THE LAMINAR FREE-CONVECTION BOUNDARY LAYER ON A VERTICAL HEATED PLATE IN THE NEIGHBOURHOOD OF A DISCONTINUITY IN PLATE TEMPERATURE

R. K. SMITH [1]

(Received 2 January 1969: revised 10 February 1969)

Communicated by A. F. Pillow

1. Introduction

If velocity and temperature profiles are known at a particular distance along a vertical heated plate, the equations of motion determine the velocity and temperature at points downstream, for a given variation of plate temperature. The problem of continuing the boundary layer solution for given initial conditions was investigated by Goldstein [2], for the isothermal case of the laminar, incompressible flow past a flat plate, with a given streamwise variation of pressure gradient outside the boundary layer. He showed that the solution is not always free from singularities and developed an expansion procedure to calculate the flow downstream when these occurred. Typical singularities occur, for instance, near the leading edge of the plate where the no-slip condition is imposed on the plate surface and near the trailing edge, where this condition is relaxed to one of zero stress along the axis of symmetry of the wake.

The method of Goldstein has been applied by Rheinboldt [6] and Watson [8] to the flow with suction or blowing of the boundary layer. It is extended here to advance the solution for the free convection boundary layer along a vertical heated plate, past the height at which a discontinuity of plate temperature occurs. It is assumed that the plate is flat and is maintained at constant temperatures T_1, T_2, below and above a height L, above its lower edge. The environment is assumed to be at rest at uniform temperature T_0 and the Prandtl number of the fluid is taken to be unity [2]. Results are obtained for the two subcases, $T_2 > T_1 > T_0$ and $T_1 > T_2 > T_0$.

[1] Present address: Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia.

[2] The theory can be applied to fluids with Prandtl numbers *of order* unity but not to those with Prandtl numbers large or small compared to one--see § 7.

A pronounced, secondary thermal layer is formed at the temperature discontinuity and grows downstream along the plate. The velocity field responds more gradually to the discontinuity since the velocity conditions on the plate are unaltered. Hence, the subsequent motions of fluid particles near the plate are determined by the buoyancy field of the new thermal layer, which establishes itself downstream at a steady rate. The solutions emphasize the speed with which the temperature field adjusts itself to the abrupt change in plate temperature.

2. The equations of motion

The laminar free-convection flow about a vertical, uniformly heated plate has been studied by a number of authors (see Ostrach [5] for references). For a rigorous derivation of the equations, the reader is referred to Ostrach [4].

Let L be a representative length along the plate; T_0 and T_1 the ambient and plate temperature and β, ν, κ, the volumetric coefficients of expansion, kinematic viscosity and thermometric conductivity of the fluid. The flow is characterised by two dimensionless parameters; the Grashof number,

$$\mathrm{Gr} = g\beta(T_1 - T_0)L^3/\nu^2,$$

comparing buoyancy to viscous terms in the equations, and the Prandtl number $\sigma = \nu/\kappa$, comparing the molecular diffusivity of momentum to that of heat. Free-convection flows occur for a range of Gr much larger than unity.

The equations of motion in non-dimensional form are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \theta + \frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2}, \tag{2.3}$$

where x, y are co-ordinates measured along and perpendicular to the plate from the leading edge; u, v, are the velocity components in these directions and θ is a scaled temperature difference given by $\theta = (T - T_0)/(T_1 - T_0)$, where T is the temperature at a general position in the flow. (Here, scales L, $(\nu^2 L/\bar{\theta})^{\frac{1}{4}}$, $(L\bar{\theta})^{\frac{1}{2}}$, $(\nu^2\bar{\theta}/L)^{\frac{1}{4}}$, have been taken, corresponding to the quantities x, y, u, v, where $\bar{\theta} = g\beta(T_1 - T_0)$, is a scale for the buoyant acceleration term.)

For a plate maintained at a uniform temperature in an otherwise still environment, the full boundary conditions are:

$$\begin{aligned} &u = \theta = 0 && \text{on} \quad x = 0, y > 0; \\ &u = v = 0,\ \theta = 1 && \text{on} \quad y = 0, x > 0; \\ &u,\ \theta \to 0 && \text{as} \quad y \to \infty, x > 0. \end{aligned} \tag{2.4}$$

Equations (2.1)—(2.3) are reduced by introducing a stream function ψ, defined by the relations

$$u = \psi_y, \quad v = -\psi_x, \tag{2.5}$$

and taking

$$\begin{aligned} \psi &= x^{\frac{3}{4}} F(\zeta), \\ \theta &= G(\zeta), \end{aligned} \tag{2.6}$$

where

$$\zeta = y x^{-\frac{1}{4}}. \tag{2.7}$$

In terms of ψ equation (2.1) is satisfied identically and equations (2.2) and (2.3) reduce to two ordinary differential equations for F and G, thus

$$F''' + \tfrac{3}{4} F F'' - \tfrac{1}{2} F'^2 + G = 0, \tag{2.8}$$

$$G'' + \tfrac{3}{4} \sigma F G' = 0. \tag{2.9}$$

Also, from (2.5), the velocity components are given by

$$\begin{aligned} u &= x^{\frac{1}{2}} F', \\ v &= \tfrac{1}{4} x^{-\frac{1}{4}} (\zeta F' - 3F). \end{aligned} \tag{2.10}$$

Equations (2.8) and (2.9) for F and G have been solved numerically subject to conditions (2.4), for a wide range of Prandtl numbers and details are given in Ostrach [4]. The computations foı $\sigma = 1$ were repeated here using a routine facility for two-point boundaıy value problems of this type on the Manchester Atlas Computer. In this case, $F''(0) = .90797$ and $G'(0) = -.40103$.

3. The continuation problem

The continuation problem outlined in § 1 is to advance the solution to equations (2.1)—(2.3), given velocity and temperature profiles at $x = 0$ and the variation of temperature along the plate; i.e.

$$
(3.1)\qquad
\begin{aligned}
&u = u_0(y), \quad \theta = \theta_0(y) && \text{on } x = 0, y > 0;\\
&\left.\begin{aligned} &u = v = 0 \quad (x \geqq 0)\\ &\theta = \Theta_0 = \Theta_1 x + \Theta_2 x^2 + \cdots \; (x > 0), \end{aligned}\right\} && \text{on } y = 0;\\
&u,\, \theta \to 0 \quad \text{as} \quad y \to \infty,\; x > 0;
\end{aligned}
$$

where

$$u_0(y) = a_1 y + a_2 y^2 + a_3 y^3 + \cdots \; (a_1 \neq 0),$$

and

$$\theta_0(y) = b_0 + b_1 y + b_2 y^2 + \cdots,$$

near $y = 0$.

(Note: We restrict ourselves here to initial velocity profiles with a single zero at the origin; if u is finite at the origin, the problem becomes one of mixed or forced convection; if $a_1 = 0$, severe complications arise and the equations are unmanageable — see Goldstein op. cit.).

If there is no singularity at $x = 0$, we can expand ψ and θ as double power series in x and y. If these series are substituted into equations (2.1)—(2.3) and the boundary conditions (3.1) are satisfied, we find that certain relations must hold between the coefficients a_i, b_i and Θ_i in (3.1). The first few are

$$
(3.2)\qquad
\begin{aligned}
&b_0 + 2!\,a_2 = 0, && b_1 + 3!\,a_3 = 0, && 18b_3 + 5!\,a_5 = 0, \cdots,\\
&b_0 = \Theta_0, && b_2 = 0, && 3!\,b_3 = a_1 \Theta_1, \cdots.
\end{aligned}
$$

Further, if these conditions hold, we have

$$
(3.3)\qquad \left(\frac{\partial\theta}{\partial y}\right)_{y=0} = a_1 + \frac{4!\,a_4}{a_1} x + \cdots
$$

and

$$
(3.4)\qquad \left(\frac{\partial\theta}{\partial y}\right)_{y=0} = b_1 + \frac{4!\,(a_1 b_4 - a_4 b_1) + a_1 \Theta_0 \Theta_1}{2a_1^2} x + \cdots
$$

The coefficient a_1, a_4, b_1, b_4, etc. are not determined by the relations (3.2); once these are specified, we can find the skin friction and local heat transfer from the plate, which are proportional to (3.3) and (3.4) respectively.

If the relations (3.2) are not satisfied, there is an algebraic singularity at $x = 0$.

4. Inner expansion

With the stream function ψ defined by equation (2.5), we make the following transformation of variables

$$\xi = x^{\frac{1}{3}}, \quad \eta = \tfrac{1}{3}yx^{-\frac{1}{3}}, \tag{4.1}$$

and take

$$\psi = \xi^2 f(\xi, \eta), \tag{4.2}$$

and

$$\theta = \tfrac{1}{27}\xi^{-1}g(\xi, \eta). \tag{4.3}$$

Then, from (2.5) and (4.2), the velocity components are given by

$$\begin{aligned} u &= \tfrac{1}{3}\xi f_\eta; \\ v &= -\tfrac{1}{3}\xi^{-1}[2f+\xi f_\xi-\eta f_\eta]. \end{aligned} \tag{4.4}$$

Further, equations (2.2) and (2.3) become

$$f_\eta^2+\xi f_{\eta\xi} f_\eta-2ff_\eta-\xi f_\xi f_{\eta\eta} = f_{\eta\eta\eta}+g, \tag{4.5}$$

$$-f_\eta g+\xi g_\xi f_\eta-2fg_\eta-\xi f_\xi g_\eta = \frac{1}{\sigma}\, g_{\eta\eta}. \tag{4.6}$$

The transformed boundary conditions from (3.1) are

$$\left.\begin{aligned} f &= f_\eta = 0, && (\xi \geqq 0), \\ g &= 27\xi[\Theta_0+\Theta_1\xi^3+\Theta_2\xi^6+\cdots) && (\xi > 0), \end{aligned}\right\} \text{ on } \eta = 0, \tag{4.7a}$$

$$\left.\begin{aligned} f_\eta &= 3\xi^{-1}(a_1(3\xi\eta)+a_2(3\xi\eta)^2+\cdots), \\ g &= 27\xi(b_0+b_1(3\xi\eta)+\cdots \end{aligned}\right\} \text{ as } \xi \to 0 \text{ and } \eta \to \infty. \tag{4.7b}$$

Guided by this form, we expand f and g as power series in ξ. Thus

$$f = f_0+\xi f_1+\xi^2 f_2+\cdots, \tag{4.8}$$

$$g = \xi g_1+\xi^2 g_2+\xi^3 g_3+\cdots \tag{4.9}$$

where f_r and g_r are functions of η only. Inserting (4.8) and (4.9) into (4.5) and (4.6), we obtain two sets of ordinary differential equations for f_r and g_r. These are:

$$\begin{aligned} & f_0'''+2f_0f_0''-f_0'^2 = 0, \\ & f_1'''+2f_0f_1''-3f_0'f_1'+3f_0''f_1 = -g_1, \\ & f_2'''+2f_0f_2''-4f_0'f_2'+4f_0''f_2 = -g_2-3f_1f_1''+2f_1'^2, \\ & f_3'''+2f_0f_3''-5f_0'f_3'+5f_0''f_3 = -g_3-4f_2f_1''+5f_2'f_1'-3f_2''f_1, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} & g_1''+2\sigma f_0g_1' = 0, \\ & g_2''+2\sigma f_0g_2'-\sigma f_0'g_2 = -3\sigma f_1g_1', \\ & g_3''+2\sigma f_0g_3'-2\sigma f_0'g_3 = \sigma(f_1'g_2-3f_1g_2'-4f_2g_1'), \\ & g_4''+2\sigma f_0g_4'-3\sigma f_0'g_4 = \sigma(2f_1'g_3+f_2'g_2-3f_1g_3'-4f_2g_2'-5f_3g_1'). \end{aligned} \tag{4.11}$$

Comparing conditions (4.7) with the series (4.8) and (4.9), we obtain the following boundary conditions for f_r and g_r,

$$f_r = f'_r = 0 \quad \text{or} \quad \eta = 0, \tag{4.12a}$$

$$g_{3r+1} = 27\Theta_r, \quad g_{3r} = g_{3r+2} = 0 \quad \text{or} \quad \eta = 0, \tag{4.12b}$$

$$\lim_{\eta\to\infty} \frac{f'_r(\eta)}{\eta^{r+1}} = 3^{r+2}a_{r+1}, \tag{4.13}$$

$$\lim_{\eta\to\infty} \frac{g_{r-1}(\eta)}{\eta^r} = 3^{r+3}b_r. \tag{4.14}$$

The solution for f_0 having a double zeıo at the origin and satisfying (4.13) is

$$f_0 = \tfrac{9}{2}\, a_1\eta^2. \tag{4.15}$$

If we take

$$z = \alpha\eta = (9a_1)^{\frac{1}{3}}\eta, \tag{4.16}$$

then

$$f_0 = \tfrac{1}{2}\alpha z^2 = A_0 z^2, \tag{4.17}$$

and these last two equations define α and A_0.

The equation for g_1 is now

$$\frac{d^2g_1}{dz^2} + \sigma z^2 \frac{dg_1}{dz} = 0. \tag{4.18}$$

This equation can be integrated directly and the solution satisfying conditions (4.12b) and (4.14) is

$$g_1 = \frac{27(b_0-\Theta_0)}{\Gamma(\frac{1}{3})}\gamma(\tfrac{1}{3}, \tfrac{1}{3}\sigma z^3) + 27\Theta_0, \tag{4.19}$$

where

$$\gamma(n, x) = \int_0^x t^{n-1}e^{-t}dt,$$

is the incomplete gamma function and

$$\Gamma(n) - \gamma(n, x) \sim e^{-x}x^{n-1}\left(1 + \frac{n-1}{x} + \frac{(n-1)(n-2)}{x^2} + \dots\right) \text{ as } x \to \infty.$$

We note in passing that

$$g'_1 = \frac{27(b_0-\Theta_0)\sigma^{\frac{1}{3}}3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})}e^{-\frac{1}{3}\sigma z^3}. \tag{4.20}$$

In terms of z, the equations for f_r and g_r can be written

$$\frac{d^3 f_r}{dz^3}+z^2\frac{d^2 f_r}{dz^2}-(r+2)z\frac{df_r}{dz}+(r+z)f_r=-\frac{g_r}{9a_1}+\alpha^{-1}F_r, \tag{4.21}$$

and

$$\frac{d^2 g_r}{dz^2}+\sigma z^2\frac{dg_r}{dz}-\sigma(r-1)zg_r=\alpha^{-1}G_r, \tag{4.22}$$

where F_r and G_r are functions involving $f_i, g_i (1 \leqq i \leqq r-1)$ and their derivatives.

Complementary functions for f_r, satisfying equation (4.21) with zero right hand side, are obtained by a power series substitution

$$f_r=\sum_{n=0}^{\infty}c_n z^n.$$

The recurrence relation, obtained by equating powers of z, is

$$c_{n+3}=-\frac{(n-1)(n-r-2)}{(n+3)(n+2)(n+1)}c_n.$$

The solutions can be conveniently expressed as generalized hypergeometric functions of the type ${}_2F_2$ and three independent complementary functions f_{ri} $(i=1, 2, 3)$, are:

$$\begin{aligned} f_{r1}&={}_2F_2(-\tfrac{1}{3}, -\tfrac{1}{3}r-\tfrac{2}{3}; \tfrac{1}{3}, \tfrac{2}{3}; -\tfrac{1}{3}z^3),\\ f_{r2}&=z,\\ f_{r3}&=z^2\,{}_2F_2(\tfrac{1}{3}, -\tfrac{1}{3}r; \tfrac{5}{3}, \tfrac{4}{3}; -\tfrac{1}{3}z^3). \end{aligned} \tag{4.23}$$

Complementary functions for g_r $(r>1)$ are found as follows. The substitution $s=z^3$ in equation (4.22) with zero right hand side gives

$$9s\frac{d^2 g_r}{ds^2}+3(2+\sigma s)\frac{dg_r}{ds}-\sigma(r-1)g_r=0.$$

Inserting

$$g_r=\sum_{n=0}^{\infty}c_n s^{\rho+n},$$

into this equation and equating powers of s, we obtain the recurrence relation

$$c_{n+1}=\frac{-\frac{1}{3}\sigma\left(n+\rho-\dfrac{r+1}{3}\right)}{(n+\rho+1)(n+\rho+2)}c_n.$$

The coefficient of z^0 gives the indicial equation

$$3\rho(3\rho-1)=0,$$

from which $\rho = 0$, and $\rho = \frac{1}{3}$, give independent solutions. These solutions, g_{ri} $(i = 1, 2)$ say, can be expressed in terms of the confluent hypergeometric function. Thus, in terms of z,

$$(4.24)\qquad \begin{aligned} g_{r1} &= {}_1F_1\left(-\frac{r-1}{3};\ \tfrac{2}{3};\ -\tfrac{1}{3}\sigma z^3\right), \\ g_{r2} &= z\,{}_1F_1\left(-\frac{r-2}{3};\ \tfrac{4}{3};\ -\tfrac{1}{3}\sigma z^3\right). \end{aligned}$$

The boundary conditions on f_r and g_r as $z \to \infty$, are

$$(4.25)\qquad \lim_{z\to\infty} \frac{f_r'(z)}{z^{r+1}} = \left(\frac{3}{\alpha}\right)^{r+2} a_{r+1},$$

and

$$(4.26)\qquad \lim_{z\to\infty} \frac{g_{r+1}(z)}{z^r} = \frac{3^{r+3}}{\alpha^r} b_r.$$

To apply these conditions, we must investigate the asymptotic behaviour of f_r and g_r, and thus the behaviour of the functions f_{ri} and g_{ri}, for large z.

Asymptotic series for f_{r1} and f_{r3} have been obtained by Goldstein op. cit.. These are quoted below.

If $r = 3n$ for some integer n, f_{r1} is a finite series. Similarly for f_{r3} if $r = 3n+1$; g_{r1} if $r = 3n+1$; g_{r2} if $r = 3n-1$.

If $r \neq 3n-1$,

$$(4.27)\qquad \begin{aligned} f_{r1} \sim &-\frac{z^{r+2}\Gamma(\frac{1}{3})}{3^{(r+2)/3}\Gamma\left(\frac{r+4}{3}\right)} \sum_{\mu=0}^{N} \frac{\left(-\frac{r}{3}\right)_\mu \left(-\frac{r+2}{3}\right)_\mu}{(r+1-3\mu)\,\mu!} \left(\tfrac{1}{3}z^3\right)^{-\mu} \\ &+ \frac{\Gamma(\frac{1}{3})\Gamma\left(-\frac{r+1}{3}\right)}{3^{\frac{1}{3}}\Gamma\left(-\frac{r+2}{3}\right)} z + O(z^{-r-3N-1}), \end{aligned}$$

and

$$(4.28)\qquad \begin{aligned} f_{r3} \sim &\frac{z^{r+2}\Gamma(\frac{5}{3})}{3^{r/3}\Gamma\left(\frac{r+5}{3}\right)} \sum_{\mu=0}^{N} \frac{\left(-\frac{r}{3}\right)_\mu \left(-\frac{r+2}{3}\right)_\mu}{(r+1-3\mu)\,\mu!} \left(\tfrac{1}{3}z^3\right)^{-\mu} \\ &+ \frac{3^{\frac{1}{3}}\Gamma(\frac{5}{3})\Gamma\left(-\frac{r+1}{3}\right)}{\Gamma\left(-\frac{r}{3}\right)} z + O(z^{-r-3N-1}). \end{aligned}$$

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

If $r = 3n-1$, the terms with $\mu = n$, in the $\sum$-expression of each series must be omitted and the last term in (4.27) replaced by

$$(4.29)\quad \frac{(-1)^n \Gamma(\frac{1}{3}) z}{3^{\frac{1}{3}} \Gamma(n+1)\Gamma(-\frac{1}{3}-n)} [\log(\tfrac{1}{3}z^3) + \psi(n+1) - \psi(\tfrac{2}{3}) - \psi(-\tfrac{1}{3})],$$

and the last term in (4.28) replaced by

$$(4.30)\quad \frac{(-1)^r 3^{\frac{1}{3}} \Gamma(\frac{5}{3}) z}{\Gamma(n+1)\Gamma(\frac{1}{3}-n)} [\log(\tfrac{1}{3}z^3) + \psi(n+1) - \psi(\tfrac{4}{3}) - \psi(\tfrac{1}{3})],$$

where $\psi(z)$ [3] is the logarithmic derivative of $\Gamma(z)$.

The asymptotic expansion of the confluent hypergeometric function with negative argument is given by

$${}_1F_1(a; b; -s) \sim \frac{\Gamma(b)}{\Gamma(b-a)} \sum_{\mu=0}^{N} \frac{(a)_\mu (1+a-b)_\mu}{\mu!} s^{-a-\mu} + O(s^{-a-N-1}),$$

(see Slater, 1964). Thus, the expansions for g_{ri} are

$$(4.31)\quad g_{r1} \sim z^{r-1} \left(\frac{\sigma}{3}\right)^{(r-1)/3} \frac{\Gamma(\frac{2}{3})}{\Gamma\left(\frac{r+1}{3}\right)} \sum_{\mu=0}^{N} \frac{\left(-\frac{r-1}{3}\right)_\mu \left(-\frac{r-2}{3}\right)_\mu}{\mu!} (\tfrac{1}{3}\sigma z^3)^{-\mu},$$

and

$$(4.32)\quad g_{r2} \sim z^{r-1} \left(\frac{\sigma}{3}\right)^{(r-2)/3} \frac{\Gamma(\frac{4}{3})}{\Gamma\left(\frac{r+2}{3}\right)} \sum_{\mu=0}^{N} \frac{\left(-\frac{r-1}{3}\right)_\mu \left(-\frac{r-2}{3}\right)_\mu}{\mu!} (\tfrac{1}{3}\sigma z^3)^{-\mu}$$

The solution for f_1. The equation for f_1 is

$$(4.33)\quad \frac{d^3 f_1}{dz^3} + z^2 \frac{d^2 f_1}{dz^2} - 3z \frac{df_1}{dz} + 3f_1 = -\frac{g_1}{9a_1}$$

For large z,

$$g_1 \sim 27b_0 + O(\exp(-\tfrac{1}{3}\sigma z^3)),$$

and therefore, a particular integral for f_1, valid for large z, is

$$(4.34)\quad f_{1p} \sim -\frac{b_0}{2a_1} z^3.$$

The asymptotic behaviours of the complementary functions f_{ri}, obtained from the series (4.27) and (4.28) are:

[3] Only in this sentence does ψ *not* denote the stream function.

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

$$
(4.35) \qquad \begin{aligned} f_{11} &= 1-\tfrac{1}{2}z^3, \\ f_{12} &= z, \\ f_{13} &\sim Az^3+Bz+C, \end{aligned}
$$

where A, B, $C(=-2A)$, are constants determined by equation (4.28) with $r = 1$. In particular,

$$
A = 2\Gamma(\tfrac{2}{3})/3^{\frac{4}{3}}.
$$

The complete solution for f_1 may be written

$$
f_1 = f_{1p}+r_1 f_{11}+s_1 f_{12}+t_1 f_{13},
$$

where r_1, s_1, t_1, are constants. Moreover, the asymptotic behaviour of this solution has the form

$$
(4.36) \qquad f_1 \sim A_1 z^3+B_1 z+C_1.
$$

The solution with a double zero at the origin and satisfying (4.25) is obtained numerically as follows. Equation (4.33) is integrated outwards with starting values $f_1(0) = f_1'(0) = 0$, $f_1''(0) = 1$, and the derivatives calculated for a large value of z ($z = 7$, is found to be sufficiently large). This solution is regarded as a particular integral for f_1, say f_{1q}. Then, the general solution with a double zero at the origin is

$$
(4.37) \qquad f_1 = f_{1q}+\lambda_1 f_{13},
$$

for any value of λ_1. Further

$$
\begin{aligned} f_1'' &= f_{1q}''+\lambda_1 f_{13}'' \\ &\sim f_{1q}''+6\lambda_1 Az \quad \text{as} \quad z \to \infty. \end{aligned}
$$

From equation (4.36) [4] we see that $f_{1q}''/z \to$ a constant as $z \to \infty$ and using condition (4.25) we find

$$
(4.38) \qquad \lim_{z\to\infty} \frac{f_1''}{z} = \frac{6a_2}{a_1} = \lim_{z\to\infty} \frac{f_{1q}''}{z} + 6\lambda_1 A.
$$

The limit f_{1q}''/z is obtained numerically and λ_1 is then deduced from (4.38). Then from (4.37), we have

$$
f_1''(0) = f_{1q}''(0)+2\lambda_1 = 1+2\lambda_1.
$$

Equation (4.33) is now solved numerically with this as the appropriate second derivative at the origin. In the asymptotic expansion for this solution, the coefficient A_1 in (4.36) has been obtained by satisfying the

[4] It can be shown that an asymptotic power series, valid in a sector of the complex plane, can be differentiated (Erdélyi, [1]).

boundary condition as $z \to \infty$, i.e. $A_1 = a_2/a_1$. The corresponding coefficient B_1 is calculated numerically using the values for f_1' and f_1''. Hence

$$B_1 = \lim_{z\to\infty} (f_1' - \tfrac{1}{2} z f_1'').$$

Also

$$C_1 = -2A_1.$$

The solution for g_2. The equation for g_2 is

$$\frac{d^2 g_2}{dz^2} + \sigma z^2 \frac{dg_2}{dz} - \sigma z g_2 = -\frac{3\sigma}{\alpha} f_1 g_1. \tag{4.39}$$

The equation has a particular integral g_{2p}, which is exponentially small for large z. The complete solution is therefore

$$g_2 = g_{2p} + m g_{21} + n g_{22}, \tag{4.40}$$

where m and n are constants and for large z,

$$g_2 \sim \left(m \left(\frac{\sigma}{3}\right)^{\frac{1}{3}} \Gamma(\tfrac{2}{3}) + n \right) z. \tag{4.41}$$

Equation (4.39) is integrated outwards with starting values $g_2(0) = 0$, $g_2'(0) = 1$, to give a particular integral g_{2q}. The general solution with a single zero at the origin is then

$$\begin{aligned} g_2 &= g_{2q} + \mu g_{22} \\ &\sim (\alpha_q + \mu) z \quad \text{as} \quad z \to \infty, \end{aligned}$$

where $\alpha_q = \lim_{z\to\infty} g_2'$, is calculated numerically and μ is any constant. Then, using (4.26)

$$\lim_{z\to\infty} \frac{g_2}{z} = \frac{3^4 b_1}{\alpha} = \alpha_q + \mu,$$

giving μ. The integration is repeated with $g_2'(0)$ replaced by $1+\mu$ to give the required solution. For this,

$$g_2 \sim \alpha_1 z = \frac{3^4 b_1}{\alpha} z \quad \text{as} \quad z \to \infty. \tag{4.42}$$

The solution for f_2. The equation for f_2 is

$$\frac{d^3 f_2}{dz^3} + z^2 \frac{d^2 f_2}{dz^2} - 4z \frac{df_2}{dz} + 4 f_2 = -\frac{g_2}{9a_1} + \frac{2f_1'^2 - 3 f_1 f_1''}{\alpha}. \tag{4.43}$$

With the asymptotic series for f_1 given by (4.36) and g_2 given by (4.42), we can find a particular integral f_{2p} to this equation such that

$$f_{2p} \sim a' z^4 + b' z^2 + c' z + d', \quad \text{as} \quad z \to \infty,$$

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

where a', b', c', d' are constants depending on A_1, B_1, C_1 and α_1. The general solution for f_2 is

$$f_2 = f_{2p}+r_2 f_{21}+s_2 f_{22}+t_2 f_{23},$$

where r_2, s_2, t_2, are constants. Then, using the asymptotic series for the functions f_{2i} given above, we find that

$$f_2 \sim A_2 z^4+B_2 z^2+C_2' z \log z+C_2 z+D_2+E_2 z^{-2}+O(z^{-5}) \quad \text{as} \quad z \to \infty. \tag{4.44}$$

Also, from equation (4.28), we have

$$f_{23} \sim \frac{3^{\frac{1}{3}}\Gamma(\frac{2}{3})}{6} z^4+O(z^2) \quad \text{as} \quad z \to \infty.$$

Hence, the general solution for f_2 with a double zero at the origin is

$$f_2 = f_{2q}+\lambda_2 f_{23},$$

and

$$f_2' \sim 4\left(A_{2q}+\lambda_2 \frac{3^{\frac{1}{3}}\Gamma(\frac{2}{3})}{6}\right) z^3+O(z) \text{ as } z \to \infty, \tag{4.45}$$

where f_{2q} is the particular integral for which $f_{2q}''(0) = 1$; A_{2q} is the leading coefficient in the asymptotic series for this solution and λ_2 is any constant. The value of A_{2q} and the next three coefficients in the series (4.44) are obtained numerically. This value of A_{2q} is used together with (4.25) and (4.47) to find λ_2. The appropriate solution for f_2 and the corresponding coefficients A_2, B_2, C_2 and C_2' are there found by integrating equation (4.43) numerically with $f''(0) = 1+2\lambda_2$.

A similar procedure can be used for calculating g_3, f_3, g_4, f_4, g_5, $\cdots$ etc., in that order. However, the labour involved increases rapidly for higher terms. We therefore restrict ourselves to calculating four terms in each of the series for f and g. These appear sufficient to enable a reasonable account to be given of the initial development of the inner layer. The asymptotic series obtained for g_3, f_3 and g_4 are given below

$$\begin{aligned} &g_3 \sim \alpha_3 z^2+\beta_3+\gamma_3 z^{-1}+O(z^{-4}),\\ &f_3 \sim A_3 z^5+B_3 z^3+C_3' z^2 \log z+C_3 z^2+D_3 z+E_3' \log z+E_3+O(z^{-1}),\\ &g_4 \sim \alpha_4 z^3+\beta_4' \log z+\beta_4+O(z^{-1}). \end{aligned} \tag{4.48}$$

5. Outer expansion

The series expansions for ψ and θ obtained in § 4, satisfy the boundary conditions on the plate ($z = 0$) and at the initial section ($x = 0$). However, for large z, the expansions have the form

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

$$\begin{aligned}\psi \sim {} & A_0\xi^2z^2+\xi^3(A_1z^3+B_1z+C_1) \\ & +\xi^4(A_2z^4+B_2z^2+C_2'z\log z+C_2z+D_2+E_2z^{-2}+\cdots) \\ & +\xi^5(A_3z^5+B_3z^3+C_3'z^2\log z+C_3z^2+D_3z+E_3'\log z+E_3+\cdots) \\ & +\cdots,\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\theta \sim \tfrac{1}{27}[& \alpha_1+\alpha_2\xi z+\xi^2(\alpha_3z^2+\beta_3+\gamma_3z^{-1}\xi+\cdots) \\ & +\xi^3(\alpha_4z^3+\beta_4'\log z+\beta_4+\cdots)+\cdots],\end{aligned} \tag{5.2}$$

and these both diverge as $z\to\infty$. The expansions are therefore valid only in a neighbourhood of the plate, in fact for $\xi z\ll 1$. To find a solution valid for large $yx^{-\frac{1}{3}}$, we seek expansions for ψ and θ in powers of ξ, which satisfy the boundary conditions on $x=0$ and as $y\to\infty$, but not necessarily on $y=0$, and which coincide with the inner solution in some region away from the plate for each ξ.

If it is assumed that rearrangement is possible, the series (5.1) and (5.2) can be written in terms of the outer variable y. Then

$$\begin{aligned}\psi \sim {} & A_0(\tfrac{1}{3}\alpha y)^2+A_1(\tfrac{1}{3}\alpha y)^3+A_2(\tfrac{1}{3}\alpha y)^4+\cdots \\ & +\xi^2(B_1(\tfrac{1}{3}\alpha y)+B_2(\tfrac{1}{3}\alpha y)^2+B_3(\tfrac{1}{3}\alpha y)^3+\cdots) \\ & +\xi^3(C_1+(C_2+C_2'\log(\tfrac{1}{3}\alpha))(\tfrac{1}{3}\alpha y)+C_2'(\tfrac{1}{3}\alpha y\log y) \\ & +(C_3+C_3'\log(\tfrac{1}{3}\alpha))(\tfrac{1}{3}\alpha y)^2+C_3'(\tfrac{1}{3}\alpha y)^2\log y+\cdots) \\ & -\xi^3\log\xi(C_2'(\tfrac{1}{3}\alpha y)+C_3'(\tfrac{1}{3}\alpha y)^2+\cdots)+\cdots,\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\theta \sim {} & \tfrac{1}{27}(\alpha_1+\alpha_2(\tfrac{1}{3}\alpha y)+\alpha_3(\tfrac{1}{3}\alpha y)^2+\cdots) \\ & +\tfrac{1}{27}\xi^2(\beta_3+\beta_4(\tfrac{1}{3}\alpha y)+\cdots) \\ & +\tfrac{1}{27}\xi^3(\gamma_3(\tfrac{1}{3}\alpha y)^{-1}+\beta_4'\log(\tfrac{1}{3}\alpha y)+\cdots) \\ & -\tfrac{1}{27}\xi^3\log\xi(\gamma_4+\cdots),\end{aligned} \tag{5.4}$$

where α is given by (4.16). We therefore assume that for sufficiently large values of $yx^{-\frac{1}{3}}$, we can write

$$\psi=\psi_0(y)+\frac{\xi^2}{2!}\psi_2(y)+\frac{\xi^3}{3!}\psi_3(y)+\frac{\xi^3\log\xi}{3!}\bar{\psi}_3(y)+O(\xi^4), \tag{5.5}$$

and

$$\theta=\theta_0(y)+\frac{\xi^2}{2!}\theta_2(y)+\frac{\xi^3}{3!}\theta_3(y)+\frac{\xi^3\log\xi}{3!}\bar{\theta}_3(y)+O(\xi^4), \tag{5.6}$$

where $\psi_0'(y)$ and $\theta_0(y)$ are given (see 3.1) and near $y=0$,

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

$$
(5.7)\qquad
\begin{aligned}
\psi_0' &= a_1y+a_2y^2+a_3y^3+a_4y^4+\cdots,\\
\psi_2 &= \tfrac{2}{3}\alpha B_1 y+\tfrac{2}{9}\alpha^2B_1y^2+\tfrac{2}{27}\alpha^3B_3y^3+\cdots,\\
\psi_3 &= 6C_1+2\alpha(C_2+C_2'\log(\tfrac{1}{3}\alpha))y+2\alpha C_2'y\log y\\
&\quad+\tfrac{2}{3}\alpha^2(C_3+C_3'\log(\tfrac{1}{3}\alpha))y^2+\tfrac{2}{3}\alpha^2C_3'y^2\log y+\cdots,\\
\bar\psi_3 &= -2\alpha C_2'y-\tfrac{2}{3}\alpha^2C_3'y^2-\cdots;
\end{aligned}
$$

and

$$
(5.8)\qquad
\begin{aligned}
\theta_0 &= b_0+b_1y+b_2y^2+b_3y^3+\cdots,\\
\theta_2 &= \tfrac{2}{27}(\beta_3+\tfrac{1}{3}\alpha\beta_4y+\cdots),\\
\theta_3 &= \tfrac{2}{3}\gamma_3(\alpha y)^{-1}+\cdots,\\
\bar\theta_3 &= \tfrac{2}{9}\gamma_4'+\cdots.
\end{aligned}
$$

If the series (5.5) and (5.6) are substituted into equations (2.2) and (2.3), making use of (2.5), and the coefficients of ξ^r and $\xi^r\log\xi$ are equated, the following sets of equations are obtained for the functions $\psi_i(y)$ and $\theta_i(y)$:

$$
(5.9)\qquad
\begin{aligned}
\psi_0'\psi_2'-\psi_2\psi_0'' &= 0,\\
\psi_0'(\psi_3'+\tfrac{1}{3}\bar\psi_3')-\psi_0''(\psi_3+\tfrac{1}{3}\bar\psi_3) &= 6(\theta_0+\psi_0'''),\\
\psi_0'\bar\psi_3-\bar\psi_3\psi_0'' &= 0,
\end{aligned}
$$

and

$$
(5.10)\qquad
\begin{aligned}
\psi_0'\theta_2-\psi_2\theta_0' &= 0,\\
\psi_0'(\theta_3+\tfrac{1}{3}\bar\theta_3)-(\psi_3+\tfrac{1}{3}\bar\psi_3)\theta_0' &= 6\theta_0''/\sigma,\\
\psi_0'\bar\theta_3-\bar\psi_3\theta_0' &= 0.
\end{aligned}
$$

The first equation in (5.9) may be integrated directly to give

$$(5.11)\qquad \psi_2 = k\psi_0',$$

where k is a constant determined by comparing the series for ψ_2 and ψ_0' in (5.7). Hence

$$(5.12)\qquad \psi_2 = \frac{2\alpha B_1}{3a_1}\psi_0'.$$

Similarly,

$$(5.13)\qquad \bar\psi_3 = -\frac{2\alpha C_2'}{a_1}\psi_0',$$

and making use of the last two equations, the first and third in (5.10) give

$$(5.14)\qquad \theta_2 = \frac{2\alpha B_1}{3a_1}\theta_0',$$

and

$$(5.15)\qquad \bar\theta_3 = -\frac{2\alpha C_2'}{a_1}\theta_0'.$$

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

Also, since $\bar{\psi}_3$ is a multiple of ψ_0', the equation for ψ_3 can be solved to give

$$\psi_3 = 6\psi_0' \int^y \frac{\theta_0 + \psi_0'''}{\psi_0'^2}\, dy. \tag{5.16}$$

The integrand can be expanded in powers of y using the series for θ_0 and ψ_0'. Hence

$$\begin{aligned}\psi_3 = &-6\left(\frac{b_0+2a_2}{a_1}\right) + y\left(ma_1 - \frac{6a_2(b_0+2a_2)}{a_1}\right)\\ &+y \log y \left(\frac{6(b_1+6a_3)}{a_1} - \frac{12a_2(b_0+2a_2)}{a_1^2}\right) + O(y^2),\end{aligned} \tag{5.17}$$

where m is a constant of integration determined by comparison of (4.17) with the series for ψ_3 in (5.7). Thus

$$m = \{\alpha(C_2 + C_2' \log(\tfrac{1}{3}\alpha))/3 + a_2(b_0+2a_2)\}/a_1. \tag{5.18}$$

The equation for θ_3 is simplified by using equation (5.15) to give

$$\theta_3 = (6\theta_0''/\sigma + \psi_3\theta_0')/\psi_0'. \tag{5.19}$$

At this stage, it is interesting to note that not all the coefficients in the asymptotic series for the functions f_r and g_r are unknown; indeed by comparing the solutions obtained for ψ_2, ψ_3, ψ_3', with the series (5.7), we see that only one coefficient in the asymptotic series for each f_r is unknown when those in its predecessors are known; this being the coefficient of z. The reason is clear; since one of the complementary functions (i.e. f_{r2}) is a multiple of z, multiples of this function occur when finding a particular integral to the equation for f_r and the appropriate factor has to be determined in each case. Moreover, when this has been done for each f_r, the asymptotic series for the functions g_r are fully determined. In particular, we find that

$$C_2' = 3(a_1(b_1+6a_3) - 2a_2(b_0+2a_2))/(\alpha a_1^2), \tag{5.20}$$

and from equations (3.2) we note that the terms containing multiples of $\log \xi$ (that is, the functions $\bar{\psi}_3$ and $\bar{\theta}_3$) vanish identically if the first two relations in (3.2) are satisfied. Further, the fact that some of the coefficients A_i, B_i, C_i, α_i, β_i, etc., are known in terms of the coefficients in the initial profiles provides a useful check on the numerical computations.

6. Uniformly heated plate with a temperature discontinuity

The general theory developed in § 4 and § 5 is now applied to the flow induced by a vertical heated plate when there is an abrupt change of temperature from T_1 to T_2 (T_1, T_2 constants) at a distance L from the leading edge. Then L provides a typical length scale for the problem.

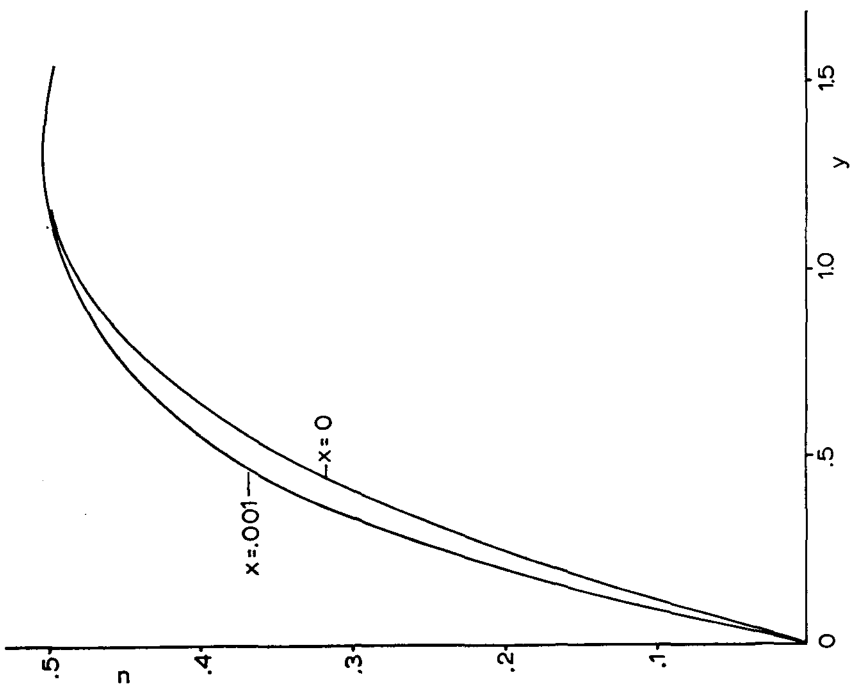

Figure 1. Velocity profiles; $\Theta_0 = 2$, $\sigma = 1$.

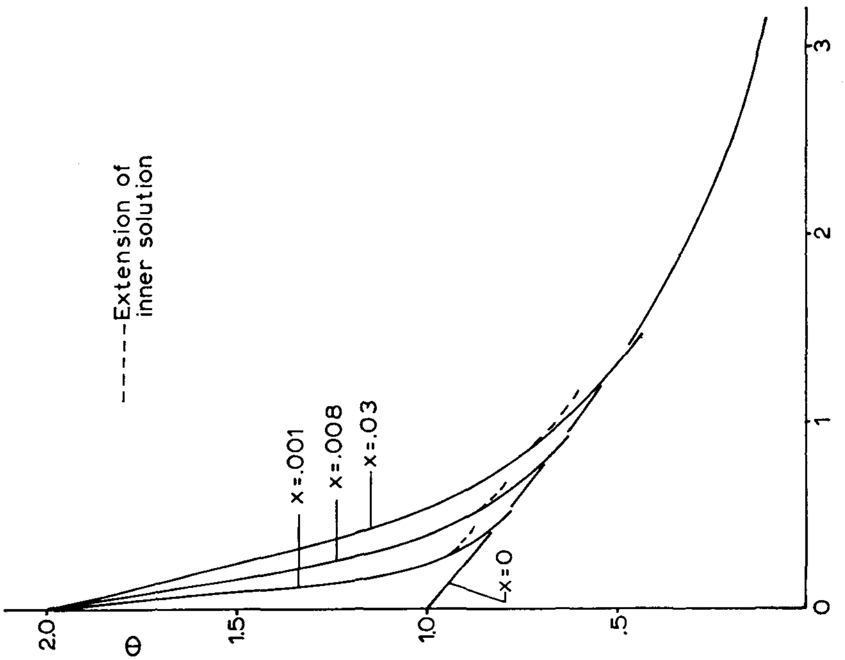

Figure 2. Temperature profiles; $\Theta_0 = 2$, $\sigma = 1$.

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

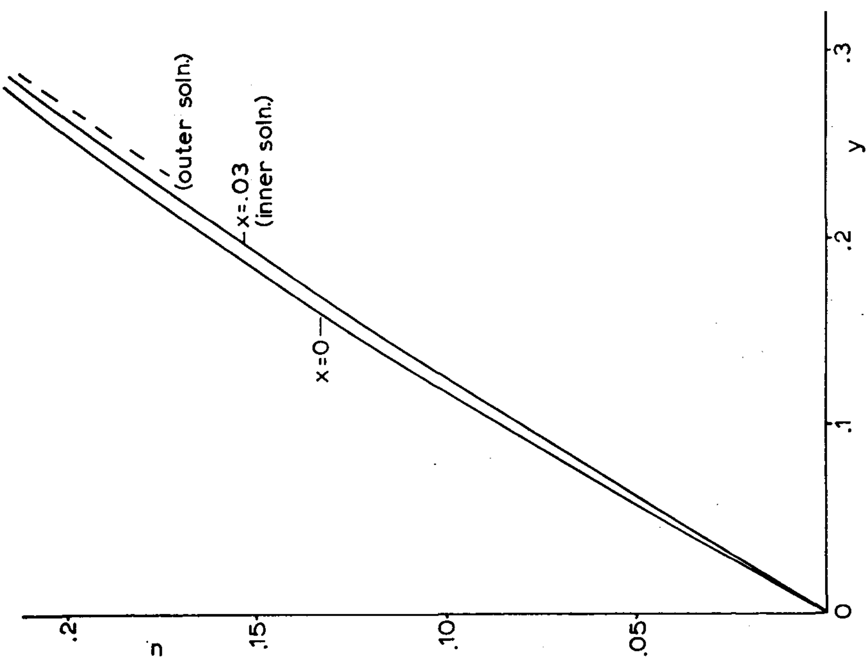

Figure 3. Velocity profiles close to the plate; $\Theta_0 = .5$, $\sigma = 1$.

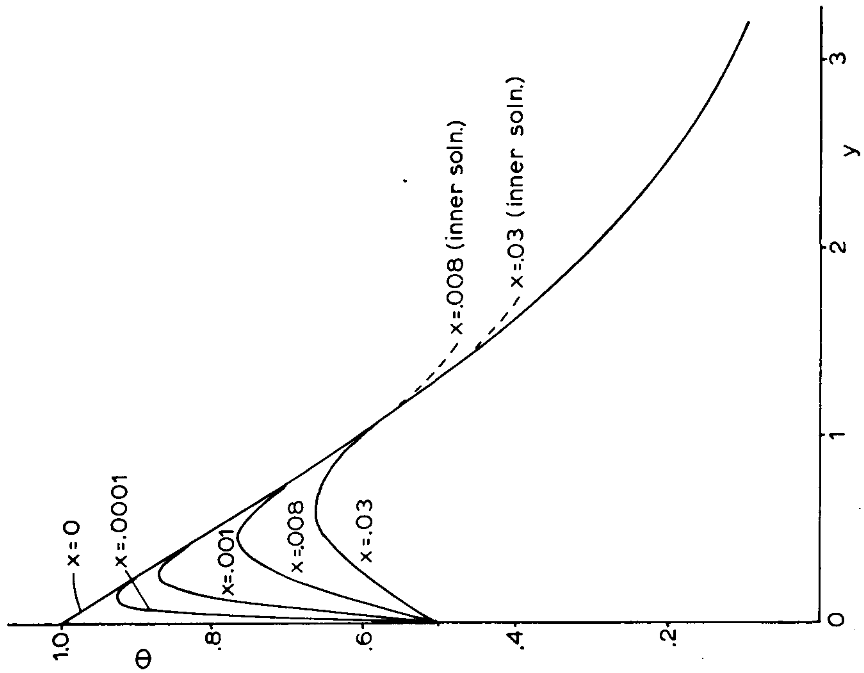

Figure 4. Temperature profiles; $\Theta_0 = .5$, $\sigma = 1$.

https://doi.org/10.1017/S1446788700006340 Published online by Cambridge University Press

Moreover, if the origin of co-ordinates is taken at the discontinuity, then the initial profiles are given by the solution to the free convection problem outlined in § 2, with x (non-dimensionalized with respect to L) taken equal to unity (then $\zeta = y$). With starting values $F''(0) = p$, $G'(0) = q$, the initial profiles are obtained by differentiating equations (2.8) and (2.9) and substituting in the Taylor series for F and G at the origin, using the boundary conditions (2.4) on $y = 0$. Hence

$$\psi_0' = py - \tfrac{1}{2}y^2 - \tfrac{1}{6}qy^3 + \tfrac{1}{96}p^2y^4 + O(y^6), \tag{6.1}$$

and

$$\theta_0 = 1 + qy - \frac{\sigma pq}{32} y^4 + O(y^5). \tag{6.2}$$

Starting values, $p(= F''(0))$ and $q(= G'(0))$ for $\sigma = 1$ are given in § 2.

Solutions are obtained for the two cases in which the temperature difference between the plate and the environment is suddenly doubled ($\Theta_0 = 2$), or halved ($\Theta_0 = 0.5$). Profiles for u and θ at various downstream positions from the singularity are shown in Figs (1)—(4) for $\sigma = 1$. These illustrate clearly the growth of the inner layer and in particular, the rapid erosion of the initial temperature profile to match the new wall condition. It is interesting to note that when the temperature difference is halved, the fluid near the wall is decelerated since this is now negatively buoyant with respect to adjacent fluid in the outer layer.

7. Discussion

The presence of a singularity (as defined in § 3) at a particular station along the plate results in the growth of an inner boundary layer from that point. As this layer spreads downstream, the original (outer) boundary layer is modified to accommodate it.

The expansions for the stream function and temperature in the inner layer (see § 4) satisfy the equations of motion and the boundary conditions on the plate, but not those as $y \to \infty$. Furthermore, these expansions also depend on the coefficient a_i, b_i and Θ_i and hence contain details of any singularity[5] at $x = y = 0$ (note, for example, that g_1 involved the difference $b_0 - \Theta_0$ and therefore describes the immediate effect on the on-coming temperature field of a temperature discontinuity on the plate at $x = y = 0$). The inner expansions are expected to be valid close to the plate, i.e. for $\eta = yx^{-\frac{1}{3}} \ll 1$ and for a range of $\xi = x^{\frac{1}{3}} < \xi_0$, say, where $\xi_0 \ll 1$.

[5] Note — the expansion method is applicable whether or not a singularity is present but the power of the technique is most evident in the former case.

The outer expansions describe the modifications of the oncoming flow *due to the presence of the inner layer. They satisfy the equations of motion* and the boundary conditions on $x = 0$ and as $y \to \infty$ for $x \geqq 0$. The forms of the expansions are obtained by the requirement that for small y, they are the same as the forms of the corresponding inner expansions for large η. Moreover, the terms in the outer expansions are determined uniquely by matching with the inner expansions as in § 5. The outer expansions are expected to be valid for a range of ξ similar to that of the inner expansions and for all but a small range of y in the neighbourhood of the plate.

Immediately downstream of the origin ($x = y = 0$), one hopes to find a range of $\xi (< \xi_1$ say), at each point of which there is a range of y (depending on ξ) in which the inner and outer expansions for ψ and θ overlap. For values of ξ slightly larger than ξ_1, it may be possible to obtain the flow profiles by interpolation between the inner and outer solutions, but as $\xi \to 1$, none of the expansions is likely to converge. Unfortunately, these series are all too complicated to allow any convergence criteria to be worked out and it is necessary to rely on an appraisal of the solutions obtained in each case. It seems reasonable to believe that the solutions obtained here are convergent at least over the range of x for which the inner and outer expansions overlap. In general, this region is that in which both the asymptotic forms of the inner solution and the series expansion about $y = 0$ of the outer solution give the same values for ψ and θ for some range of y.

In the above discussion, it has been tacitly assumed that the Prandtl number of the fluid is of order unity. If the Prandtl number is either large or small compared to one, the thermal and momentum layers thicken at widely different rates and the inner expansions (and outer expansions) for each layer would be valid in different regions in the plane. It is not then clear how one might represent the coupling term (i.e. the buoyancy term) in the inner and outer solutions. As far as the author is aware, the case of extreme Prandtl number remains unsolved.

The continuation method described in this paper could be used in principle to advance the boundary layer solution indefinitely, in a step-by-step fashion. However, the convergence of the inner and outer expansions appears to be too slow for this to be practical and with high speed computing facilities available, a direct numerical solution of the equations seems a more preferable means of continuation. A finite difference procedure has been developed to this end, for free convection boundary layers, by Merkin [3]. The present theory is primarily intended to overcome the difficulties encountered at a singularity, where numerical methods break down. For this purpose the method of matched co-ordinate expansions is a powerful one, as is clearly illustrated by the solutions to the above problems. Moreover, the expansions for ψ, and θ obtained here may be used to provide initial conditions for a numerical solution, once the singularity has been transversed.

Acknowledgement

The author wishes to express his gratitude to Mr. E. J. Watson for suggesting the problem and for a number of useful discussions.

References

[1] A. Erdelyi, *Asymptotic Expansions*, Dover, (1956), p. 21.
[2] S. Goldstein, Concerning some solutions of the boundary layer equations in hydrodynamics, *Proc. Camb. Phil. Soc.* 26 (1930), 1—30.
[3] J. Merkin, Ph. D. Thesis (Manchester University), (1968).
[4] S. Ostrach, An analysis of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force, NACA Rep. No. 1111, (1953), 17 pp.
[5] S. Ostrach, Laminar flows with body forces, Section F of Theory of Laminar Flows, (Ed. F. K. Moore), O.U.P., (1964).
[6] W. Rheinboldt, Zur Berechnung Stationärer Grenzschichten bei kontinuierlicher Absaugung mit unstetig veränderlicher Absaugegeschwindigkeit, *J. Rat. Mech. Anal.*, 5 (1956), 539—604.
[7] L. J. Slater, *The Confluent Hypergeometric Function*, C.U.P., (1960), p. 59.
[8] E. J. Watson, The Continuation problem for boundary layer flow with suction, (1967), unpublished manuscript).

The University of Manchester
Manchester, U.K.