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Nagoya Math. J.
Vol. 36 (1969), 65–81

GAUSSIAN MEASURE ON A BANACH SPACE AND ABSTRACT WINER MEASURE

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In this paper, we shall show that any Gaussian measure on a separable or reflexive Banach space is an abstract Wiener measure in the sense of L. Gross [1] and, for the proof of that, establish the Radon extensibility of a Gaussian measure on such a Banach space. In addition, we shall give some remarks on the support of an abstract Wiener measure.

An abstract Wiener measure is a σ-extension in a Banach space X of the canonical Gaussian cylinder measure $\mu_{\mathfrak{X}}$ of a real separable Hilbert space $\mathfrak{X}$ which is contained in X densely. The idea of the abstract Wiener measure coincides with that of *the White Noise* (T. Hida [13]) and plays an important role not only in the theory of probability but in the theory of functional analysis (T. Hida [13], Y. Umemura [12], I.E. Segal [4, 5], L. Gross [3] and Yu. L. Daletskii [16]).

We shall show first that any Gaussian measure on a separable or reflexive Banach space can be extended to a Radon measure on the strong topological σ-algebra (Theorem 1). With the same idea of the proof of Theorem 1, we can prove that this result is true for any probability measure on a Banach space, the finite dimensional distribution of which is Radon.

Utilizing the above result, we shall restrict the support of a Gaussian measure to a separable subspace which is explicitly constructed. Furthermore, constructing a suitable Hilbert subspace of the support, we shall show that any Gaussian measure on such a Banach space is an abstract Wiener measure (Theorem 2). L. Gross [1] showed that there exists and abstract Wiener measure on any separable Banach space. Our result shows that any *given* Gaussian measure on a separable or reflexive Banach space is an abstract Wiener measure. This means that the study of a Gaussian measure

Received Oct. 9, 1968.

on such a Banach space can be reduced to that of an abstract Wiener measure on a separable Banach space, and clears a new way for the investigation of a Gaussian measure on a Banach space, and makes the study of an abstract Wiener measure more meaningful.

As a corollary of Theorem 2, we shall show that the canonical Gaussian cylinder measure of a *nonseparable* Hilbert space can not be extended to a σ-additive measure in any Banach space.

Before stating the remaining results in this paper, we establish terminology and notation.

Let X be a real Banach space, X^* be its topological dual space and $\xi(x)$, $(\xi \in X^*,\ x \in X)$, be the natural linear form. A cylinder set in X is a set of the form

$$C = \{x \in X\colon\ (\xi_1(x),\ \cdots,\ \xi_n(x)) \in D\}$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are in X^* and D is a Borel set in the n-dimensional Euclidean space R_n. $\mathfrak{A}_X$ is the family of all cylinder sets in X and $\overline{\mathfrak{A}}_X$ is the minimal σ-algebra including $\mathfrak{A}_X$. τ_X is the weak topological σ-algebra in X and $\hat{\tau}_X$ is the strong topological σ-algebra in it. Evidently we have

$$\mathfrak{A}_X \subset \overline{\mathfrak{A}}_X \subset \tau_X \subset \hat{\tau}_X$$

and if X is separable, then $\overline{\mathfrak{A}}_X = \hat{\tau}_X$ (E. Mourier [8]).

Let $\mathfrak{X}$ be a real Hilbert space. *The canonical Gaussian cylinder measure* $\mu_{\mathfrak{X}}$ of $\mathfrak{X}$ is a finitely additive nonnegative set function on $(\mathfrak{X}, \mathfrak{A}_{\mathfrak{X}})$ such that

$$\mu_{\mathfrak{X}}[x \in \mathfrak{X}\colon\ \xi(x) \leqslant \alpha] = \frac{1}{\sqrt{2\pi}\ |\xi|} \int_{-\infty}^{\alpha} \exp\Big[-\frac{u^2}{2|\xi|^2}\Big] du, \qquad (1.\ 1)$$

for any $\xi \in \mathfrak{X}^*$ and real number α, where $|\xi|$ is the norm in $\mathfrak{X}^*$. It is well-known that $\mu_{\mathfrak{X}}$ does not have σ-additive extension to $(\mathfrak{X}, \overline{\mathfrak{A}}_{\mathfrak{X}})$, (see Corollary of Lemma 6).

Let $\|x\|$ be a continuous norm on $\mathfrak{X}$, and X be the Banach space obtained by the completion of $\mathfrak{X}$ in the norm $\|x\|$. Since we may consider X^* as a subspace of $\mathfrak{X}^*$ through the natural imbedding, $\mu_{\mathfrak{X}}$ induces a Gaussian cylinder measure μ on $(X, \mathfrak{A}_X)$ as follows. If $\xi_1, \xi_2, \cdots, \xi_n$ are in X^* and D is a Borel set in R_n, define

$$\begin{aligned} &\mu[x \in X;\ (\xi_1(x),\ \cdots,\ \xi_n(x)) \in D] \\ &= \mu_{\mathfrak{X}}[x \in \mathfrak{X};\ (\xi_1(x),\ \cdots,\ \xi_n(x)) \in D]. \end{aligned} \qquad (1.\ 2)$$

μ is well-defined. Furthermore, if μ has a σ-additive extension on $(X, \overline{\mathfrak{A}}_X)$, then we call it *the σ-extension of $\mu_{\mathfrak{X}}$ on the Banach space X* and the norm $\|x\|$ *admissible* on $\mathfrak{X}$. If a norm on $\mathfrak{X}$ is induced by an inner product, namely, a continuous symmetric bilinear form on $\mathfrak{X}$, then we call it *Hilbertian*. A measurable norm is defined by L. Gross [1, 2] as follows. A norm $\|x\|_1$ on $\mathfrak{X}$ is a *measurable norm* if for every positive real number ε there exists a finite dimensional projection P_0 of $\mathfrak{X}$ such that for every finite dimensional projection P orthogonal to P_0 we have

$$\mu_{\mathfrak{X}}[x \in \mathfrak{X}: \ \|Px\|_1 > \varepsilon] < \varepsilon.$$

L. Gross [1] showed that the measurable norm is admissible.

In the last section, we shall give some remarks on the admissible norm. We shall give a necessary and sufficient condition for a Hilbertian norm to be admissible (Theorem 3) and show that there exists a measurable norm such that there is no Hilbertian admissible norm stronger than it (Example 2). This means that as a support of an abstract Wiener measure we can choose a Banach subspace which includes no Hilbert subspace of full measure. We shall also show that there exists an admissible norm which is not a measurable norm. This means that for a norm to be an admissible norm it is not necessary to be a measurable norm.

2. Gaussian measure and Radon measure.

Let X be a Banach space with norm $\|x\|$ and X^* be the topological dual for X with norm $|||\xi|||$. A probability measure μ on $(X, \overline{\mathfrak{A}}_X)$ is *Gaussian* if for every $\xi \in X^*$, $\xi(x)$ is a Gaussian random variable with mean zero on the probability space $(X, \overline{\mathfrak{A}}_X, \mu)$. In other words, for every $\xi \in X^*$ and real number α,

$$\mu[x \in X: \ \xi(x) \leqslant \alpha] = \frac{1}{\sqrt{2\pi v(\xi)}} \int_{-\infty}^{\alpha} \exp\Big[-\frac{u^2}{2v(\xi)}\Big] du, \tag{2.1}$$

where $v(\xi)$ is the variance of $\xi(x)$.

Theorem 1. *Every Gaussian measure μ on a separable or seflexive Banach space $(X, \overline{\mathfrak{A}}_X)$ can be extended to a Radon measure on (X, τ_X).*

Proof. If X is separable, $\overline{\mathfrak{A}}_X = \tau_X$ and the proof is trivial. Let X be a reflexive Banach space and let X^{**} be the topological dual space of X^*. Let $\overline{\mathfrak{A}}^*$ be the minimal σ-algebra of subsets of X^{**} with respect to which

https://doi.org/10.1017/S002776300001312X Published online by Cambridge University Press

all the functions $\xi(x)$, $\xi \in X^*$, are measurable, where $\xi(x)$ ($\xi \in X^*$, $x \in X^{**}$) denotes the continuous linear form and τ^* is the topological σ-algebra with respect to X^*-topology in X^{**} (W. Dunford and J.T. Schwartz [15], p. 419). Define a measure μ^* on $(X^{**}, \overline{\mathfrak{A}}^*)$ as follows:

$$\mu^*[x \in X^{**}: (\xi_1(x), \cdots, \xi_n(x)) \in D]$$
$$= \mu[x \in X: (\xi_1(x), \cdots, \xi_n(x)) \in D]. \tag{2. 2}$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are in X^* and D is a Borel set in R_n. The measure μ^* is well defined and is Gaussian. Since all the open sets in $\overline{\mathfrak{A}}^*$ form an open basis which determines X^*-topology and since X^{**} is the topological dual for the Banach space X^*, μ^* can be extended to a Radon measure $\tilde{\mu}^*$ on (X^{**}, τ^*) uniquely (Yu. V. Prohorov [10], Theorem 1, Lemma 3 and Example 1). Since X is reflexive, we have $X = X^{**}$ and $\tau^* = \tau_X$. Therefore $\tilde{\mu}^*$ is a Gaussian Radon measure on $(X, \hat{\tau}_X)$. Since X is a Banach space, the weak Radon measure $\tilde{\mu}^*$ can be extended to a strong Radon measure $\tilde{\mu}$ on $(X, \hat{\tau}_X)$ and, it is easy to see from (2. 2), that $\tilde{\mu}$ is an extension of μ. Thus we have proved the theorem.

Remark. Without any change in the proof, we can prove Theorem 1 not only for a Gaussian measure but for any probability measure on a Banach space, the finite dimensional distribution of which is Radon.

We can therefore consider a Gaussian measure on a Banach space X as a Radon measure on $(X, \hat{\tau}_X)$.

3. Gaussian measure and abstract Wiener measure.

Let μ be a Gaussian measure on a separable or reflexive Banach space X. We use the same notations used in Section 2. Choose the maximal subset $\{\xi_\alpha; \alpha \in \Lambda\}$ of X^* such that

$$\begin{aligned} &\xi_\alpha \in X^* \text{ and } |||\xi_\alpha||| = 1, \quad \alpha \in \Lambda \\ &\int_X \xi_\alpha(x)\xi_\beta(x)d\mu(x) = 0 \quad \text{if} \quad \alpha \neq \beta,\ \alpha,\ \beta \in \Lambda. \end{aligned} \tag{3. 1}$$

LEMMA 1. *Let $\Lambda_0 = \{\alpha \in \Lambda;\ v(\xi_\alpha)) \neq 0\}$, then Λ_0 is an at most countable subset of Λ.*

Proof. Let $\{\alpha_n\}_{n=1,2,\cdots}$, be an arbitrary countable subset of Λ. Since it holds that

$$\sup_n |\xi_{\alpha_n}(x)| \leqslant \sup_{\substack{|||\xi|||=1 \\ \xi\in X^*}} |\xi(x)| = \|x\| < +\infty, \text{ for every } x \in X, \tag{3. 2}$$

we can choose a positive number M such that

$$\mu[x \in X: \sup_n |\xi_{\alpha_n}(x)| \leqslant M] > \frac{1}{2}. \tag{3. 3}$$

On the other hand, we have

$$\begin{aligned} &\mu[x \in X; \sup_n |\xi_{\alpha_n}(x)| \leqslant M] \\ &= \lim_{N\to+\infty} \mu[x \in X: \sup_{1\leqslant n\leqslant N} |\xi_{\alpha_n}(x)| \leqslant M] \\ &= \lim_{N\to+\infty} \mu[\bigcap_{1\leqslant n\leqslant N} \{x \in X: |\xi_{\alpha_n}(x)| \leqslant M\}]. \end{aligned}$$

Since the collection $\{\xi_{\alpha_n}(x)\}$ is Gaussian, from (3. 1), $\xi_{\alpha_n}(x)$ and $\xi_{\alpha_m}(x)$ are mutually independent if $n \neq m$. Therefore,

$$\begin{aligned} &\mu[x \in X: \sup_n |\xi_{\alpha_n}(x)| \leqslant M] \\ &= \lim_{N\to+\infty} \prod_{1\leqslant n\leqslant N} \mu[x \in X; |\xi_{\alpha_n}(x)| \leqslant M] \\ &= \lim_{N\to+\infty} \prod_{1\leqslant n\leqslant N} \frac{1}{\sqrt{2\pi v(\xi_{\alpha_n})}} \int_{-M}^{M} \exp\left[-\frac{u^2}{2v(\xi_{\alpha_n})}\right] du \\ &= \lim_{N\to+\infty} \prod_{1\leqslant n\leqslant N} \frac{1}{\sqrt{2\pi}} \int_{-\frac{M}{\sqrt{v(\xi_{\alpha_n})}}}^{\frac{M}{\sqrt{v(\xi_{\alpha_n})}}} \exp\left[-\frac{u^2}{2}\right] du. \end{aligned}$$

Together with (3. 3), we have

$$\lim_{N\to+\infty} v(\xi_{\alpha_n}) = 0. \tag{3. 4}$$

Since the choice of the countable subset $\{\alpha_n\}$ is arbitrary, the set

$$\Lambda_N = \left\{\alpha \in \Lambda; \ v(\xi_{\alpha_n}) \geqslant \frac{1}{N}\right\}$$

must be a finite subset of Λ for every positive integer N. Otherwise we have a contradiction to (3. 4). Therefore,

$$\Lambda_0 = \bigcup_{N=1}^{+\infty} \Lambda_N$$

must be a countable subset of Λ.

LEMMA 2. *Define* X_α, $\alpha \in \Lambda$, *by*

$$X_\alpha = \{x \in X;\ \xi_\alpha(x) = 0\}, \quad \alpha \in \Lambda,$$

and set $\tilde{X} = \bigcap_{\alpha \in \Lambda - \Lambda_0} X_\alpha$. *Then we have*

$$\mu[\tilde{X}] = 1. \tag{3. 5}$$

Proof. Let Γ be the family of all finite subsets of $\Lambda - \Lambda_0$ and define $X_J = \bigcap_{\alpha \in J} X_\alpha$; $J \in \Gamma$. Obviously X_J is a strongly closed linear subspace of X and the family $\{X_J\colon J \in \Gamma\}$ is directed. Since $v(\xi_\alpha) = 0$, $\xi_\alpha(x)$ is a Dirac measure for every $\alpha \in \Lambda - \Lambda_0$, we have

$$\mu[X_J] = 1 \qquad \text{for every } J \in \Gamma.$$

Therefore,

$$\mu[\tilde{X}] = \mu[\bigcap_{J \in \Gamma} X_J]$$
$$= \inf_{J \in \Gamma} \mu[X_J] = 1,$$

(L. Schwartz [11]). Thus we have proved the lemma.

This lemma means that the measure μ is concentrated in some closed linear subspace $\tilde{X}$. $\tilde{X}$ is also a Banach space with the norm $\|x\|$. Let $\mathfrak{C}$ be the closed linear manifold spanned by $\{\xi_\alpha;\ \alpha \in \Lambda - \Lambda_0\}$. Then the topological dual $\tilde{X}^*$ for $\tilde{X}$ is isomorphic to $X^*/\mathfrak{C}$.
It is easy to see that in $\tilde{X}^*$

$$v(\xi) = 0 \quad \text{implies} \quad \xi = 0. \tag{3. 6}$$

Let $|||\xi|||$ be the norm in $\tilde{X}^*$ again.

Hereafter, we restrict the measure μ to $\tilde{X}$. For every $\xi, \eta \in X^*$ define

$$(\xi, \eta) = \int_{\tilde{X}} \xi(x)\eta(x)d\mu(x), \tag{3. 7}$$

$$|\xi| = \sqrt{(\xi, \xi)} = \sqrt{v(\xi)} \tag{3. 8}$$

Then, according to (3. 6),

$$|\xi| = 0 \text{ if and only if } |||\xi||| = 0, \tag{3. 9}$$

in $\tilde{X}^*$. Therefore the bilinear form (ξ, η) is an inner product and $|\xi|$ is a norm on $\tilde{X}^*$. Next we shall show that the norm $|\xi|$ is continuous.

LEMMA 3. *There exists a positive constant C such that*

$$|\xi| \leqslant C\,|||\xi||| \quad \textit{for every } \xi \in \tilde{X}^*. \tag{3. 10}$$

Proof. It is sufficient to show

$$C = \sup_{\substack{|||\xi|||=1 \\ \xi\in\tilde{X}^*}} |\xi| < +\infty.$$

Suppose not, then there exists a sequence $\{\xi_n\}$ in $\tilde{X}^*$ such that

$$|||\xi_n||| = 1, \quad n = 1, 2, 3, \cdots$$

$$\lim_{n\to+\infty} |\xi_n| = +\infty.$$

By choosing a sufficiently large number M, we have

$$\mu[x \in \tilde{X}: \sup_n |\xi_n(x)| \leqslant M] > \frac{1}{2}, \tag{3. 11}$$

(see the proof of Lemma 1). On the other hand,

$$\begin{aligned}
&\mu[x \in \tilde{X}: \sup_n |\xi_n(x)| \leqslant M] \\
&= \lim_{n\to+\infty} \mu[x \in \tilde{X}: \sup_{1\leqslant\nu\leqslant n} |\xi_\nu(x)| \leqslant M] \\
&\leqslant \lim_{n\to+\infty} \mu[x \in \tilde{X}: |\xi_n(x)| \leqslant M] \\
&= \lim_{n\to+\infty} \frac{1}{\sqrt{2\pi}\,|\xi_n|} \int_{-M}^{M} \exp\Big[-\frac{u^2}{2|\xi_n|^2}\Big] du \\
&= \lim_{n\to+\infty} \frac{1}{\sqrt{2\pi}} \int_{-\frac{M}{|\xi_n|}}^{\frac{M}{|\xi_n|}} \exp\Big[-\frac{u^2}{2}\Big] du = 0.
\end{aligned}$$

This contradicts (3. 11) and concludes the proof.

Let $\mathfrak{X}^*$ be the Hilbert space obtained by the completion of $\tilde{X}^*$ with respect to the inner product (ξ, η), and let $\mathfrak{X}$ be its topological dual space. By the definition (3. 8) of the norm $|\xi|$, the relation (1. 2) is valid for μ and the canonical Gaussian cylinder measure $\mu_{\mathfrak{X}}$ of the Hilbert space $\mathfrak{X}$.

This means that μ is a σ-extension of $\mu_{\mathfrak{X}}$ in $\tilde{X}$. On the other hand, it is easy to see that the system $\{\xi_\alpha/|\xi_\alpha|: \alpha \in \Lambda_0\}$ is a C.O.N.S. (complete orthonormal system) in $\mathfrak{X}^*$. Since Λ_0 is at most countable, $\mathfrak{X}$ is a separable Hilbert space.

LEMMA 4. *$\mathfrak{X}$ is a subspace of $\tilde{X}$.*

Proof. The measure μ extends to a Gaussian measure μ^* on $\tilde{X}^{**}$ by (2. 2), where $\tilde{X}^{**}$ is the topological dual for $\tilde{X}^*$. Then $\tilde{X}$ is a measurable subset of $\tilde{X}^{**}$ and $\mu^*(\tilde{X}^{**}) = \mu^*(\tilde{X}) = 1$ is true (see the proof of Theorem 1). Since $\tilde{X}^*$ is included in $\mathfrak{X}^*$ its dual $\mathfrak{X}$ is included in $(\tilde{X}^*)^* = \tilde{X}^{**}$. The relation (1. 2) is also valid for μ^* and $\mu_{\mathfrak{X}}$. Therefore, by identifying $\mathfrak{X}^*$ and $\mathfrak{X}$, for every $x_0 \in \mathfrak{X}\,(= \mathfrak{X}^*)$

$$\mu^*[\tilde{X} + x_0] = \mu^*[\tilde{X}] = 1, \tag{3. 12}$$

due to the fact that μ^* is quasi-invariant. (Y. Umemura [12]). On the other hand, if $\mathfrak{X}$ is not a subspace of $\tilde{X}$, namely, if there exists x_0 in $\mathfrak{X}$ which is not in $\tilde{X}$, then we have

$$[\tilde{X} + x_0] \cap \tilde{X} = \phi. \tag{3. 13}$$

For, if there exists y in $[\tilde{X} + x_0] \cap \tilde{X}$, then there exists y' in $\tilde{X}$ such that $y = y' + x_0$. Since $\tilde{X}$ is a linear space, $x_0 = y - y'$ is in $\tilde{X}$. This is a contradiction to the assumption on x_0 and (3. 13) is true. Thus we have

$$1 = \mu^*[\tilde{X}^{**}] \geqslant \mu^*[[\tilde{X} + x_0] \cup \tilde{X}] = 2.$$

This contradicts (3. 12), which proves the lemma.

LEMMA 5. *$\mathfrak{X}$ is dense in $\tilde{X}$.*

Proof. Let $\bar{\mathfrak{X}}$ be the closure of $\mathfrak{X}$ in $\tilde{X}$. If there exists x_0 in $\tilde{X} - \bar{\mathfrak{X}}$, then, by the Hahn-Banach theorem, there exists $\xi \neq 0$ in $\tilde{X}^*$ such that $\xi(x) = 0$ on $\mathfrak{X}$. On the other hand, let $|x|_0$ be the norm on $\mathfrak{X}$. Then we have

$$|\xi| = \sup_{\substack{|x|_0 = 1 \\ x \in \mathfrak{X}}} |\xi(x)| = 0.$$

According to (3. 9), this means $\xi = 0$ in $\tilde{X}^*$ and contradicts the choice of ξ. Therefore $\tilde{X} = \bar{\mathfrak{X}}$, that is, $\mathfrak{X}$ is dense in $\tilde{X}$.

COROLLARY. *$\tilde{X}$ is separable.*

Proof. The space $\mathfrak{X}$ is a separable Hilbert space and, by Lemma 5, is dense in $\tilde{X}$. Furthermore, the norm $|x|_0$ on $\mathfrak{X}$ is stronger than that on $\tilde{X}$. Therefore $\tilde{X}$ is separable.

Summing up these results, we can derive the following theorem.

THEOREM 2. (*A*). *Let μ be a Gaussian measure on a separable or reflexive Banach space. Then there exists a separable closed linear subspace $\tilde{X}$ such that $\mu[\tilde{X}] = 1$ and* (3. 6) *is valid in $\tilde{X}^*$.*

(*B*). *Let μ be a Gaussian measure on a separable Banach space $\tilde{X}$, and assume that* (3. 6) *is valid in $\tilde{X}^*$. Then there exists a dense Hilbert subspace $\mathfrak{X}$ of $\tilde{X}$ such that μ is an abstract Wiener measure, that is, μ is a σ-extension in $\tilde{X}$ of the canonical Gaussian cylinder measure $\mu_{\mathfrak{X}}$ of $\mathfrak{X}$. The norm $\|x\|$ is admissible on $\mathfrak{X}$.*

COROLLARY. *There is no admissible norm on a nonseparable Hilbert space $\mathfrak{X}$.*

Proof. Suppose that a norm $\|x\|$ on $\mathfrak{X}$ is admissible, X be the completion of $\mathfrak{X}$ in the norm $\|x\|$, and let μ be the σ-extension in X of the canonical Gaussian cylinder measure $\mu_{\mathfrak{X}}$ of $\mathfrak{X}$. Since $\mathfrak{X}$ is dense in X and $\|x\| = 0$ implies $x = 0$ in $\mathfrak{X}$, we can show that X^* is a dense subspace of $\mathfrak{X}^*$ and (3. 6) is valid in X^* in the manner similar to that used in the proof of Lemma 5. Therefore, we can choose a C.O.N.S. $\{\xi_\alpha^0 \colon \alpha \in \Lambda\}$ of $\mathfrak{X}^*$ from X^*. Λ is an uncountable set since $\mathfrak{X}^*$ is nonseparable. Let $\xi_\alpha = \xi_\alpha^0 / \|\|\xi_\alpha^0\|\|$; $\alpha \in \Lambda$. Then (3. 1) is valid for $\{\xi_\alpha \colon \alpha \in \Lambda\}$. On the other hand, considering (3. 6), $v(\xi_\alpha) = \frac{1}{\|\|\xi_\alpha^0\|\|} \neq 0$ for every $\alpha \in \Lambda$. This contradicts Lemma 1.

4. Admissible norm.

Let $\mathfrak{X}$ be a separable Hilbert space with norm $|x|$ and inner product (x, y). We study the condition under which a Hilbertian norm on $\mathfrak{X}$ is admissible.

LEMMA 6(*). *Let H be a separable Hilbert space and let μ be a Gaussian cylinder measure on $(H, \mathfrak{A}_H)$, that is, for every $\xi \in H^*$, $\xi(x)$ is a Gaussian random variable on $(H, \mathfrak{A}_H, \mu)$ with mean $m(\xi)$ and variance $v(\xi)$. (In this lemma, we do not assume zero mean.)*

(*) This lemma was suggested by Prof. K. Ito.

https://doi.org/10.1017/S002776300001312X Published online by Cambridge University Press

Then μ has a σ-additive extension to $(H, \overline{\mathfrak{A}}_H)$ if and only if the characteristic functional of μ is of the form

$$\int_H e^{i\xi(x)} d\mu(x) = \exp\left[i\langle\xi, m\rangle - \frac{1}{2}\|S\xi\|^2\right], \quad \xi \in H^*, \tag{4. 1}$$

where m is an element of H, S is a nonnegative self-adjoint Hilbert-Schmidt operator and $\|\xi\|$ is the norm on H^*.

Proof. The sufficiency is derived from V.V. Sazonav [6].

We have only to prove the necessity. Assume that there exists a σ-additive extension to $(H, \overline{\mathfrak{A}}_H)$ and denote it by μ again. Identify H^* and H and let $\langle\cdot, \cdot\rangle$ be its inner product and $\|\cdot\|$ be its norm. Then $\langle\xi, x\rangle$; $\xi \in H^*(= H)$, $x \in H$ denotes the natural linear form.

Let $\{\xi_n\}$ be a sequence in H convergent to zero. Then $\langle\xi_n, x\rangle$ converges to zero for all x in H. Since $\{\langle\xi_n, x\rangle\}$ is a Gaussian random sequence on $(H, \overline{\mathfrak{A}}_H, \mu)$,

$$m(\xi_n) = \int_H \langle\xi_n, x\rangle \, d\mu(x) \tag{4. 2}$$

converges to zero (§33, Lemma 1 of K. Ito [14]). Therefore $m(\xi)$ is a continuous linear functional on H^* and there exists $m \in H$ such that

$$m(\xi) = \langle\xi, m\rangle \quad \text{for any} \quad \xi \in H. \tag{4. 3}$$

Next, let $\{\varphi_j\}$ be a C.O.N.S. in H, and, for m and for every ξ, x in H, set

$$\begin{aligned} m_j &= \langle\varphi_j, m\rangle, \\ x_j &= x_j(x) = \langle\varphi_j, x\rangle, \quad j = 1, 2, 3, \cdots, \\ \xi_j &= \xi_j(\xi) = \langle\varphi_j, \xi\rangle. \end{aligned} \tag{4. 4}$$

Then obviously

$$\mu[x \in H: \sum_{j=1}^{+\infty} x_j(x)^2 < +\infty] = \mu[H] = 1. \tag{4. 5}$$

On the other hand, let

$$\begin{aligned} \xi^N &= \sum_{j=1}^{N} \xi_j \varphi_j, \quad N = 1, 2, 3, \cdots \\ v_{ij} &= \int_H (x_j(x) - m_j)(x_i(x) - m_i) d\mu(x), \\ &\qquad i, j = 1, 2, 3, \cdots. \end{aligned} \tag{4. 6}$$

Then

$$\int_H \exp[i\langle\xi^N, x\rangle]\, d\mu(x)$$

$$= \int_H \exp\Big[i \sum_{j=1}^{N} \xi_j x_j(x)\Big] d\mu(x)$$

$$= \exp\Big[i \sum_{j=1}^{N} m_j \xi_j - \frac{1}{2} \sum_{k,j=1}^{N} v_{kj}\xi_k\xi_j\Big]. \tag{4. 7}$$

Averaging both sides of (4. 7) with respect to the measure

$$(2\pi)^{-\frac{N}{2}} \exp\Big[-\frac{1}{2} \sum_{j=1}^{N} \xi_j^2\Big] d\xi_1 d\xi_2 \cdots d\xi_N,$$

we have

$$\int_H \exp\Big[-\frac{1}{2} \sum_{j=1}^{N} x_j(x)^2\Big] d\mu(x) \leqslant \frac{1}{\sqrt{1+\sum_{j=1}^{N} v_{jj}}}. \tag{4. 8}$$

If $\sum_{j=1}^{+\infty} v_{jj}$ is divergent, then from (4. 8) we have

$$\int_H \exp\Big[-\frac{1}{2} \sum_{j=1}^{+\infty} x_j(x)^2\Big] d\mu(x) = 0,$$

and

$$\exp\Big[-\frac{1}{2} \sum_{j=1}^{+\infty} x_j(x)^2\Big] = 0, \quad \text{a.e.}\,.$$

Therefore

$$\mu\Big[\sum_{j=1}^{+\infty} x_j(x)^2 = +\infty\Big] = 1.$$

This contradicts (4. 5) and we have,

$$\sum_{j=1}^{+\infty} v_{jj} < +\infty. \tag{4. 9}$$

Define a linear operator V on H by

$$\langle V\varphi_i, \varphi_j\rangle = v_{ij}, \quad i, j = 1, 2, 3, \cdots. \tag{4. 10}$$

Then V is a nonnegative self-adjoint operator on H and further, it is nuclear, since

$$\sum_{j=1}^{+\infty} \langle V\varphi_j, \varphi_j\rangle = \sum_{j=1}^{+\infty} v_{jj} < +\infty.$$

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Let S be $\sqrt{V}$. Then it is easy to see that S is the required Hilbert-Schmidt operator. Thus we have proved the lemma.

COROLLARY 1. *The canonical Gaussian cylinder measure $\mu_{\mathfrak{X}}$ on a Hilbert space $\mathfrak{X}$ does not have a σ-additive extension to $(\mathfrak{X}, \overline{\mathfrak{A}}_{\mathfrak{X}})$.*

Proof. The characteristic functional of $\mu_{\mathfrak{X}}$ is

$$\int_{\mathfrak{X}} \exp[i\xi(x)]d\mu_{\mathfrak{X}}(x) = \exp\left[-\frac{1}{2}|\xi|^2\right]$$
$$= \exp\left[-\frac{1}{2}|I\xi|^2\right], \tag{4. 11}$$

where $|\xi|$ is the norm on $\mathfrak{X}^*$ and I is the identity. But I is not of Hilbert-Schmidt type. Therefore, by Lemma 6, $\mu_{\mathfrak{X}}$ does not have a σ-additive extension to $(\mathfrak{X}, \overline{\mathfrak{A}}_{\mathfrak{X}})$.

COROLLARY 2. *In Lemma* 6, *if μ has a σ-additive extension to $(H, \overline{\mathfrak{A}}_H)$ and mean zero, then for every $\xi, \eta \in H^*(=H)$*

$$\int_H \xi(x)\eta(x)d\mu(x) = \langle S\xi, S\eta\rangle, \tag{4. 12}$$

where S is the Hilbert-Schmidt operator determined by (4. 1).

Utilizing Lemma 6, we have the following theorem.

THEOREM 3. *A Hilbertian norm $\|x\|$ on a separable Hilbert space $\mathfrak{X}$ is admissible if and only if there exists a one to one Hilbert-Schmidt operator S_0 such that*

$$\|x\| = |S_0x|, \qquad x \in \mathfrak{X}, \tag{4. 13}$$

where $|x|$ is the initial norm on $\mathfrak{X}$.

Proof. The sufficiency is well-known (for example, see Y. Umemura [12]).

We prove the necessity. Let $\|x\|$ be a Hilbertian admissible norm induced by an inner product $\langle x, y\rangle$ on $\mathfrak{X}$ and let H be the completion of $\mathfrak{X}$ in the norm $\|x\|$. Then H is also a Hilbert space with the inner product $\langle x, y\rangle$. Let μ be the σ-extension in H of the canocial Gaussian cylinder measure $\mu_{\mathfrak{X}}$ of $\mathfrak{X}$. Then μ is a Gaussian measure on the Hilbert space H. Therefore, by Lemma 6, there exists a nonnegative Hilbert-Schmidt operator S on H^* determined by (4. 1). Since we are assuming mean zero, (4. 12) is also valid (Corollary 2 of Lemma 6).

Identifying $\mathfrak{X}$ and $\mathfrak{X}^*$, and remembering H^* is a subspace of $\mathfrak{X}^*(=\mathfrak{X})$, we have

$$|||S\xi|||^2 = \int_H \xi(x)^2 d\mu(x)$$

$$= \int_{\mathfrak{X}} (\xi, x)^2 d\mu_{\mathfrak{X}}(x) = |\xi|^2, \tag{4. 14}$$

for every ξ in H^* where $|||\xi|||$ is the norm on H^*. Consequently,

$$|||S\xi||| = |\xi|, \quad \text{for every } \xi \in H^*. \tag{4. 15}$$

Since $\|x\|=0$ implies $x=0$ in $\mathfrak{X}$ and so $|\xi|=0$ implies $\xi=0$ in H^*. Therefore, by (4. 15), $S\xi=0$ implies $\xi=0$ in H^* and S is a one to one operator.

Let $\{\lambda_j\}$ and $\{\varphi_j\}$ be eigenvalues and eigenvectors of S, respectively. Then $\lambda_j>0$, $j=1, 2, \cdots$, and $\sum_{j=1}^{+\infty} \lambda_j^2 < +\infty$ because S is a one to one Hilbert-Schmidt operator.

Further, since μ is the σ-extension of the canonical Gaussian cylinder measure $\mu_{\mathfrak{X}}$, we have

$$(\varphi_i, \varphi_j) = \int_{\mathfrak{X}} \varphi_j(x)\varphi_i(x) d\mu_{\mathfrak{X}}(x)$$

$$= \int_H \langle \varphi_i, x\rangle \langle \varphi_j, x\rangle \, d\mu(x)$$

$$= \langle S\varphi_i, S\varphi_j\rangle = \lambda_i \lambda_j \delta_{ij},$$

$$i, j = 1, 2, 3, \cdots.$$

Let $\phi_j = \lambda_j^{-1}\varphi_j$, $j=1, 2, 3, \cdots$. Then $\{\phi_j\}$ is a C.O.N.S. in $\mathfrak{X}$ and

$$\sum_{j=1}^{+\infty} |S\phi_j|^2 = \sum_{j=1}^{+\infty} |||S^2\phi_j|||^2 = \sum_{j=1}^{+\infty} |||\lambda_j \varphi_j|||^2$$

$$= \sum_{j=1}^{+\infty} \lambda_j^2 < +\infty.$$

Therefore S can be extended to a Hilbert-Schmidt operator on $\mathfrak{X}^*(=\mathfrak{X})$ and we denote it by S again. Let S_0 be the dual operator of S in $\mathfrak{X}$. Then S_0 is the required operator. In fact, since SH^* is dense in H^* and H^* is dense in $\mathfrak{X}^*(=\mathfrak{X})$, for every x in $\mathfrak{X}(\subset H)$,

$$\begin{aligned}\|x\| &= \sup_{\substack{|||\xi|||=1\\ \xi\in H^*}} |\xi(x)| = \sup_{\substack{|||S\xi|||=1\\ \xi\in H^*}} |(S\xi)(x)| \\ &= \sup_{\substack{|\xi|=1\\ \xi\in H^*}} |(S\xi, x)| = \sup_{\substack{|\xi|=1\\ \xi\in H^*}} |(\xi, S^*x)| \\ &= \sup_{\substack{|\xi|=1\\ \xi\in \mathfrak{X}^*}} |(\xi, S_0x)| = |S_0x|.\end{aligned}$$

The proof is now complete.

COROLLARY. *Let $\|x\|$ be an admissible norm on $\mathfrak{X}$. If there exists a Hilbertian admissible norm stronger than $\|x\|$ then for any C.O.N.S. $\{\varphi_j\}$ in $\mathfrak{X}$ we have*

$$\sum_{j=1}^{+\infty} \|\varphi_j\|^2 < +\infty. \tag{4. 16}$$

Proof. Suppose that $\|x\|'$ is a Hilbertian admissible norm stronger than $\|x\|$, say, $\|x\| \leqslant \|x\|'$. By Theorem 3, there exists a Hilbert-Schmidt operator S such that $\|x\|' = |Sx|$, $x \in \mathfrak{X}$. Then for any C.O.N.S. $\{\varphi_j\}$ in $\mathfrak{X}$,

$$\begin{aligned}\sum_{j=1}^{+\infty} \|\varphi_j\|^2 &\leqslant \sum_j \|\varphi_j\|'^2 \\ &= \sum_j |S\varphi_j|^2 < +\infty.\end{aligned}$$

This was to be proved.

Next we give some examples of admissible norms on a separable Hilbert space $\mathfrak{X}$.

EXAMPLE 1. Define

$$\|x\|_1 = |Sx|, \qquad x \in \mathfrak{X},$$

where S is a one to one Hilbert-Schmidt operator on $\mathfrak{X}$. Then $\|x\|_1$ is a measurable norm (Section 1). Therefore, by Theorem 3, every Hilbertian admissible norm is a measurable norm.

EXAMPLE 2. Define

$$\|x\|_2 = \sup_n \frac{1}{\sqrt{n}} |(\varphi_n, x)|, \qquad x \in \mathfrak{X}$$

where $\{\varphi_n\}$ is a C.O.N.S. in $\mathfrak{X}$. Then $\|x\|_2$ is a measurable norm but there is no Hilbertian admissible norm stronger than $\|x\|_2$.

In fact it is evident that $\|x\|_2$ is a norm on $\mathfrak{X}$. To prove that $\|x\|_2$ is a measurable norm, we imbed $\mathfrak{X}$ in a measurable space $(\Omega, \overline{\mathfrak{A}})$ in which $\mathfrak{X}$ is an $\overline{\mathfrak{A}}$-measurable subspace and all functions (φ_n, x), $n = 1, 2, 3, \cdots$ are extended to $\overline{\mathfrak{A}}$-measurable functions on Ω, further, there exists a σ-additive extension μ of $\mu_{\mathfrak{X}}$. As an example of such a space, we can choose the space of all sequences.

Then since μ is a σ-extension of $\mu_{\mathfrak{X}}$, we have

$$\begin{aligned}
&\mu[x \in \Omega\colon \|x\|_2 < +\infty] \\
&= \mu\Big[x \in \Omega\colon \sup_n \frac{1}{\sqrt{n}} |(\varphi_n, x)| < +\infty\Big] \\
&= \lim_{N\to+\infty} \lim_{M\to+\infty} \mu\Big[x \in \Omega\colon \sup_{1\leqslant n\leqslant N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| \leqslant M\Big] \\
&= \lim_{N\to+\infty} \lim_{M\to+\infty} \mu_{\mathfrak{X}}\Big[x \in \mathfrak{X}\colon \sup_{1\leqslant n\leqslant N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| \leqslant M\Big] \\
&= \lim_N \lim_M \prod_{n=1}^{N} \Big\{\frac{1}{\sqrt{2\pi}} \int_{-M\sqrt{n}}^{M\sqrt{n}} \exp\Big[-\frac{u^2}{2}\Big] du\Big\} \\
&= \lim_N \lim_M \prod_{n=1}^{N} \Big\{1 - \sqrt{\frac{2}{\pi}} \int_{M\sqrt{n}}^{+\infty} \exp\Big[-\frac{u^2}{2}\Big] du\Big\} \\
&\geqslant \lim_N \lim_M \prod_{n=1}^{N} \Big\{1 - \sqrt{\frac{2}{\pi}} \frac{1}{M\sqrt{n}} \exp\Big[-\frac{M^2}{2} n\Big]\Big\} \\
&\geqslant \lim_N \lim_M \prod_{n=1}^{N} \Big\{1 - \exp\Big[-\frac{M^2}{2} n\Big]\Big\} \\
&\geqslant \lim_N \lim_M \Big\{1 - \sum_{n=1}^{N} \exp\Big[-\frac{M^2}{2} n\Big]\Big\} \\
&= \lim_N \lim_M \left\{1 - \frac{1 - \exp\Big[-\frac{M^2}{2} N\Big]}{1 - \exp\Big[-\frac{M^2}{2}\Big]} \exp\Big[-\frac{M^2}{2}\Big]\right\} \\
&= 1,
\end{aligned}$$

and for any positive number ε

$$\begin{aligned}
&\mu[x \in \Omega\colon \|x\|_2 < \varepsilon] \\
&= \lim_{N\to+\infty} \mu_{\mathfrak{X}}\Big[x \in \mathfrak{X}\colon \sup_{1\leqslant n\leqslant N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| < \varepsilon\Big] \\
&\geqslant \prod_{n=1}^{+\infty} \Big\{1 - \frac{1}{\varepsilon\sqrt{n}} \exp\Big[-\frac{\varepsilon^2}{2} n\Big]\Big\} > 0,
\end{aligned}$$

https://doi.org/10.1017/S002776300001312X Published online by Cambridge University Press

because

$$\sum_{n=1}^{+\infty} \frac{1}{\varepsilon\sqrt{n}} \exp\left[-\frac{\varepsilon^2}{2} n\right] \leqslant \frac{1}{\varepsilon} \frac{\exp\left[-\frac{\varepsilon^2}{2}\right]}{1-\exp\left[-\frac{\varepsilon^2}{2}\right]} < +\infty.$$

Therefore, by Corollary 4. 5 of L. Gross [2], $\|x\|_2$ is a measurable norm.

While for the C.O.N.S. $\{\varphi_n\}$ in $\mathfrak{X}$

$$\sum_{n=1}^{+\infty} \|\varphi_n\|_2{}^2 = \sum_{n=1}^{+\infty} \left\{\sup_\nu \frac{1}{\sqrt{\nu}} |(\varphi_\nu, \varphi_n)|\right\}^2$$

$$= \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty.$$

By Corollary of Theorem 3, there is no Hilbertian admissible norm stronger than $\|x\|_2$. This means that there is no Hilbert space of full measure which is included in the Banach space obtained by the completion of $\mathfrak{X}$ in the norm $\|x\|_2$.

EXAMPLE 3. Define

$$\|x\|_3 = \left[\sup_n \frac{1}{n} \sum_{\nu=1}^{n} |(\varphi_\nu, x)|^2\right]^{\frac{1}{2}}, \quad x \in \mathfrak{X}$$

where $\{\varphi_n\}$ is a C.O.N.S. in $\mathfrak{X}$. Then $\|x\|_3$ is an admissible norm on $\mathfrak{X}$ but not a measurable norm.

Proof. Imbed $\mathfrak{X}$ in the measurable space $(\Omega, \overline{\mathfrak{A}}, \mu)$ as in Example 2. Then by the law of large number, we have

$$\mu[x \in \Omega\colon \|x\|_3 < +\infty]$$

$$\geqslant \mu\left[x \in \Omega\colon \limsup_n \frac{1}{n} \sum_{\nu=1}^{n} |(\varphi_\nu, x)|^2 = 1\right] = 1.$$

Therefore $\|x\|_3$ is an admissible norm; but, according to Corollary 4. 5 of L. Gross [2], it is not a measurable norm. This means that for a norm on a separable Hilbert space to be admissible, it is not necessary to be a measurable norm in the sense of L. Gross [1].

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