$ and $a_n \to 0[B, \alpha, \beta]_r$, then $s_n \to s[B, \alpha, \beta]_r$.*

Proof. By Lemma 2(i), $s_n \to s(B, \alpha, \mu)$. The result now follows by [9, Theorem 3] and the note following [9, Theorem 3].

THEOREM 2. *Let $p \geq 1$. If $s_n \to s[B, \alpha, \beta+\varepsilon]_p$ for some $\varepsilon > 0$ and $s_n = 0(1)[B, \alpha, \beta]_p$, then $s_n \to s[B, \alpha, \beta+\delta]_p$ for every $\delta > 0$.*

Proof. We can suppose without loss of generality that $s = 0$. Then $s_n \to 0(B, \alpha, \beta+\varepsilon)$ and $s_n = 0(1)(B, \alpha, \beta)$ by Lemma 2(i) and Lemma 6(i). Hence $s_n \to 0(B, \alpha, \beta+\delta)$ by [5, Theorem 2] for $\delta > 0$. Also $s_n = 0(1)[B, \alpha, \beta+\delta]_p$ by Lemma 6(ii) or (iii). Therefore, letting $f(x) = \alpha s_{\alpha, \beta+\delta-1}(x)$, we have that

$$e^{-x}\int_0^x f(t)\,dt = S_{\alpha,\beta+\delta}(x) = o(1)$$

and

$$e^{-x}\int_0^x e^{(1-p)t}\,|f(t)|^p\,dt = e^{-x}\int_0^x e^t\,|S_{\alpha,\beta+\delta-1}(t)|^p\,dt = 0(1)$$

using Lemma 4(i), and consequently,

$$e^{-x}\int_0^x e^t\,|S_{\alpha,\beta+2\delta-1}(t)|^p\,dt = e^{-x}\int_0^x e^{(1-p)t}\,|f_\delta(t)|^p\,dt = o(1)$$

using Lemma 4(i) and Lemma 8, i.e. $s_n \to 0[B, \alpha, \beta+2\delta]_p$. This establishes the desired result.

THEOREM 2*. *Let $p \geq 1$. If $\sum_0^\infty a_n = s[B, \alpha, \beta+\varepsilon]_p$ for some $\varepsilon > 0$ and $a_n = 0(1)[B, \alpha, \beta]_p$, then $\sum_0^\infty a_n = s[B, \alpha, \beta+\delta]_p$ for every $\delta > 0$.*

Proof. By Lemma 1(i), $a_n \to 0[B, \alpha, \beta+\varepsilon]_p$ and thus, by Theorem 2, $a_n \to 0[B, \alpha, \beta+\delta]_p$ for every $\delta > 0$. The result now follows by Theorem 1.

A real-valued function $g(x)$, with domain $[0, \infty)$, is slowly decreasing if for every $\varepsilon > 0$ there exist positive numbers X, δ such that $g(x) - g(y) > -\varepsilon$ whenever $x \geq y \geq X$ and $x - y \leq \delta$. The following result is [5, Theorem 3]: *If $s_n \to s(B, \alpha, \beta+\varepsilon)$ for some $\varepsilon > 0$ and $S_{\alpha,\beta}(x)$ is slowly decreasing, then $s_n \to s(B, \alpha, \beta)$.* We now show that there is no analogue to this result for the $[B, \alpha, \beta]_p$ method.

Let $\{s_n\}$ be the sequence defined by $\sum_{n=0}^\infty s_n(x^n/n!) = e^x \sin e^x$ (cf. [7, p. 183]). Then $S_{1,1}(x) = \sin e^x$ where we choose $N = 0$. Thus, using Lemma 4(i),

$$S_{1,2}(x) = e^{-x}\int_0^x e^t \sin e^t\,dt = e^{-x}(\cos 1 - \cos e^x) = o(1)$$

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and therefore $s_n \to 0(B,1,2)$. (In fact, by [5, Theorem 2], $s_n \to 0(B,1,1+\delta)$ for every $\delta>0$.) Hence, by Lemma 2(ii), $s_n \to 0[B,1,3]_r$, for every $r>0$. Furthermore,

$$e^{-x}\int_0^x e^t\,|S_{1,1}(t)-0|^r\,dt = e^{-x}\int_0^x e^t\,|\sin e^t|^r\,dt$$

$$= e^{-x}\int_1^{e^x} |\sin u|^r\,du \to \frac{L(r)}{\pi}$$

as $x\to\infty$ where $L(r)=\int_0^\pi |\sin u|^r\,du$. Therefore $s_n \not\to 0[B,1,2]_r$, $s_n \to 0[B,1,3]_r$ and both $e^{-x}\int_0^x e^t S_{1,1}(t)\,dt$ and $e^{-x}\int_0^x e^t\,|S_{1,1}(t)|^r\,dt$ are slowly decreasing (since they both tend to a limit as $x\to\infty$).

THEOREM 3. *Let $p\ge 1$. If $s_n \to s[B,\alpha,\mu]_p$ and*

(i) *$s_n \ge -K$ for all $n\ge 0$, or*

(ii) *$a_n \ge -K$ for all $n\ge 0$, or*

(iii) *$S_{\alpha,\mu}(z)$ is of exponential type in H_δ, or*

(iv) *$A_{\alpha,\mu}(z)$ is of exponential type in H_δ, or*

(v) *$|a_n|\le K^n$ for all $n\ge 0$,*

where K, δ are positive constants, then

$$s_n \to s[B,\alpha,\beta]_r$$

for every $r>0$.

Proof. By Lemma 2(i), $s_n \to s(B,\alpha,\mu)$. Hence, by [5, Theorem 5, 5*, 6, 6*, or 7], $s_n \to s(B,\alpha,\beta-1)$. The result now follows by Lemma 2(ii).

4. **Tauberian theorems for absolute Borel-type summability.** We first show that the scale in Theorem A(iii) is proper in the sense that for each β there is a sequence $\{s_n\}$ which is summable $|B,\alpha,\beta|$ but is not summable $|B,\alpha,\beta-1|$.

Choose an integer m such that $\alpha m>1$ and let P be the smallest integer such that $mP\ge N$. Let

$$x^P e^{-x}\sin e^x = \sum_{n=P}^{\infty} b_n x^n$$

and let

$$s_n = \begin{cases} \Gamma(\alpha n+\beta)b_k & \text{if } n=mk,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$S_{\alpha,\beta}(x) = \alpha x^{\alpha m P+\beta-1} e^{-x} e^{-x^{\alpha m}} \sin e^{x^{\alpha m}} = o(1)$$

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and

$$\begin{aligned}S'_{\alpha,\beta}(x) = \alpha(\alpha mP+\beta-1)x^{\alpha mP+\beta-2}e^{-x}e^{-x^{\alpha m}}\sin e^{x^{\alpha m}}\\ -\alpha x^{\alpha mP+\beta-1}e^{-x}e^{-x^{\alpha m}}\sin e^{x^{\alpha m}}\\ -\alpha(\alpha m)x^{\alpha mP+\alpha m+\beta-2}e^{-x}e^{-x^{\alpha m}}\sin e^{x^{\alpha m}}\\ +\alpha(\alpha m)x^{\alpha mP+\alpha m+\beta-2}e^{-x}\cos e^{x^{\alpha m}}\end{aligned}$$

so that $S'_{\alpha,\beta}(x)=o(1)$ and $S'_{\alpha,\beta}(x)\in L_1[0,\infty)$ since $\alpha mP+\beta-2\geq\alpha N+\beta-2\geq 0$ by our choice of N. Hence $s_n\to 0\,|B,\alpha,\beta|$. However

$$S''_{\alpha,\beta}(x)=f(x)-\alpha(\alpha m)^2x^{\alpha mP+2\alpha m+\beta-3}e^{-x}e^{x^{\alpha m}}\sin e^{x^{\alpha m}}$$

where $f(x)\in L_1[0,\infty)$ and therefore $S''_{\alpha,\beta}(x)\notin L_1[0,\infty)$ since $\alpha m>1$. Thus, since

$$S_{\alpha,\beta-1}(x)=S_{\alpha,\beta}(x)+S'_{\alpha,\beta}(x)$$

and

$$S'_{\alpha,\beta-1}(x)=S'_{\alpha,\beta}(x)+S''_{\alpha,\beta}(x),$$

we have that

$$s_n\to 0(B,\alpha,\beta-1)\quad\text{but}\quad s_n\not\to 0\,|B,\alpha,\beta-1|.$$

THEOREM 4. *If* $s_n\to s\,|B,\alpha,\mu|$ *and* $a_n\to 0\,|B,\alpha,\beta|$, *then* $s_n\to s\,|B,\alpha,\beta|$.

Proof. By [5, Theorem 1], $s_n\to s(B,\alpha,\beta)$. Thus it remains only to show that $S_{\alpha,\beta}(x)\in BV_x[0,\infty)$. Let k be a positive integer. Then, in view of Theorem A(iii), $A_{\alpha,\beta+(k-1)\alpha}(x)\in BV_x[0,\infty)$. Moreover, by Lemma 4(ii),

$$S_{\alpha,\beta+(k-1)\alpha}(x)=A_{\alpha,\beta+(k-1)\alpha}(x)+S_{\alpha,\beta+k\alpha}(x)+\alpha e^{-x}s_{N-1}\frac{x^{\alpha N+\beta-1}}{\Gamma(\alpha N+\beta)}.$$

Therefore $S_{\alpha,\beta+(k-1)\alpha}(x)\in BV_x[0,\infty)$ if $S_{\alpha,\beta+k\alpha}(x)\in BV_x[0,\infty)$. Since, in view of Theorem A(iii), $S_{\alpha,\beta+k\alpha}(x)\in BV_x[0,\infty)$ when $\beta+k\alpha\geq\mu$, it readily follows that $S_{\alpha,\beta}(x)\in BV_x[0,\infty)$.

If $\{s_n\}$ is the sequence described in the paragraph preceding Theorem 3, then, using Lemma 4(i),

$$S_{1,3}(x)=e^{-x}\int_0^x(\cos 1-\cos e^t)\,dt$$

and thus it is readily seen that $s_n\to 0\,|B,1,3|$ and $s_n\not\to 0\,|B,1,2|$. Hence there is also no immediate absolute summability analogue to [5, Theorem 3].

Our final results are extensions of a result due to Gaier (see [6]).

THEOREM 5. *If* $s_n\to s\,|B,\alpha,\mu|$ *and* $S_{\alpha,\mu}(z)$ *is of exponential type in* H_δ *for some* $\delta>0$, *then* $s_n\to s\,|B,\alpha,\beta|$.

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Proof. Let k be a positive integer such that $\mu-k\le\beta$. By [5, Theorem 6] we have that $s_n\to s(B,\alpha,\mu-k)$. Furthermore, since

$$S_{\alpha,\mu-1}(z)=S_{\alpha,\mu}(z)+S^{(1)}_{\alpha,\mu}(z),$$

it is readily seen that

$$S_{\alpha,\mu-k}(z)=S_{\alpha,\mu}(z)+\sum_{j=1}^{k}\binom{k}{j}S^{(j)}_{\alpha,\mu}(z).$$

Since $S_{\alpha,\mu}(z)$ is of exponential type in H_δ and since $S_{\alpha,\mu}(x)\in BV_x[0,\infty)$ by hypothesis, we have, by Lemma 10, that $S^{(j)}_{\alpha,\mu}(x)\in BV_x[\delta,\infty)$ for $j=1,\ldots,k$; also, since we choose N so that $\alpha N+\mu-k\ge 1$, we have that $S^{(j)}_{\alpha,\mu}(x)\in BV_x[0,\delta]$ for $j=1,\ldots,k$. Therefore, $S^{(j)}_{\alpha,\mu}(x)\in BV_x[0,\infty)$ for $j=1,\ldots,k$ and, consequently, $S_{\alpha,\mu-k}(x)\in BV_x[0,\infty)$. Hence $s_n\to s\,|B,\alpha,\mu-k|$ and, by Theorem A(iii), $s_n\to s\,|B,\alpha,\beta|$.

THEOREM 5*. *If $s_n\to s\,|B,\alpha,\mu|$ and $A_{\alpha,\mu}(z)$ is of exponential type in H_δ for some $\delta>0$, then $s_n\to s\,|B,\alpha,\beta|$.*

Proof. By Lemma 1(ii), $a_n\to 0\,|B,\alpha,\mu|$ and thus, by Theorem 5, $a_n\to 0\,|B,\alpha,\beta||$. The result now follows by Theorem 4.

THEOREM 6. *If $s_n\to s\,|B,\alpha,\mu|$ and $|a_n|\le K^n$ for all $n\ge 0$ where K is a positive constant, then $s_n\to s\,|B,\alpha,\beta|$.*

Proof. Since $|a_n|\le K^n$ for all $n\ge 0$, we have that

$$|A_{\alpha,\mu}(z)|\le Ae^{K^{1/\alpha}|z|}$$

for some positive constant A. The desired result now follows by Theorem 5*.

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