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FINITE GROUPS WHOSE CHARACTER CODEGREES ARE CONSECUTIVE INTEGERS

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Abstract

Let G be a finite group. We investigate the structure of finite groups whose irreducible character codegrees are consecutive integers.

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1. Introduction

Throughout this paper, G always denotes a finite group. As usual, $\mathrm{Irr}(G)$ denotes the set of complex irreducible characters of G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the set of character degrees. A number of papers, such as [5–7], have studied the influence of the set $\mathrm{cd}(G)$ on the structure of G. In particular, Huppert in [3, Theorem 32.1] considered finite groups whose irreducible character degrees are consecutive integers and showed that if $\mathrm{cd}(G) = \{1, 2, \ldots, k-1, k\}$, then G is solvable if and only if $k \leqslant 4$, and that if $k > 4$, then $k = 6$ and $G = HZ(G)$, where $H \cong \mathrm{SL}(2, 5)$.

Inspired by these results, we consider the analogous problem related to the character codegrees. The concept of character codegrees was first introduced by Qian *et al.* in [9] as follows. For $\chi \in \mathrm{Irr}(G)$, the *codegree* of χ is defined to be

$$\mathrm{cod}\,\chi = \frac{|G : \ker\chi|}{\chi(1)}.$$

Recently many papers have studied character codegrees (see, for instance, [4, 8, 10]). Let $\mathrm{Cod}(G) = \{\mathrm{cod}\,\chi \mid \chi \in \mathrm{Irr}(G)\}$ be the set of irreducible character codegrees of G. The aim of this paper is to investigate finite groups whose irreducible character codegrees are consecutive integers. We have the following result.

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THEOREM 1.1. *Let G be a group with* $\mathrm{Cod}(G) = \{1, 2, \ldots, n-1, n\}$, *where n is a positive integer. Then $n \leqslant 3$ and one of the following holds:*

(1) *if $n = 1$, then $G = 1$;*
(2) *if $n = 2$, then G is an elementary abelian 2-group;*
(3) *if $n = 3$, then $G = N \rtimes H$ is a Frobenius group with an elementary abelian 3-group as its kernel, $N = G'$ and H is cyclic of order 2.*

2. Preliminaries

We begin with the following basic lemma concerning character codegrees, which will be used frequently in our proofs.

LEMMA 2.1 [9, Lemma 2.1]. *Let G be a group and $\chi \in \mathrm{Irr}(G)$.*

(1) *If N is a normal subgroup of G, then $\mathrm{Cod}(G/N) \subseteq \mathrm{Cod}(G)$.*
(2) *If N is subnormal in G and $\phi \in \mathrm{Irr}(N)$ is a constituent of χ_N, then $\mathrm{cod}\,\phi \mid \mathrm{cod}\,\chi$.*

Next we recall the concept of the codegree graph, which was first introduced in [9]. The codegree graph $\Gamma(G)$ is a graph whose vertex set $V(G)$ is the set of all primes dividing $\mathrm{cod}\,\chi$ for some $\chi \in \mathrm{Irr}(G)$ and there is an edge between two distinct primes p and q if pq divides $\mathrm{cod}\,\chi$ for some $\chi \in \mathrm{Irr}(G)$. We present the following facts on the codegree graph $\Gamma(G)$.

LEMMA 2.2 [9, Theorems A and E]. *Let G be a group and $\pi(G)$ be the set of prime divisors of $|G|$.*

(1) *$\pi(G)$ coincides with $V(G)$, the vertex set of $\Gamma(G)$.*
(2) *For any subset $\Delta \subseteq \pi(G)$ with $|\Delta| \geqslant 3$, there are two distinct primes $p, q \in \Delta$ so that there is an edge between p and q.*
(3) *$\Gamma(G)$ is not connected if and only if G is a Frobenius group or a 2-Frobenius group.*

3. Proof of Theorem 1.1

We start by proving the following result concerning number theory, which plays a very important role in determining the integer n when $\mathrm{Cod}(G) = \{1, 2, \ldots, n-1, n\}$.

PROPOSITION 3.1. *Let n be an integer and r, q, p be three consecutive primes so that $2 < r < q < p \leqslant n$ and p is the largest prime less than or equal to n. Then $n < 2p$ and $n < rq$.*

PROOF. Assume that $n \geqslant 2p$. Then $p < 2p \leqslant n$. By Bertrand's postulate, there exists a prime, say s, so that $p < s < 2p$. This contradicts the hypothesis that p is the largest prime less than or equal to n. Hence, $n < 2p$.

Now assume that $n \geqslant rq$. Applying Bertrand's postulate again, we see that $q < p < 2q$ and so $q < p < 2q < 3q \leqslant rq \leqslant n$. By [1, Theorem 1.3], there is a prime between $2q$ and $3q$. This is a contradiction. Thus, $n < rq$. □

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Proposition 3.1 enables us show that the integer n will not be too large.

PROPOSITION 3.2. *Let G be a group with* $\mathrm{Cod}(G) = \{1, 2, \ldots, n-1, n\}$, *where n is a positive integer. Then $n \leqslant 6$ and $n \neq 5$.*

PROOF. Assume that $n \geqslant 7$. Then there are three consecutive primes r, q, p as defined in Proposition 3.1. Thus, $n < 2p$ and $n < rq$. Consider the subset $\Delta = \{r, p, q\} \subseteq V(G) = \pi(G)$. By Lemma 2.2(2), there exists $\chi \in \mathrm{Irr}(G)$ so that $pq \mid \mathrm{cod}\,\chi$, $rq \mid \mathrm{cod}\,\chi$ or $rp \mid \mathrm{cod}\,\chi$. It follows from Proposition 3.1 that $n \geqslant \mathrm{cod}\,\chi \geqslant \min\{pq, rq, rp\} > n$. This is a contradiction. Thus, $n \leqslant 6$. Similarly, by Lemma 2.2(2), $n \neq 5$. □

With the above proposition, to prove Theorem 1.1, we only need to classify the groups when $1 \leqslant n \leqslant 3$ and show that $n \neq 4, 6$. Notice that if $n \leqslant 6$, the codegree graph $\Gamma(G)$ is not connected. Then by Lemma 2.2(3), G is a Frobenius group or a 2-Frobenius group. So we need to understand the structure of Frobenius groups. In particular, we give the following proposition.

PROPOSITION 3.3. *Let $G = N \rtimes H$ be a Frobenius group with kernel N. Suppose that $\pi(N) = \{p_1, p_2, \ldots, p_s\}$. Then the following statements hold.*

(1) *If $\phi \in \mathrm{Irr}(N)$, then $\mathrm{cod}\,\phi \mid \mathrm{cod}\,\chi$ for some $\chi \in \mathrm{Irr}(G)$. In particular, $\prod_{i=1}^{s} p_i \mid \mathrm{cod}\,\chi$ for some $\chi \in \mathrm{Irr}(G)$.*
(2) $\mathrm{Cod}(G) = \mathrm{Cod}(G/N) \bigcup \{\mathrm{cod}(\phi^G) \mid 1_N \neq \phi \in \mathrm{Irr}(N)\}$. *Furthermore,* $\mathrm{cod}(\phi^G)$ *divides* $|N|$ *if* $1_N \neq \phi \in \mathrm{Irr}(N)$.

PROOF. (1) The first part follows from Lemma 2.1(2) immediately. Notice that N is nilpotent. Then $N = P_1 \times P_2 \times \cdots \times P_s$, where P_i is a Sylow p_i-subgroup of N. Let $1_{P_i} \neq \lambda_i \in \mathrm{Irr}(P_i)$ and set $\phi = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_s$. Then $\phi \in \mathrm{Irr}(N)$ and $\mathrm{cod}\,\phi = \prod_{i=1}^{s} \mathrm{cod}\,\lambda_i$ with $p_i \mid \mathrm{cod}\,\lambda_i$. Hence, by Lemma 2.1(2), $\prod_{i=1}^{s} p_i \mid \mathrm{cod}\,\chi$ for some $\chi \in \mathrm{Irr}(G)$, as required.

(2) It is well known that $\mathrm{Irr}(G) = \mathrm{Irr}(G/N) \bigcup \{\phi^G \mid 1_N \neq \phi \in \mathrm{Irr}(N)\}$. Thus, the first part is true. Notice that $\phi^G(1) = |G : N|\phi(1)$. Then

$$\mathrm{cod}(\phi^G) = \frac{|G : N||N|}{|G : N|\phi(1)|\ker \phi^G|} = \frac{|N|}{\phi(1)|\ker \phi^G|}$$

divides $|N|$, as required. □

PROPOSITION 3.4. *Let G be a group with $|\pi(G)| = 3$. Suppose that* $\mathrm{Cod}(G) \subseteq \{1, 2, 3, 4, 5, 6\}$. *Then G is not a Frobenius group.*

PROOF. We work by contradiction. Assume that $G = N \rtimes H$ is a Frobenius group with kernel N. By Lemma 2.2(1) and (2), $\pi(G) = \{2, 3, 5\}$ and $6 \in \mathrm{Cod}(G)$. First we consider the case when $|\pi(N)| = 2$. Since N is nilpotent, it follows from Proposition 3.3(1) that $\pi(N) = \{2, 3\}$ and N is a direct product of an elementary abelian 2-group and an elementary abelian 3-group. Notice that the complement H is a cyclic 5-group and $\mathrm{Cod}(H) \subseteq \mathrm{Cod}(G)$. Then H must be cyclic of order 5. Since there is $\phi \in \mathrm{Irr}(N)$

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so that $\mathrm{cod}(\phi^G) = |N|/|\ker\phi^G| = 6$, we have $\ker\phi^G < N$ and so $G/\ker\phi^G = N/\ker\phi^G \rtimes H\ker\phi^G/\ker\phi^G \cong C_6 \rtimes C_5 \cong C_{30}$. Hence, $30 \in \mathrm{Cod}(G)$, a contradiction.

Assume now that $|\pi(N)| = 1$. By Proposition 3.3(2), $\pi(N) = \{5\}$ and so N is elementary abelian. Since there is $\phi \in \mathrm{Irr}(N)$ so that $\mathrm{cod}(\phi^G) = |N|/|\ker\phi^G| = 5$, we have $\ker\phi^G < N$. Let $\overline{G} = G/\ker\phi^G$. Then $\overline{G} = \overline{N} \rtimes \overline{H}$ is a Frobenius group with kernel $\overline{N} \cong C_5$ and $\overline{H} \cong H$. Let $\overline{Q}$ be a Sylow 3-subgroup of $\overline{H}$. Then $\overline{Q}$ is cyclic of order 3. It follows that $\overline{NQ}$ is a Frobenius group of order 15. This is a contradiction since such a group does not exist.

Both cases are impossible. The proof is completed. □

For convenience, here we introduce the notation of 2-Frobenius groups. If G is a 2-Frobenius group, then there are normal subgroups N, M of G so that G/N is a Frobenius group with kernel M/N, and M is a Frobenius group with kernel N. We write $G = \mathrm{Frob}_2(G, M, N)$ to denote such a 2-Frobenius group.

Proof of Theorem 1.1. We first introduce two basic facts.

(A) $\mathrm{cod}\,\chi > \chi(1)$ if $1_G \neq \chi \in \mathrm{Irr}(G)$.
(B) If G is abelian, then $\mathrm{cod}\,\chi$ is equal to the order of χ in the group $\mathrm{Irr}(G) \cong G$.

There is nothing to prove when $n = 1$. Assume that $n \geqslant 2$. Applying fact (A), together with $2 \in \mathrm{Cod}(G)$, we see that there exists a linear character $\chi \in \mathrm{Irr}(G)$ such that $\mathrm{cod}\,\chi = 2$ and hence $\chi \in \mathrm{Irr}(G/G')$. Then it follows from fact (B) that $2 \mid |G : G'|$.

If $n = 2$, then by facts (A) and (B), G is an elementary abelian 2-group and (2) follows.

Assume that $n = 3$. Then by Lemma 2.2(1) and (3), $\pi(G) = \{2, 3\}$ and G is a Frobenius group or a 2-Frobenius group. First suppose that $G = N \rtimes H$ is a Frobenius group with kernel N. Since $2 \mid |G : G'|$, it follows that H is a 2-group and N is a 3-group. By Proposition 3.3, $\mathrm{Cod}(N) = \{1, 3\}$ and $\mathrm{Cod}(G/N) = \mathrm{Cod}(H) = \{1, 2\}$. Therefore, N is an elementary abelian 3-group and H is an elementary abelian 2-group. Notice that the complement H must be cyclic or a generalised quaternion group (see [2, Theorem 9.2.10]). Hence, H is cyclic of order 2. Since $6 \notin \mathrm{Cod}(G)$, we have $G' = N$. To complete the proof of (3), we only need to show that G is not a 2-Frobenius group. Assume that $G = \mathrm{Frob}_2(G, M, N)$ is a 2-Frobenius group. It follows from Proposition 3.3 that $\mathrm{Cod}(G/N) = \mathrm{Cod}(M) = \{1, 2, 3\}$. Similarly, $G/N \cong C_3^s \rtimes C_2$ and $M \cong C_3^t \rtimes C_2$ for some positive integers s and t. This is a contradiction. Hence, (3) follows.

Now we show that $n \neq 4$. If $n = 4$, then $\pi(G) = \{2, 3\}$ and G is a Frobenius group or a 2-Frobenius group. First assume that $G = N \rtimes H$ is a Frobenius group with kernel N. Then by a proof similar to that above, N is an elementary abelian 3-group and H is a 2-group with $\mathrm{Cod}(H) = \{1, 2, 4\}$. Together with the fact that the complement H must be cyclic or a generalised quaternion group, we have $H \cong C_4$ or Q_8. Notice that there exists $\phi \in \mathrm{Irr}(N)$ so that $\mathrm{cod}(\phi^G) = |N|/|\ker\phi^G| = 3$. It is obvious that $\ker\phi^G < N$. Then $G/\ker\phi^G = N/\ker\phi^G \rtimes H\ker\phi^G/\ker\phi^G \cong C_3 \rtimes C_4$ or $C_3 \rtimes Q_8$ is a Frobenius group, which is a contradiction. Thus, G cannot be a Frobenius group.

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Assume that $G = \mathrm{Frob}_2(G, M, N)$ is a 2-Frobenius group. Since G/N is Frobenius and $\mathrm{Cod}(G/N) \subseteq \mathrm{Cod}(G)$, we have $\mathrm{Cod}(G/N) = \{1, 2, 3\}$. Similarly, $\mathrm{Cod}(M) = \{1, 2, 3\}$. This cannot happen by statement (2) of this theorem. Hence, $n \neq 4$.

By Proposition 3.2, it remains to show that $n \neq 6$. If $n = 6$, then $\pi(G) = \{2, 3, 5\}$ and G is a Frobenius group or a 2-Frobenius group. It follows by Proposition 3.4 that G is not a Frobenius group. We may assume that $G = \mathrm{Frob}_2(G, M, N)$ is a 2-Frobenius group. By Proposition 3.3(1) and (2), $\mathrm{Cod}(G/N) \subseteq \mathrm{Cod}(G)$ and $\mathrm{Cod}(M/N) \subseteq \mathrm{Cod}(M) \subseteq \mathrm{Cod}(G)$. Since both G/N and M are Frobenius groups, it follows by Proposition 3.4 that $|\pi(G/N)| = |\pi(M)| = 2$. Write $\overline{G} = G/N$ and then $\overline{G} = \overline{M} \rtimes \overline{K}$, where $\overline{K}$ is the Frobenius complement. First consider the case when $\pi(\overline{K}) \neq \{2\}$. As $\overline{K}$ is cyclic and $\mathrm{Cod}(\overline{K}) \subseteq \mathrm{Cod}(G)$, we have $G' \leqslant M$ and $\overline{K}$ is cyclic of order 3 or 5. Notice that $2 \mid |G : G'|$. Then $\overline{K} \cong C_3$ and $\overline{M}$ is a 2-group. Since M is Frobenius and $\overline{M}$ is isomorphic to its complement, it follows that $\overline{M}$ is cyclic or a generalised quaternion group and hence $\overline{M} \cong C_2, C_4$ or Q_8. But $\overline{G} \cong \overline{M} \rtimes C_3$ is Frobenius, which is impossible. Assume now that $\pi(\overline{K}) = \{2\}$. It is obvious that $\pi(M) = \{3, 5\}$. Let $M = N \rtimes H$ be a Frobenius group with kernel N. By a similar proof, there exists $\phi \in \mathrm{Irr}(N)$ so that $G/\ker\phi^G \cong N/\ker\phi^G \rtimes H \cong C_{15}$. It follows that $15 \in \mathrm{Cod}(G)$, a contradiction. Hence, $n \neq 6$. The proof is completed. □

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